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# STRONG UNICITY VERSUS MODULUS OF CONVEXITY

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We show that a Banach space has modulus of convexity of power type $p$ if and only if best approximants to points from straight lines are uniformly strongly unique of order $p$. Assuming that the space is smooth, we derive a characterisation of the best simultaneous approximant to two elements, and use the characterisation to prove that $p$-type modulus of convexity implies order $p$ strong unicity of the simultaneous approximant.

The study of strong unicity, initiated by Newman and Shapiro [9], resulted from a concern with the conditioning of the best approximation problem. In the present paper, the relation between strong unicity and modulus of convexity is explored. Throughout our discussion, we suppose that $(X, \|\cdot\|)$ is a Banach space with modulus of convexity $\delta > 0$, and that $V$ is a closed subspace of $X$. Bjornestal [2] showed that if $x_0 \in S(X)$ and if $\|x_0 - y\| \geqslant \|x_0\|$ for every $y \in V$, then, for every $y \in V$ with $\|y\| \leqslant 3$,

$$\|x_0 - y\| - \|x_0\| \geqslant \delta(\|y\|/4),$$

that is, the vector 0 is a *strongly unique* best approximation to $x_0$ from $V$. We shall now show that Bjornestal's inequality characterises the modulus of convexity in a certain context, and that an analogue of this inequality holds in the non-uniformly convex space which underlies the theory of simultaneous approximation. We hope that these results will contribute to a deeper understanding of uniform convexity, as well as to the more "practical" goal of finding conditions under which the best approximation operator is continuous.

We begin with precise definitions of the relevant concepts. Suppose that $f \in X$ is fixed. An element $z \in V$ is said to be a *best* $\|\cdot\|$-*approximant* to $f$ from $V$ if $\|z - f\| = \inf\{\|w - f\| : w \in V\}$. Since $X$ is uniformly convex, there is exactly one such $z$, and we denote it by $f^*$. We say that $f^*$ is $(M,\gamma)$-*strongly unique of order* $p$ if, for some $M = M(f) > 0$, there exists a $\gamma = \gamma(f, M) > 0$ such that, for all $v \in V$ with $\|v - f^*\| \leqslant M$

$$\|v - f\| \geqslant \|f^* - f\| + \gamma \|v - f^*\|^p.$$

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Two facts which demonstrate the importance of the concept of strong unicity were pointed out in [6]: Suppose $f^*$ and $g^*$ are $(M,\gamma)$-strongly unique best approximants to $f$ and $g$ respectively. If $v \in V$ and $d(f,V) \leqslant \|f - v\| \leqslant d(f,V)+\varepsilon$, then $\|v - f^*\| \leqslant (\varepsilon/\alpha)^{1/p}$, that is, the order of strong unicity measures how close a good approximation is to the best approximant. Also, $\|f^* - g^*\| \leqslant (2\|f-g\|/\gamma)^{1/p}$, that is, the nearest-point projection is locally Lipschitz of order $1/p$.

Let $S = S(X)$ be the unit sphere, $\{x \in X : \|x\| = 1\}$. The *modulus of convexity* of $X$ is the function $\delta : (0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|x+y\|/2 : x, y \in S, \|x-y\| \geqslant \varepsilon\}.$$

In some sense the modulus of convexity measures the flatest spot on the unit ball. For $p > 1$, we say that $X$ has modulus of convexity of *power type* $p$ if there exists $k \in (0,\infty)$ such that $\delta(\varepsilon) \geqslant k\varepsilon^p$. An example of such a space is $L_p[0,1]$, $1 < p < \infty$, which has modulus of convexity of power type $\max(2,p)$ [4]. A result similar to that of Bjornestal was recently published by Lin [8].

THEOREM 1. *Suppose $X$ is a uniformly convex Banach space with modulus of convexity of power type $p$, $V$ is a closed subspace of $X$, and $f \in X$. Then $f^*$ is strongly unique of order $p$.*

It is our first goal to show that Lin's theorem is essentially a characterisation. Let $\mathcal{V}$ consist of all straight lines in $X$ and, given $V \in \mathcal{V}$, let $A(V)$ be the cylindrical shell $\{x \in X : \inf_{v \in V}\|x - v\| \in [1/3, 1]\}$.

THEOREM 2. *Let $X$, $\mathcal{V}$, and $A(V)$ be as specified above. Then $X$ has modulus of convexity of power type $p$ if and only if there exist $M$ and $\gamma$ such that for every $V \in \mathcal{V}$ and for every $f \in A(V)$, $f^*$ is $(M,\gamma)$-strongly unique of order $p$.*

PROOF: Suppose that $X$ has modulus of convexity of power type $p$ and $f \in A(V)$. No generality is lost in assuming that $f^* = 0$. Our proof that $f^*$ is $(M,\gamma)$-strongly unique is essentially the same as that of [8, Theorem 1]. If $y \in V \cap S$ and $0 < a \leqslant 2\|f\|$, then

$$3\|f\| \geqslant \|f + ay\| \geqslant \|f + ay/2\| \geqslant \|f\|,$$

where the last two inequalities follow from the fact that 0 is the best approximant to $f$. Given $z \in X$, let $n(z) = z/\|f + ay\|$. By [5, Remark 1, p.125] there exists $k > 0$ such that

$$1 - \|n(f) + n(f+ay)\|/2 \geqslant k\|n(f+ay) - n(f)\|^p$$

which implies, as in [8], that $\|f + ay\| \geqslant \|f\| + ka^p3^{-p}\|f\|^{1-p}$. Thus, $f^*$ is strongly unique of order $p$, with

(i) $$M = 2\|f\| \quad \text{and} \quad \gamma = k3^{-p}\|f\|^{1-p}.$$

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Since $\|f\| \in [1/3, 1]$, $f^*$ is $(2/3, k3^{-p})$-strongly unique of order $p$.

Conversely, suppose that there exist $M$ and $\gamma$ such that $f^*$ is $(M, \gamma)$-strongly unique for every $f \in A(V)$, $V \in \mathcal{V}$. Let $E = \min(M, 2/3)$. Suppose that $x, y \in S$ and $\|x - y\| \leqslant E$. Let $W$ be the straight line containing $x$ and $y$ and let $f^*$ be the best approximant to 0 from $W$. Since $\|x - f^*\| \leqslant \|x - y\| < 2/3$ and $\|x\| = 1$, the triangle inequality shows that $\|f^*\| > 1/3$. By hypothesis, $1 - \|f^*\| \geqslant \gamma \|f^* - y\|^p$ whenever $\|x - y\| \leqslant E$. We may assume without loss of generality that $\|f^* - y\| \geqslant \|f^* - x\|$. Then $\|f^* - y\| \geqslant \|x - y\|/2$ and there exists an $\alpha \in [1/2, 1]$ such that

$$\begin{aligned} 1 - \|x + y\|/2 &= 1 - \|\alpha f^* + (1 - \alpha) y\| \\ &\geqslant 1 - \alpha \|f^*\| - (1 - \alpha) \\ &\geqslant \alpha\gamma \|f^* - y\|^p \\ &\geqslant \gamma 2^{-p-1} \|x - y\|^p. \end{aligned}$$

Thus, for $\varepsilon \leqslant E$, $\delta(\varepsilon) \geqslant k\varepsilon^p$, with $k = \gamma 2^{-k-1}$.

By [5, Remark 2, p.125] $\delta$ is an increasing function on $(0, 2]$. If $E < \varepsilon \leqslant 2$, then $\delta(\varepsilon) \geqslant \delta(E) = (\delta(E)/2^p) 2^p \geqslant (\delta(E)/2^p)\varepsilon^p$. Thus for every $\varepsilon \in (0, 2]$,

$$\delta(\varepsilon) \geqslant \min\left\{k, \frac{\delta(E)}{2^p}\right\} \varepsilon^p.$$

This concludes the proof of Theorem 2. □

The interval $[1/3, 1]$ in Theorem 2 can be generalised to $[a, b] \subset (0, \infty)$ but, by (2.i), the cylinder must have positive inner radius and finite outer radius.

We now turn our attention to the problem of strong unicity of simultaneous approximants. Suppose that $X$ is smooth and $f$ and $g$ are fixed elements of $X$. The most commonly used measure of simultaneous approximation is the vectorial norm on $X \times X$ defined by $|||(x, y)||| = \max(\|x\|, \|y\|)$. Let $\Phi : X \to \mathbb{R}$ be defined by $\Phi(v) = |||(f - v, g - v)|||$. An element $z \in V$ is said to be a *best simultaneous* $\|\cdot\|$-*approximant* of $f$ and $g$ from $V$ if $\Phi(z) = \inf\{\Phi(v) : v \in V\}$. See [7] for a proof that such a $z$ exists and is unique. We shall denote this $z$ by $(fg)^*$. We say that $(fg)^*$ is *strongly unique of order* $\alpha$ if, for some $M > 0$, there exists a $\gamma = \gamma(f, g, M) > 0$ such that, for all $v \in V$ with $\|v - (fg)^*\| \leqslant M$

$$\Phi(v) \geqslant \Phi((fg)^*) + \gamma \|v - (fg)^*\|^\alpha. \tag{3}$$

We now show that the conditioning of the simultaneous approximation problem is no worse than that of the related single approximation problems. This will be accomplished with the aid of a characterisation theorem related to a necessary condition discovered by Amir and Ziegler [1]. A loose statement of the Amir-Ziegler condition is that $(fg)^*$ is either a *relative midpoint* of $f$ and $g$ (that is, $\|f-(fg)^*\| = \|g-(fg)^*\|$) or $(fg)^*$ is either $f^*$ or $g^*$. Another necessary condition is that the product of the Gateaux derivatives of $\|f-w\|$ and $\|g-w\|$ must be nonpositive at $w=(fg)^*$. These two conditions combine to characterise the best simultaneous approximant to $f$ and $g$.

The following theorem describes this characterisation. It is in some sense a generalisation of the fact that a differentiable convex function, $u$, on $\mathbb{R}$, has a minimum at $t_0$ if and only if $u'(t_0)=0$. The discussion will be facilitated by notation for the cross-sectional distance functions and their derivatives: If $w \in V$, $h \in S$, and $t \in \mathbb{R}$, let $F_{w,h}(t) := \|f-(w+th)\|$ and $G_{w,h}(t) := \|g-(w+th)\|$. By [3], the derivatives $f_{w,h} := F'_{w,h}(0)$ and $g_{w,h} := G'_{w,h}(0)$ both exist. Note that

$$f_{w,-h} = -f_{w,h} \quad \text{and} \quad g_{w,-h} = -g_{w,h}. \tag{4}$$

THEOREM 5. *Consider the following conditions.*

(i) $f_{w,h}g_{w,h} \leqslant 0$ *for every* $h \in S$,

(ii) $\|f-w\| = \|g-w\|$,

(iii) $\|g-w\| < \|f-w\|$ *and* $w = f^*$,

(iv) $\|f-w\| < \|g-w\|$ *and* $w = g^*$.

*In the above context,* $w = (fg)^*$ *if and only if* (i) *and exactly one of* (ii), (iii) *or* (iv) *hold.*

PROOF: Suppose first that $w = (fg)^*$. If $w$ does not satisfy (i), then, by (4), there exist $h \in S$ and $\alpha > 0$ such that $f_{w,h} < 0$ and $g_{w,h} < 0$, so $\Phi(w+\alpha h) < \Phi(w)$. That exactly one of (ii), (iii) or (iv) holds is proven in [1].

For the converse, suppose that (i) and (ii) hold. There are several possibilities for the sign of $f_{w,h}$ and $g_{w,h}$. The case $[f_{w,h} < 0,\ g_{w,h} < 0]$ is ruled out by (i), as is the case $[f_{w,h} > 0,\ g_{w,h} > 0]$. If $f_{w,h} > 0$, and $g_{w,h} < 0$, then, since $F_{w,h}$ is convex and $f_{w,h} > 0$, it must be that $F_{w,h}(t) > F_{w,h}(0)$ for every $t > 0$. Similarly, $G_{w,h}(t) > 0$ for every $t < 0$. Thus, $\max(F_{w,h}(t), G_{w,h}(t))$ is minimal at $t = 0$. Clearly the case $[f_{w,h} < 0,\ g_{w,h} > 0]$ is symmetric to the case $><$. Suppose that $f_{w,h} = 0$. Since $\|\cdot\|$ is uniformly convex, $F_{w,h}$ is strictly convex, so it has a unique minimum, that is, $F_{w,h}(t) > F_{w,h}(0)$ for every $t \neq 0$. The case $g_{w,h} = 0$ is treated similarly. Since this applies to every direction in $S$, it follows that $\Phi$ has its minimum at $w$.

If (iii) holds, then the result follows from the fact that, for every $v \in S$, $\|f-(f^*+tv)\|$ has its unique minimum at $t=0$. A similar argument applies if (iv) holds. ∎

We now use the characterisation in Theorem 5 to prove that $p$-type modulus of convexity implies order $p$ strong unicity of the simultaneous approximant.

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**THEOREM 6.** *Suppose $X$ is a smooth uniformly convex Banach space with modulus of convexity of power type $p$ and $V$ is a closed subspace of $X$. Then, for any $f, g \in X$, $(fg)^*$ is strongly unique of order $p$.*

PROOF: Suppose first that $(fg)^* = f^*$. By Theorem 1, there exist $M > 0$ and $\gamma > 0$ such that

$$\text{(i)} \qquad \|g - v\| \geqslant \|(fg)^* - g\| + \gamma \|v - (fg)^*\|^p$$

whenever $\|v - (fg)^*\| \leqslant M$. By Theorem 5, $\|g - (fg)^*\| \geqslant \|f - (fg)^*\|$. Thus (i) implies (3). A symmetric argument applies if $(fg)^* = g^*$.

For the remainder of this proof, we shall suppose that $(fg)^*$ is neither $f^*$ nor $g^*$. In this case, Theorem 5 entails that $\|f - (fg)^*\| = \|g - (fg)^*\|$. We shall assume without loss of generality that $(fg)^* = 0$.

For every $0 \neq h \in K$, either $f_{0,h} \geqslant 0$ or $g_{0,h} \geqslant 0$. We will assume without loss of generality that $f_{0,h} \geqslant 0$. For every $h \in K$ with $0 < \|h\| \leqslant 2\|f\|$ it follows from the definition of $f_{0,h}$ and the convexity of the norm that

$$3\|f\| \geqslant \|f - h\| \geqslant \|f - h/2\| \geqslant \|f\|.$$

Then, as in the proof of Theorem 1 in [8] (with $x$, $y$, and $a$ replaced by $f$, $h/\|h\|$ and $-\|h\|$), we have

$$\|f - h\| \geqslant \|f\| + \gamma \|h\|^p$$

where $\gamma = k3^{-p}\|f\|^{1-p}$. Since $\|f\| = \|g\|$,

$$\begin{aligned} \max(\|f - h\|, \|g - h\|) &\geqslant \|f - h\| \\ &\geqslant \|f\| + \gamma \|h\|^p \\ &= \max(\|f\|, \|g\| + \gamma \|h\|^p) \\ &= \max(\|f\|, \|g\|) + \gamma \|h\|^p. \end{aligned}$$

This concludes the proof. ∎

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