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$H = \lambda G$ AND THE PALM TRANSFORMATION

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Abstract

We show that the stationary version of the queueing relation $H = \lambda G$ is equivalent to the basic Palm transformation for stationary marked point processes.

CONSERVATION LAWS; CAMPBELL'S FORMULA; PALM INVERSION FORMULA; STATIONARY MARKED POINT PROCESS

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1. Introduction and summary

The purpose of this note is to show that the stationary version of the queueing relation $H = \lambda G$ is actually equivalent to the fundamental Palm transformation for stationary marked point processes, as discussed in Franken et al. (1981), Rolski (1981), Baccelli and Brémaud (1987), Chapter 7 of Walrand (1988), and Brandt et al. (1990). Drawing on Franken (1976), Stidham (1979), (1982) first showed that a stationary version of $H = \lambda G$ follows from Campbell's formula; see Theorem 6.3.1 of Brandt, et al. (1990) and Theorem 6.2 of Whitt (1991). (A variant of this result was also established by Miyazawa (1979).) Below we show that the stationary version of $H = \lambda G$ also implies Campbell's formula and that Campbell's formula implies the full Palm transformation.

The history of the queueing relations $L = \lambda W$ and $H = \lambda G$ is reviewed in Whitt (1991). The derivation of results related to $H = \lambda G$ from Campbell's formula in Franken (1976), Section 3 of Miyazawa (1979) and Section 4.2 of Franken et al. (1981) are for the special case of the $G/G/s$ queue with the first-come first-served (FCFS) discipline. However, as noted (evidently for the first time) on p. 238 of Whitt (1991), the general model considered by Stidham (1979), (1982), Brandt et al. (1990) and Whitt (1991) can always be regarded as a special case of the $G/G/\infty$ model by simply interpreting the time in system of each customer as his service time. A major contribution of Stidham (1979), (1982) was to emphasize the full generality.

Establishing the reverse implication (that $H = \lambda G$ implies Campbell's formula), which we do here, is interesting because it helps reveal a greater unity in the overall theory. Previous contributions toward establishing greater unity are contained in Brémaud (1991), Miyazawa (1990) and Sigman (1991). Brémaud (1991) showed that the rate conversation law (RCL) of Miyazawa (1983), (1985) is equivalent to the Palm inversion formula. (The RCL was originally established by Miyazawa as an elementary consequence of the Palm inversion formula.) Miyazawa (1990) then showed directly that $L = \lambda W$ and Mecke's formula (or the generalized Campbell formula) are consequences of the RCL. (These general relations can also be deduced from Brémaud (1991), because Mecke's formula can be deduced from the Palm inversion formula; for example, see Section 3.2 of Rolski (1981).) Finally, Sigman (1991) established an equivalence between the sample-path-average versions of $H = \lambda G$ and the RCL. Given these results, our result should not be considered surprising. Indeed, an alternative to our proof is a stationary analog of Sigman (1991), which can be done, combined with Brémaud (1991), but the direct proof here is different. Sigman (personal communica-

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tion) has recently also shown that the Palm inversion formula can easily be deduced from $H=\lambda G$.

All these results show that there is a greater unity in the overall theory than previously realized. This unity is further emphasized in a new paper by Brémaud (1993). Brémaud (1993) establishes generalizations of the basic Palm formulae similar to extensions of the sample-path version of $H=\lambda G$ in Glynn and Whitt (1989). Brémaud shows that the equivalence established here can also be demonstrated by directly constructing $H=\lambda G$ as an application of the generalized Campbell formula instead of the Campbell formula. Indeed, he points but that $H=\lambda G$ can essentially be identified with the generalized Campbell formula.

In Section 2 we relate Campbell's formula to the basic Palm transformation and in Section 3 we relate $H=\lambda G$ to Campbell's formula.

2. The Palm transformation and Campbell's formula

Let $(A, M)\equiv\{(A_k, M_k):-\infty<k<\infty\}$ be a *synchronous* (*discrete-time*) *stationary marked point process,* with A_k being the kth point, $A_0=0$ and M_k being the kth mark. We assume that the point process A has *finite intensity* and is *simple* ($A_k<A_{k+1}$ for all k with probability 1). Let (A', M') be the associated (continuous-time) *stationary marked point process* with $A'_0<0<A_1$. Let $(A, M)+t=\{(A_k-t, M_k):-\infty<k<\infty\}$, corresponding to moving the origin to t. (This notation follows Walrand (1988).)

In this framework the *basic Palm transformation* can be expressed as

$$\lambda l(I)\boldsymbol{P}((A, M)\in B)=\boldsymbol{E}\sum_{k=-\infty}^{\infty}1((A', M')+A'_k\in B, A'_k\in I), \tag{2.1}$$

where λ is the intensity and $l(I)$ is the Lebesgue measure of the (measurable) set I, which we can take to be $[0, 1]$; see (3.1.1) on p. 9 of Baccelli and Brémaud (1987). As indicated on p. 11 of Baccelli and Brémaud (1987), (2.1) is equivalent to the *generalized Campbell formula* (or Mecke's formula), which in turn implies *Campbell's formula.*

$$\lambda\boldsymbol{E}\int_{-\infty}^{\infty}f(t, M_0)\,dt=\boldsymbol{E}\sum_{k=-\infty}^{\infty}f(A'_k, M'_k), \tag{2.2}$$

which is valid for any non-negative measurable real-valued function f.

However, as is evidently reasonably well known, we can also use (2.2) to derive (2.1). To do so, we consider new marks. The new kth mark is the entire synchronous marked point process centered at A_k; i.e. we let

$$\tilde{M}_k=(A, M)+A_k\quad\text{for each } k. \tag{2.3}$$

Then we apply (2.2) to the synchronous marked point process $(A, \tilde{M})=\{(A_k, \tilde{M}_k):-\infty<k<\infty\}$. We obtain (2.1) from (2.2) by using the function

$$f(a, \tilde{m})=1\{\tilde{m}\in B, a\in I\}. \tag{2.4}$$

3. $H=\lambda G$ and Campbell's formula

For the stationary version of $H=\lambda G$, we have a synchronous marked point process $(A, C)\equiv\{(A_k, C_k):-\infty<k<\infty\}$, where the kth mark C_k is a stochastic process (which we can take to be in the function space $D(-\infty, \infty)$ of right-continuous real-valued functions with left limits for regularity purposes), i.e. $C_k\equiv\{C_k(t-A_k):-\infty<t<\infty\}$. Again let primes denote the associated continuous-time stationary process. The quantities of interest are

$$T_k=\int_{-\infty}^{\infty}C_k(t-A_k)\,dt \tag{3.1}$$

and

$$N(t)=\sum_{k=-\infty}^{\infty}C_k(t). \tag{3.2}$$

Then Campbell's formula (2.2) implies $H = \lambda G$; i.e. if c is a possible sample path of a function C_k and

$$g(a, c) = c(a), \tag{3.3}$$

then

$$\begin{aligned} H \equiv \boldsymbol{E}N'(0) &= \boldsymbol{E} \sum_{k=-\infty}^{\infty} C_k'(0) = \boldsymbol{E} \sum_{k=-\infty}^{\infty} g(A_k', C_k') \\ &= \lambda \int_{-\infty}^{\infty} \int g(a, c) \boldsymbol{P}(C_0 \in dc)\, da \\ &= \lambda \int_{-\infty}^{\infty} \int c(a) \boldsymbol{P}(C_0(\cdot - A_0) \in dc)\, da \\ &= \lambda \boldsymbol{E} \int_{-\infty}^{\infty} C_0(a)\, da = \lambda \boldsymbol{E} T_0 \equiv \lambda G, \end{aligned} \tag{3.4}$$

as shown in p. 201 of Brandt et al. (1990) and Sections 3 and 6.1 of Whitt (1991). Of course, the canonical example arises when $C_k(t)$ is 1 when customer k is in the system and 0 otherwise; i.e. then $C_k(t)$ typically is the indicator function of the interval $[A_k, D_k]$ for some $D_k \geqq A_k$. Then T_k in (3.1) is the time customer k spends in the system and $N(t)$ is the number of customers in the system at time t. In this canonical example (3.4) produces the stationary version of $L = \lambda W$.

Now we want to derive Campbell's formula in the form (2.2) from $H = \lambda G$ here. Given the synchronous marked point process (A, M) associated with (2.2), let the marks for $H = \lambda G$ be defined by

$$C_k(t) = f(A_k + t, M_k), \qquad -\infty < t < \infty. \tag{3.5}$$

Then $(A, C) \equiv \{(A_k, C_k(\cdot - A_k)) : -\infty < t < \infty\}$ is a synchronous marked point process. We obtain (2.2) from (3.4) because

$$H \equiv \boldsymbol{E}N'(0) = \boldsymbol{E} \sum_{k=-\infty}^{\infty} C_k'(0) = \boldsymbol{E} \sum_{k=-\infty}^{\infty} f(A_k', M_k') \tag{3.6}$$

and

$$\lambda G \equiv \lambda \boldsymbol{E} T_0 = \lambda \boldsymbol{E} \int_{-\infty}^{\infty} C_0(a)\, da = \lambda \boldsymbol{E} \int_{-\infty}^{\infty} f(a, M_0)\, da. \tag{3.7}$$

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