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REFLEXIVE MODULES OVER QF-3 RINGS

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ABSTRACT. We give a characterization of reflexive modules over QF-3 rings generalizing the concept of linearly compact modules. Further, we study necessary and sufficient conditions for left QF-3 rings to be right QF-3.

1. **Introduction.** Let R be a ring with identity. R is called *left* QF-3 (cf. [10]), if R has a unique minimal faithful left module, which is isomorphic to a direct summand of every faithful left module. (In the following QF-3 rings mean left and right QF-3.) Especially, if R is an injective cogenerator as a left R-module, R is said to be a *left PF-ring*.

On the other hand, a left module M over a ring R is said to be *linearly compact*, if every finitely solvable system of congruences $\{x \equiv x_i \bmod M_i\}_{i \in I}$, where M_i is a submodule of M, is solvable. It is to be noted that Artinian modules are linearly compact (see [11]). From a result of Müller [5] a left PF ring R is right PF, iff R is linearly compact as a left R-module. Furthermore, reflexive left modules over (two-sided) PF-rings are precisely linearly compact left modules. The purpose of this paper is to study the property of QF-3 rings generalizing this concept of linearly compact modules. In Theorem 3 we shall give a characterization of reflexive modules over QF-3 rings extending the above results of Müller, so that we are able to study the left-right symmetry of one sided QF-3 rings. It is known that semi-primary left QF-3 rings with *acc* (and hence *dcc*) on annihilator left ideals are right QF-3 (see [1], [4]). Concerning this result in Theorem 5 we shall obtain necessary and sufficient conditions for left QF-3 rings to be right QF-3.

2. **Reflexive modules.** Throughout this paper every ring has an identity and every homomorphism between modules will be written on the opposite side of scalars. We denote by $I({}_RM)$ the injective hull of the left module M over a ring R. A left ideal D of R is called *dense*, if $\mathrm{Hom}\big({}_RR/D, {}_RI({}_RR)\big) = 0$. In the following let us denote by $Q(\ )$ the localization functor with respect to the Lambek torsion theory, which is cogenerated by $I({}_RR)$ (cf. [9]).

LEMMA 1. *Let U be a faithful left module over a ring R such that $I({}_RU)$ is torsionless and $T = \mathrm{End}({}_RU)$. Let M be a left R-module, B a right T-submodule of* $\mathrm{Hom}({}_RM, {}_RU)$ *and $\psi \in \mathrm{Hom}(B_T, U_T)$. If $g_1, g_2, \ldots, g_n \in B$, then there exists a dense left ideal D of R such that for every $d \in D$ we can select $m \in M$ satisfying $d\psi(g_i) = (m)g_i$, $1, 2, \ldots, n$.*

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PROOF. Put $K = \{((x)g_1, (x)g_2, \ldots, (x)g_n); x \in M\}$, which is a submodule of the left R-module $\oplus_{i=1}^n U$. Since U is a Lambek torsion free R-module, $K \subseteq Q(K) \subseteq \oplus_{i=1}^n I(_RU)$. Suppose that $(\psi(g_1), \ldots, \psi(g_n))$ is not contained in $Q(K)$. Since $_RU$ is faithful and $I(_RU)$ is torsionless, we can easily see that $I(_RR)$ is U-torsionless (i.e., it is embedded in a direct product of copies of U), and then so is $\oplus_{i=1}^n U/[Q(K) \cap \oplus_{i=1}^n U]$. Therefore, there exists $t: \oplus_{i=1}^n U \longrightarrow U$ which vanishes on K but not on $(\psi(g_1), \ldots, \psi(g_n))$. Let $t_i \in T$ be the canonical mapping induced from t such that $t = \sum_{i=1}^n t_i$. Then, we have a contradiction, since $\psi(\sum_{i=1}^n g_i t_i) \neq 0$ and $(x)[\sum_{i=1}^n g_i t_i] = 0$ for every $x \in M$. Hence the consequence is immediate.

LEMMA 2. *Assume U is the same as in Lemma 1 and there exists an idempotent $f \in R$ such that RfR is a minimal dense left ideal of R. Assume M is a U-torsionless left R-module such that for any finitely solvable system of congruences $\{x \equiv fx_i \operatorname{Mod} M_i\}_{i \in I}$, where M_i is a submodule of M, is solvable. Let $M^* = \operatorname{Hom}(_RM, {}_RU)$ and $M^{**} = \operatorname{Hom}(M_T^*, U_T)$. Then, $_RM$ is embedded in $_RM^{**}$ canonically satisfying $\operatorname{Hom}(_RM^{**}/M, {}_RI(_RR)) = 0$.*

PROOF. Let $\psi \in M^{**}$. Since f is contained in every dense left ideal, by Lemma 1 for every $g \in M^*$ there exists $m_g \in M$ such that $f\psi(g) = (m_g)g$. We may assume $m_g = fm_g$. Let $g_1, \ldots, g_n \in M^*$. Then by Lemma 1, again, there exists $a \in M$ such that $f\psi(g_i) = (a)g_i = (m_{g_i})g_i$, $i = 1, \ldots, n$. It follows $\{x \equiv fm_g \operatorname{Mod} \ker g\}_{g \in M^*}$ is a finitely solvable system of congruences and hence solvable. Thus, there exists $x \in M$ such that $f\psi(g) = (x)g$ for all $g \in M^*$ and this implies $(RfR)M^{**}$ is embedded in M. This completes the proof.

It is well known that a minimal faithful (injective) left module over a left QF-3 ring R is isomorphic to a left ideal Re, where $e^2 = e \in R$. In this case, ReR is a minimal dense right ideal of R (see [7], Theorem 1.4). In the following, a submodule M of a left module P over a ring R is said to be *R-closed* in P, if P/M is torsionless.

THEOREM 3. *Let R be a QF-3 ring with a minimal faithful right module fR, with $f^2 = f \in R$. Then, the following conditions are equivalent for a left R-module M.*

(i) M is reflexive.

(ii) (a) M is embedded into $\prod_{\lambda \in \Lambda} R^{(\lambda)}$, a direct product of copies of the left R-module R, as an R-closed submodule.

(ii) (b) Any finitely solvable system of congruences $\{x \equiv fx_i \operatorname{Mod} M_i\}_{i \in I}$, where M_i is a submodule of M, is solvable.

PROOF. (i) $\Rightarrow$ (ii). By ([3], Lemma 1.5) (a) is evident, since $M \cong \operatorname{Hom}(M_R^*, R_R)$, where $M^* = \operatorname{Hom}(_RM, {}_RR)$. Let $\{x \equiv fx_i \operatorname{Mod} M_i\}_{i \in I}$ be a finitely solvable system, $M_i \subseteq M$. Put $K_i = Q(M_i) \cap M$ and $N_i = \{g \in M^*; (K_i)g = 0\}$. Assume $n \in \sum_{i \in I} N_i$ and A is a finite subset of I such that $n = \sum_{i \in A} n_i$, where $n_i \in N_i$. It is evident that $\{x \equiv fx_i \operatorname{Mod} K_i\}_{i \in I}$ is a finitely solvable system. Then there is a well defined R-homorphism $\theta: \sum_{i \in I} N_i \longrightarrow fR$ by $\theta(n) = \sum_{i \in A}(fx_i)n_i$. Since fR_R is injective, θ is extended to an element of $M^{**} = \operatorname{Hom}(M_R^*, R_R)$. Now, M is reflexive, then there exists $x \in M$ such that $(x)n_i =$

$\theta(n_i) = (fx_i)n_i$ for any $n_i \in N_i$, $i \in I$. Suppose there exists $j \in I$ such that $fx_j - x$ is not contained in K_j. Since M/K_j is torsionless, we can select $n_j \in N$ such that $(fx_j - x)n_j \neq 0$. This is a contradiction. Thus, we have $fx_i - x \in K_i$, $i \in I$. As f is contained in a minimal dense left ideal of R, $fx_i - fx = f(fx_i - x) \in M_i$ and thus (b) holds.

(ii) $\Rightarrow$ (i). Let M^* and M^{**} be the same as in the proof of (i) $\Rightarrow$ (ii). Let $p_\lambda : M \longrightarrow R$ be the canonical mapping such that $\bigcap_{\lambda\in\Lambda} \ker p_\lambda = 0$ in (b). Define an R-homomorphism $\psi : M^{**} \longrightarrow \prod_{\lambda\in\Lambda} R^{(\lambda)}$ by $(x)\psi \cdot p_\lambda = x(p_\lambda)$, $x \in M^{**}$. Suppose $\ker\psi \neq 0$, i.e., there exists $0 \neq x \in M^{**}$ such that $x(p_\lambda) = 0$ for all $\lambda \in \Lambda$. If we put $U = R$ in Lemma 2, it follows $\mathrm{Hom}\big({}_RM^{**}/M, {}_RI({}_RR)\big) = 0$. Since RfR is a minimal dense left ideal, there exists $fr \in fR$ and $0 \neq m \in M$ such that frx is identified with m. We have a contradiction, since $0 = frx(p_\lambda) = (m)p_\lambda$ for all $\lambda \in \Lambda$. Therefore, it follows $\prod_{\lambda\in\Lambda} R^{(\lambda)} \supseteq M^{**} \supseteq M$ and hence M is reflexive. Because, the fact that $\prod_{\lambda\in\Lambda} R^{(\lambda)}/M$ is torsionless implies $M = Q(M) \cap \prod_{\lambda\in\Lambda} R^{(\lambda)}$. This completes the proof.

A left module M over a ring R is said to be *$I({}_RR)$-dominant dimension $M \geqq 2$*, if $Q(M) = M$ (cf. [10]). Then, we have the following

COROLLARY 4. *Let M be a finitely generated left module over a QF-3 ring R. Then, (i) $\Longleftrightarrow$ (ii) $\Leftarrow$ (iii), where*

(i) M is reflexive.

(ii) M is embedded into $\prod_{\lambda\in\Lambda} R^{(\lambda)}$ as an R-closed submodule.

(iii) $I({}_RR)$-domi. dim. $M \geqq 2$.

Especially, if R is a maximal quotient ring (of itself), these three conditions are equivalent.

PROOF. Since a finite direct sum of copies of the left R-module R is reflexive, it is easily checked that for a finitely generated left R-module M every finitely solvable system of congruences $\{x \equiv fx_i \operatorname{Mod} M_i\}_{i\in\Lambda}$, where $M_i \subseteq M$, is solvable. Hence (i) $\Longleftrightarrow$ (ii) and (ii) $\Leftarrow$ (iii) are evident. Moreover, if R is a maximal quotient ring, (i) implies (iii) in view of [3, Lemma 1.5].

3. **Left QF-3 rings which are right QF-3.** In [6, (2.1)] it is proved that QF-3 maximal quotient rings are precisely the endomorphism rings of modules which are linearly compact, generator and cogenerator. On the other hand, if R is a left QF-3 ring with a minimal faithful left R-module Re, every simple right eRe-module is embedded in Re (cf. [7]).

Now we are able to prove the following:

THEOREM 5. *Let R be a left QF-3 ring. Then, the following conditions are equivalent.*

(i) R is right QF-3.

(ii) There exists an idempotent $f \in R$ such that

(a) RfR is a minimal dense left ideal.

(b) Every finitely solvable system of congruences $\{x \equiv fx_i \operatorname{Mod} L_i\}_{i\in I}$, where L_i is a left ideal of R, is solvable.

(iii) There exist orthogonal idempotents $f_1, f_2, \ldots, f_n$ such that

(a) $\{Rf_i/Jf_i; i = 1, \ldots, n\}$ is a representative system of all minimal non-isomorphic left ideals of R, where J is the Jacobson radical.

(b) If we put $f = f_1 + \cdots + f_n$, every finitely solvable system of congruences $\{x \equiv fx_i \bmod A_i\}_{i \in I}$, where A_i is an annihilator left ideal of R, is solvable.

PROOF. (i) $\Rightarrow$ (iii). By a well known property of QF-3 rings (cf. [10]) we can take a minimal faithful right R-module fR such that $f = f_1 + \cdots + f_n$, a finite sum of local idempotents satisfying (a) in (iii). Furthermore, since $_RR$ is reflexive, (b) is evident from Theorem 3.

(iii) $\Rightarrow$ (ii). For every dense left ideal D we see $Rf_i + D/D$ is a Lambek torsion module and hence $f_i \in D$, otherwise $Rf_i + D/D$ has a unique simple homorphic image Rf_i/Jf_i, which is torsion free. Let Re be a minimal faithful left R-module such that $e = e_1 + \cdots + e_n$, a finite sum of local idempotents, and the socle S_i of Re_i is isomorphic to Rf_i/Jf_i $(i = 1, \ldots, n)$. Let r be a non-zero element of R. There exists $a \in R$ and e_i such that $0 \neq rae_i$ and hence $b \in R$ $0 \neq brae_i \in S_i$ i.e., $Rbrae_i = S_i$. Since $f_iS_i \neq 0$, $RfRr \neq 0$ and then RfR is a minimal dense left ideal. Let $\{x \equiv fx_i \operatorname{Mod} L_i\}_{i \in I}$ be a finitely solvable system, where L_i is a left ideal. Put $A_i = Q(L_i) \cap R$. Since A_i is an annihilator left ideal, $\{x \equiv fx_i \operatorname{Mod} A_i\}_{i \in I}$ is solvable, i.e., there exists $x \in R$ such that $fx_i - x \in A_i$, $i \in I$. So $fx_i - fx = f(fx_i - x) \in L_i$.

(ii) $\Rightarrow$ (i). Let Re be the same as in the proof of (iii) $\Rightarrow$ (ii). Put $_RU_T = {}_RRe_{eRe} = {}_RM$ in Lemma 1. Let B be a right ideal of T $(= \operatorname{Hom}(_RM, {}_RU))$ and $\psi \in \operatorname{Hom}(B_T, U_T)$. If we exchange in the proof of Lemma 2 M^* for B and M^{**} for $\operatorname{Hom}(B_T, U_T)$, there exists $x \in M(= U)$ such that $f\psi(g) = (x)g$ for any $g \in B$. Then, fU is an injective right T-module. Let S be a simple right T-module, which is embedded in U. Then, $fRS \neq 0$ implies S is embedded in fU and hence U_T is cogenerator. Put $_RU_T = {}_RRe_{eRe}$ and $_RM = {}_RR$ in Lemma 2. Let $Q = \operatorname{End}(U_T)$. Since $M^{**} = Q$, $\operatorname{Hom}\big(_RQ/R, {}_RI(_RR)\big) = 0$. Hence the injective R-module U is also Q-injective and this implies U_T is linearly compact from the proof of [5, Lemma 3]. Since U is generator and cogenerator, Q is a QF-3 maximal (two-sided) quoteint ring of itself. Furthermore, $\operatorname{Hom}\big(_RQ/R, {}_RI(_RR)\big) = 0$ implies Q is a maximal left quotient ring of R. Since R is left QF-3, Q becomes a maximal two-sided quotient ring of R (cf. [10, Proposition 4.6]). Let K be a minimal faithful right Q-module contained in Q, such that $K = I(M_1) \oplus \cdots \oplus I(M_n)$, a finite direct sum of injective hulls of minimal right ideals of Q. Then, M_iReR is a minimal right ideal of R, because ReR is a minimal dense right ideal of R and for every $0 \neq a \in M_iReR$ we have $M_iReR = (aQ)ReR \subseteq aR$. On the other hand, $0 \neq RfRx \subseteq R$ for every non-zero element $x \in I(M_i)$ implies $I(M_i)$ is embedded in R, since $I(M_i)$ contains a minimal right ideal of R. Consequently, K is a minimal faithful right R-module. This completes the proof.

A ring R is called a *left linearly compact ring*, if R is linearly compact as a left R-module. In this case R is a semi-perfect ring (see [2], [8])). Therefore, we have

COROLLARY 6. *Every left linearly compact and left QF-3 ring is right of QF-3.*

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EXAMPLE. The equivalent conditions in Theorem 5 do not imply that R is a left linearly compact ring. Let Z be integers and Q rationals. Put

$$R = \begin{pmatrix} Q & 0 & 0 \\ Q & Z & 0 \\ Q & Q & Q \end{pmatrix}.$$

Clearly, R is a QF-3 ring. Let e_{ij} be the canonical matrix unit and P the set of all primes in Z. Then, $\{x \equiv pe_{22} \operatorname{Mod}[pZe_{22} + Qe_{32}]\}_{p\in P}$ is a finitely solvable system, but is not solvable.

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