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ON CERTAIN PROBLEMS IN THE THEORY OF SEQUENCES

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1. **Introduction.** We are well-acquainted with the theorem about sequences which states that, the existence of

$$\lim_{n\to\infty}\sum_{k=0}^{n} a_k \tag{1}$$

is sufficient to imply $\lim_{k\to\infty} a_k=0$. Partially out of a growing interest in the theory of regularly varying sequences ([1]), and probably as an interesting problem, in and of itself, some mathematicians have tried to find conditions weaker than (1) that would guarantee $\lim_{k\to\infty} a_k=0$. This was the subject of a previous paper (See [3]), in which I proved the following main theorem:

THEOREM 1. *Let* (a_k) *be a sequence of complex numbers, such that*

$$\lim_{n\to\infty}\sum_{k=[\lambda n]+1}^{n} a_k \tag{2}$$

exists for $\lambda=\xi$ *and* $\lambda=1-\xi$, *where* ξ *is an irrational number in* $(0, 1)$. *Then* $\lim_{k\to\infty} a_k=0$.

In this paper, we ask under what conditions on a set E of real numbers will the sequence (a_k) converge to zero if

$$\lim_{n\to\infty}\sum_{k=n+1}^{[\lambda n]} a_k \tag{3}$$

is equal to zero, for every fixed $\lambda\in E$?

Interestingly, (3) can hold for every $\lambda\in Z^+$, but the sequence (a_k) need not converge to zero. The counterexample which verifies this assertion is based on a construction of J. Galambos and E. Seneta ([2]). They define a sequence (b_n) as follows: For each $n\geq 2$, let $b_n=w(n)+(\log\log n)^{1/2}$, where $w(n)$ denotes the number of prime divisors of n.

Using the fact that there exists a subsequence (p_{j_n}) of primes, such that $w(p_{j_n}-1)\sim\log\log p_{j_n} (n\to\infty)$, it is asserted in [2] that

$$\lim_{n\to\infty} b_{p_{j_n}}/b_{p_{j_n}-1} \tag{4}$$

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is equal to zero. Moreover, for $k \geq 1$, it is shown that

$$\lim_{n\to\infty} b_{nk}/b_n \tag{5}$$

is equal to 1.

From the above sequence (b_n), it is easy to define a sequence (a_n), such that (3) holds for every $n \in Z^+$ and $\lim a_n \neq 0$. Define (a_n) as follows: For each $n \geq 2$, let $a_n = \log(b_n/b_{n-1})$.

Then, by (5), we see that

$$\lim_{n\to\infty} \sum_{j=n+1}^{nk} a_j = \lim_{n\to\infty} \log(b_{nk}/b_n) = 0.$$

On the other hand, by (4), we have

$$\lim_{n\to\infty} a_{p_{j_n}} = \lim_{n\to\infty} \log(b_{p_{j_n}}/b_{p_{j_n}-1}) = -\infty$$

Since $(a_{p_{j_n}})$ is a subsequence of (a_n), we have $\lim_{k\to\infty} a_n \neq 0$. Therefore, the sequence (a_n) is a counterexample.

It is true, however, that if (3) holds for every λ in E, a 2nd category subset of $(1, \infty)$, then $\lim_{k\to\infty} a_n$ does equal zero.

We can state this result more precisely as the following theorem:

THEOREM 2. *Let (a_k) be a sequence of complex numbers, such that*

$$\lim_{n\to\infty} \sum_{k=n+1}^{[\lambda n]} a_k = \phi(\lambda) \tag{6}$$

for every fixed λ in a 2nd category subset E of $(1, \infty)$. *If ϕ is continuous on E, then* $\lim_{k\to\infty} a_k = 0$.

The proof of Theorem 2 can be modified to deduce the following theorem as well:

THEOREM 3. *Let (a_k) be a sequence of complex numbers, such that*

$$\lim_{n\to\infty} \sum_{k=[\lambda n]+1}^{n} a_k = \phi(\lambda) \tag{7}$$

for every fixed λ in a 2nd category subset E of $(0, 1)$. *If ϕ is continuous on E, then* $\lim_{k\to\infty} a_k = 0$.

2. **Proof of Theorem** 2. Throughout this argument, $A_n(\lambda)$ will denote

$$\sum_{k=n+1}^{[\lambda n]} a_k$$

Let $\varepsilon > 0$. For every positive integer N, define the sets S_N as follows:

$$S_N = \left\{\lambda:\quad \text{for all}\quad n > N,\ |A_n(\lambda) - \phi(\lambda)| \leq \frac{\varepsilon}{2}\right\}$$

Then, by hypothesis, $S = \cup_{N\in Z^+} S_N$ is a 2nd category subset of $(1, \infty)$.

By Baire's Category Theorem, S cannot be the countable union of nowhere dense sets. Hence, there exists a positive integer N', such that $\text{int}(\overline{S_{N'}})\neq\varnothing$. Let α_0 be an irrational number in $\text{int}(\overline{S_{N'}})$. Choose $\delta>0$ so small that the interval $I=(\alpha_0-\delta, \alpha_0+\delta)$ is contained in $\overline{S_{N'}}$.

We assert that every irrational number $\alpha \in I$ is also an element of $S_{N'}$. To see this, let $\alpha \in I$ be irrational. Since $I\subset \overline{S_{N'}}$, there exists a sequence $(a_m)\subset S_{N'}$, such that $\lim_{m\to\infty} a_m=\alpha$. Let n be any integer greater than N'. Clearly, $\lim_{m\to\infty} a_m n=\alpha n$. Since the greatest integer function [ ] is discontinuous only at integers and αn is irrational, [ ] is continuous at αn. Hence, for m sufficiently large, we have $[a_m n]=[\alpha n]$. This implies

$$\begin{aligned}|A_n(\alpha)-\phi(\alpha)| &= \lim_{m\to\infty} |A_n(a_m)-\phi(\alpha)|\\ &\leq \lim_{m\to\infty} (|A_n(a_m)-\phi(a_m)|+|\phi(a_m)-\phi(\alpha)|)\\ &\leq \frac{\varepsilon}{2}\end{aligned}$$

Therefore, $\alpha \in S_{N'}$.

Choose N'' so large that $\alpha_0/(N''-1)$ is less than δ. Let $\bar{N}=\max(N', N'')+1$. For $n>\bar{N}$, α_0 and $\alpha_0+\varepsilon_n$ are irrational numbers in I, where $\varepsilon_n=\alpha_0/(n-1)$. Hence, α_0 and $\alpha_0+\varepsilon_n$ are in $S_{N'}$. Therefore,

$$|A_n(\alpha_0)-\phi(\alpha_0)| \leq \frac{\varepsilon}{2}$$

and

$$|A_{n-1}(\alpha_0+\varepsilon_n)-\phi(\alpha_0+\varepsilon_n)| \leq \frac{\varepsilon}{2}.$$

Since

$$|a_n| = |A_{n-1}(\alpha_0+\varepsilon_n)-A_n(\alpha_0)|.$$

we have, by the triangular inequality,

$$|a_n| \leq |A_{n-1}(\alpha_0+\varepsilon_n)-\phi(\alpha_0+\varepsilon_n)|+|A_n(\alpha_0)-\phi(\alpha_0)|+|\phi(\alpha_0+\varepsilon_n)-\phi(\alpha_0)|.$$

Therefore,

$$\limsup_{n\to\infty} |a_n| \leq \varepsilon.$$

which proves our theorem.

References

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