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# BOUNDS ON THE DIMENSION OF MANIFOLDS WITH INVOLUTION FIXING $F^n \cup F^2$

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**Abstract.** Let $M^m$ be a closed smooth manifold with an involution having fixed point set of the form $F^n \cup F^2$, where $F^n$ and $F^2$ are submanifolds with dimensions $n$ and 2, respectively, where $n \geq 4$ is even $(n < m)$. Suppose that the normal bundle of $F^2$ in $M^m$, $\mu \to F^2$, does not bound, and denote by $\beta$ the stable cobordism class of $\mu \to F^2$. In this paper, we determine the upper bound for $m$ in terms of the pair $(n, \beta)$ for many such pairs. The similar question for $n$ odd $(n \geq 3)$ was completely solved in a previous paper of the authors. The existence of these upper bounds is guaranteed by the famous 5/2-theorem of Boardman, which establishes that, under the above hypotheses, $m \leq 5/2n$.

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**1. Introduction.** Let $F$ be a disjoint (finite) union of smooth and closed manifolds, $F = \bigcup_{j=0}^{n} F^j$, with $F^j$ denoting the union of those components of $F$ having dimension $j$ and thus $n$ being the dimension of the components of $F$ of largest dimension. Suppose that $M^m$ is an $m$-dimensional, smooth and closed manifold equipped with a smooth involution $T : M^m \to M^m$ whose fixed point set is $F$. It is well known, from equivariant bordism theory, that if $(M^m, T)$ is non-bounding then $n$ cannot be too small with respect to $m$. This fact was evidenced from an old result of Conner and Floyd (Theorem 27.1 of [**3**]), which stated: for each natural number $n$, there exists a number $\varphi(n)$ with the property that, if $m > \varphi(n)$, then $(M^m, T)$ bounds equivariantly. Later, this was explicitly confirmed by the famous 5/2-Theorem of Boardman [**1**]: if $M^m$ is nonbounding, then $m \leq 5/2n$. A strengthened version of this fact was obtained by Kosniowski and Stong [**6**]: if $(M^m, T)$ is a non-bounding involution, which is equivalent to the fact that the normal bundle of $F$ in $M^m$ is not a boundary (see [**3**]), then $m \leq 5/2n$. In particular, if $F$ is non-bounding (which means that at least one $F^j$ is nonbounding), then $m \leq 5/2n$. The generality of this last result allows the possibility that fixed components of all dimensions $j$, $0 \leq j \leq n$, occur; in this way, it is natural to ask whether there exists

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a better upper bound for $m$ when we omit some components of $F$. This is inspired by the fact that, if $F$ has constant dimension $n$ and if $m > 2n$, then $(M^m, T)$ bounds equivariantly; this result was proved by Kosniwoski and Stong [**6**]. In particular, if $F = F^n$ with constant dimension $n$ is nonbounding and if $(M^m, T)$ fixes $F$, then $m \leq 2n$. This bound is best possible, as can be seen by taking the involution $(F^n \times F^n, T)$, where $F^n$ is any non-bounding $n$-dimensional manifold (with the exception of $n = 1$ and $n = 3$) and $T$ switches coordinates. That is, $m \leq 5/2n$ can be improved to $m \leq 2n$, in a best possible way, if all $F^j$ with $j < n$ are omitted.

Once the case $F = F^n$ with constant dimension $n$ is established, the next natural step is to consider fixed sets of the form $F = F^n \cup F^j$, $j < n$. If the normal bundle of $F^j$ in $M^m$ is a boundary, it can be equivariantly removed to give a new involution, equivariantly cobordant to $(M, T)$ and with fixed point set $F^n$ (see [**3**]); that is, this case reduces to the constant dimension case. Thus, it is reasonable to suppose that the normal bundle over $F^j$ does not bound. In this case, we showed in [**8**] that for $j = n - 1$ we also have $m \leq 2n$ and that this bound is best possible. For $j = 0$, $F = F^n \cup F^0$ reduces to $F = F^n \cup \{point\}$, and the normal bundle over $F^j$ is nonbounding automatically. Concerning this case, Pergher and Stong proved [**10**] that, for each natural number $n$, $m \leq m(n)$, where the bounds $m(n)$ are described as follows: writing $n = 2^p q$, where $p \geq 0$ and $q$ is odd,

$$m(n) = \begin{cases} 2n + p - q + 1, & \text{if } p \leq q + 1 \\ 2n + 2^{p-q}, & \text{if } p \geq q. \end{cases}$$

Further, they constructed, for each $n \geq 1$, special involutions $(V^{m(n)}, T_n)$ with the fixed point set having the form $F = F^n \cup \{point\}$, thus showing that the bounds $m(n)$ cannot be improved. For $q = 1$, $m(n)$ is precisely the Boardman bound, but for $q > 1$ it is a smaller bound. The bounds $m(n)$ have a special feature: for some other values of $j$, the corresponding bounds are related to $m(n)$. In fact, in [**4**] and [**5**], Kelton studied bounds for $m$ when $F$ has the special form $F = F^n \cup RP^j$, where $RP^j$ is the $j$-dimensional real projective space, and a consequence of the obtained results is that, for $F = F^n \cup F^1$, $m \leq m(n-1) + 1$ if $n$ is odd and $m \leq m(n-1) + 2$ if $n$ is even. In addition, these bounds are best possible. In [**8**], we considered the case $F = F^n \cup F^2$, and showed that $m \leq m(n-2) + 4$ is the best possible bound in this case (it is also interesting to note the following relation between the case $j = n - 1$ and the numbers $m(n)$: $m(n - j) + 2j = m(1) + 2n - 2 = 2 + 2n - 2 = 2n$).

If $\eta \to F^n$ and $\mu \to F^j$ are the normal bundles of $F^n$ and $F^j$ in $M^m$, and if $\mu \to F^j$ and $\mu' \to F'^j$ are cobordant as bundles over $j$-dimensional and closed manifolds, that is, represent the same element in the cobordism group $\mathcal{N}_j(BO(m-j))$, then there exists an involution $(N^m, T')$, cobordant to $(M^m, T)$, and with fixed data $(\eta \to F^n) \cup (\mu' \to F'^j)$ (see [**3**]). Thus $\mu$ must be considered up to cobordism; that is, when looking for bounds, it suffices to consider the non-zero classes of $\mathcal{N}_j(BO(m-j))$ for $m > n$. Since $\mathcal{N}_j(BO(k)) \cong \mathcal{N}_j(BO(k+1))$ if $k \geq j$ and $\mathcal{N}_j(BO(k)) \subset \mathcal{N}_j(BO(k+1))$ if $k < j$, these non-zero classes are concentrated in $\mathcal{N}_j(BO(j))$ up to stability. In the cases $j = 0$ and $j = 1$, one has a unique such non-zero stable cobordism class: the class of the trivial bundle when $j = 0$ and the class of the canonical line bundle over $RP^1$ when $j = 1$. However, as it was seen in [**7**] and [**8**], in the case $j = 2$ one has seven such classes. Each one of these classes is identified by a non-zero list of three mod 2 characteristic numbers, $(a_1, a_2, a_3)$, coming from the list of characteristic classes $(w_1^2, (F^2), v_2(F^2), v_1^2(F^2))$. Specifically, we used $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ and $\beta_7$ to denote the stable cobordism classes corresponding to the lists $(1, 0, 0)$, $(1, 0, 1)$, $(1, 1, 0)$, $(1, 1, 1)$, $(0, 0, 1)$, $(0, 1, 0)$ and $(0, 1, 1)$, respectively (see Lemma 2.1 of [7] for an explicit description of these classes). The bound $m \leq m(n-2)+4$ works for any $n > 2$ and any $\beta_i$ and it was shown to be best possible via an example with $\mu \to F^2$ representing $\beta_4$. Hence this suggests the question of improving this bound for specific values of $n$ and $\beta_i$. Inspired in this setting, we define the number

$\varphi(n, \beta_i) =$ maximum $\{m \mid$ there exists an involution $(M^m, T)$ having fixed set of the form $F = F^n \cup F^2$ and such that $\mu \mapsto F^2$ represents $\beta_i\}$.

In [7], we completely solved this question for $n$ odd, showing that $\varphi(n, \beta_i) = m(n—2)+2 = n+1$ if either $n \equiv 3 \bmod 4$ and $\mu$ represents $\beta_2, \beta_3, \beta_5$ or $\beta_6$, or $n \equiv 1 \bmod 4$ and $\mu$ represents $\beta_1, \beta_2, \beta_6$ or $\beta_7$ and that $\varphi(n, \beta_i) = m(n-2)+4 = n+3$ in all the remaining cases.

This paper considers the case $n$ even. We completely solve the case in which $n \equiv 0 \bmod 4$. We also calculate $\varphi(n, \beta_i)$ in the cases:

(i) $i = 3, 5$ or $7$ and $n$ satisfies the fact that $n-2 = 2^p q$ where $q$ is odd and $p \leq q$. Note that if $n \equiv 0 \bmod 4$ then $n$ is of this form.

(ii) $i = 1$ or $4$ and $n > 2$ is any even.

Precisely, the results are summarized in the following table:

| $\beta_i$ | $n$ even | $\varphi(n, \beta_i)$ |
|---|---|---|
| $\beta_1$ | every $n$ even | $m(n-2)+2$ |
| $\beta_2$ | $n \equiv 0 \bmod 4$ | $m(n-2)+4$ |
| $\beta_3$ | $n-2 = 2^p q$, where $p \leq q$ | $m(n-2)+2$ |
| $\beta_4$ | every $n$ even | $m(n-2)+4$ |
| $\beta_5$ | $n-2 = 2^p q$, where $p \leq q$ | $m(n-2)+2$ |
| $\beta_6$ | $n \equiv 0 \bmod 4$ | $m(n-2)+4$ |
| $\beta_7$ | $n-2 = 2^p q$, where $p \leq q$ | $m(n-2)+2$ |

The cases where $n \equiv 2 \bmod 4$ and $i = 2$ or $6$, and where $n-2 = 2^p q$ with $p > q$ and $i = 3, 5$ or $7$, are left open. The difficulty in these cases consists in finding suitable maximal examples.

Section 2 shows, via a characteristic number calculation, that the bound $\varphi(n, \beta_i) \leq m(n-2)+4$ can be improved to $\varphi(n, \beta_i) \leq m(n-2)+2$ for $i = 1, 3, 5$ and $7$ (not necessarily in the best possible way). Section 3, the key point of the paper, is devoted to the construction of suitable maximal examples, which give the results of the table above; in comparison with the odd case, the examples for $n$ even require more sophistication.

**2. An improvement for the bound $\varphi(n, \beta_i) \leq m(n-2)+4$.** First we establish some notations and facts. As in Section 1, take an involution $(M^m, T)$ with fixed data of the form $(\eta \to F^n) \cup (\mu \to F^2)$, where $n > 2$ is even and $\mu \to F^2$ does not bound. Write $W(F^n) = 1 + \theta_1 + \cdots + \theta_n$, $W(\eta) = 1 + u_1 + \cdots + u_k$, $W(F^2) = 1 + w_1 + w_2$ and $W(\mu) = 1 + v_1 + v_2$ for the Stiefel–Whitney classes of $F^n$, $\eta$, $F^2$ and $\mu$, respectively. As described in Section 1, the cobordism class $\beta_i = [\mu \to F^2] \in \mathcal{N}_2(BO(m-2))$ is determined by the non-zero list of characteristic numbers $(a_1, a_2, a_3) = (w_1^2[F^2], v_2[F^2], v_1^2[F^2])$, where $[F^2]$ is the fundamental homology class of $F^2$.

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LEMMA 2.1. *If* $m > m(n-2)+2$, *then* $w_1^2 = v_1^2$.

As a consequence, one has

THEOREM 2.2. $\varphi(n, \beta_i) \leq m(n-2)+2$ *for* $i = 1, 3, 5$ *and* 7.

The following basic fact from **[3]** is needed to prove Lemma 2.1: the projective space bundles $RP(\eta)$ and $RP(\mu)$, with the standard line bundles $\lambda \to RP(\eta)$ and $\nu \to RP(\mu)$, are cobordant as elements of the bordism group $\mathcal{N}_{m-1}(BO(1))$. Then any class of dimension $m-1$, given by a product of the classes $w_i(RP(\eta))$ and $w_1(\lambda)$, evaluated on the fundamental homology class $[RP(\eta)]$, gives the same characteristic number as the one obtained by the corresponding product of the classes $w_i(RP(\mu))$ and $w_1(\nu)$, evaluated on $[RP(\mu)]$. In this setting, a very special class plays a crucial role. This class, denoted by $X$, was introduced by Pergher and Stong **[10]** to find bounds in the case $F^n \cup \{point\}$. With this same general aim, $X$ was also used in **[4]** and **[8]**. We proceed with the description of $X$ and the proof of Lemma 2.1. Write $W(\lambda) = 1 + c$. From **[2]** one knows that

$$W(RP(\eta)) = (1 + \theta_1 + \cdots + \theta_n)\{(1+c)^k + (1+c)^{k-1}u_1 + \cdots + (1+c)u_{k-1} + u_k\},$$

where we are suppressing bundle maps. For any integer $r$, one lets

$$W[r] = \frac{W(RP(\eta))}{(1+c)^{k-r}}.$$

Note that each class $W[r]_j$ is a polynomial in the classes $w_i(RP(\eta))$ and $c$. Further, these classes satisfy the following special properties (see **[10]**, Section 2):

$$\begin{aligned} W[r]_{2r} &= \theta_r c^r + \text{terms with smaller } c \text{ powers},\\ W[r]_{2r+1} &= (\theta_{r+1} + u_{r+1})c^r + \text{ terms with smaller } c \text{ powers}. \end{aligned}$$

Write $n - 2 = 2^p q$, where $p \geq 1$ and $q$ is odd and suppose first that $p < q+1$. In this case, the class $X$ is

$$X = W[2^p - 1]^{q+1-p}_{2^{p+1}-1}.W[r_1]_{2r_1}.W[r_2]_{2r_2} \cdots W[r_p]_{2r_p},$$

where $r_i = 2^p - 2^{p-i}$ for $1 \leq i \leq p$. If $p \geq q+1$, $X$ is

$$X = W[r_1]_{2r_1}.W[r_2]_{2r_2} \cdots W[r_{q+1}]_{2r_{q+1}},$$

where $r_i = 2^p - 2^{p-i}$ for $1 \leq i \leq q+1$. An easy calculation shows that $X$ has dimension $m(n-2)$; also, by using the properties of the classes $W[r]_j$ above listed, it can be proved that $X$ has the form

$$X = A_l.c^{m(n-2)-l} + \text{ terms with smaller } c \text{ powers},$$

where $A_l$ is a cohomology class of dimension $l \geq n-1$ and comes from the cohomology of $F^n$ (see **[8]** or **[10]**). Now $W[0]_1 = \theta_1 + u_1$, and so $\dim(W[0]_1^2.A_l) = \dim((\theta_1^2 + u_1^2).A_l) \geq n+1$ which comes from the cohomology of $F^n$. Therefore $W[0]_1^2.X$ is a class in $H^{m(n-2)+2}(RP(\eta), Z_2)$ with each one of its terms having a factor of dimension at least $n+1$ from $F^n$. Thus $W[0]_1^2.X = 0$. Since $m > m(n-2)+2$, one can form the

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class

$$W[0]_1^2.X.c^{m-1-(m(n-2)+2)},$$

which yields the zero characteristic number

$$W[0]_1^2.X.c^{m-1-(m(n-2)+2)}[RP(\eta)].$$

Our next task is to analyse the class associated to $\nu \to RP(\mu)$ which corresponds to $W[0]_1^2.X.c^{m-1-(m(n-2)+2)}$. Setting $W(\nu) = 1 + d$, this class is

$$W[n-2]_1^2.(RP(\mu)).Y.d^{m-1-(m(n-2)+2)},$$

where $Y$ is obtained from $X$ by replacing each $W[r]_i$ by $W[n+r-2]_i$. The Stiefel–Whitney class of $RP(\mu)$ is

$$\begin{aligned} W(RP(\mu)) &= (1 + w_1 + w_2)\{(1+d)^{n+k-2} + (1+d)^{n+k-3}v_1 + (1+d)^{n+k-4}v_2\} \\ &= (1+d)^{n+k-4}\{(1 + w_1 + w_2)\{(1+d)^2 + (1+d)v_1 + v_2\}\}. \end{aligned}$$

Then

$$W[n-2](RP(\mu)) = (1+d)^{n-4}\{(1 + w_1 + w_2)\{(1+d)^2 + (1+d)v_1 + v_2\}\}.$$

Since $n \geq 4$ is even, $(1+d)^{n-4}$ has no terms of dimension 1, and thus $W[n-2]_1^2(RP(\mu)) = (w_1 + v_1)^2 = w_1^2 + v_1^2$. Denote by $\mathcal{I}$ the ideal of $H^*(RP(\mu), Z_2)$ generated by the classes coming from $F^2$ and with positive dimension. Then $W[n-2]_1^2(RP(\mu)).v = 0$ for each $v \in \mathcal{I}$. This means that, in the computation of $Y$, one needs to consider only $W(RP(\mu)) \equiv (1+d)^{n+k-2} \bmod \mathcal{I}$ and, for each integer $l$, $W[l] \equiv (1+d)^l \bmod \mathcal{I}$. For $r_i = 2^p - 2^{p-i}$, $i = 1, 2, \ldots, p$, set $l_i = n + r_i - 2 = 2^p q + 2 + 2^p - 2^{p-i} - 2 = 2^p q + 2^p - 2^{p-i}$. Then

$$W[l_i]_{2r_i} \equiv \binom{2^p q + 2^p - 2^{p-i}}{2^{p+1} - 2^{p-i+1}} d^{2r_i} \bmod \mathcal{I}.$$

Also, if $r = 2^p - 1$, $l = n + r - 2 = 2^p q + 2^p - 1$ and

$$W[l]_{2r+1} \equiv \binom{2^p q + 2^p - 1}{2^{p+1} - 1} d^{2r+1} \bmod \mathcal{I}.$$

The lesser term of the 2-adic expansion of $2^p q + 2^p$ is $2^{p+1}$. Using the fact that a binomial coefficient $\binom{a}{b}$ is non-zero modulo 2 if and only if the 2-adic expansion of $b$ is a subset of the 2-adic expansion of $a$, we conclude that the above binomial coefficients are non-zero modulo 2. It follows that all classes $W[r]_i$ occurring in $Y$ satisfy $W[r]_i \equiv d^i \bmod \mathcal{I}$, which implies that $Y \equiv d^{m(n-2)} \bmod \mathcal{I}$. Since $H^*(RP(\mu), Z_2)$ is the free $H^*(F^2, Z_2)$-module on $1, d, d^2, \ldots, d^{n+k-3}$, we then have

$$\begin{aligned} &W[n-2]_1^2(RP(\mu)).Y.d^{m-1-(m(n-2)+2)}[RP(\nu)] \\ &\quad = \left(w_1^2 + v_1^2\right).d^{m-3}[RP(\nu)] = \left(w_1^2 + v_1^2\right)[F^2]. \end{aligned}$$

Putting together with the previous calculations on $F^n$, we get $w_1^2 = v_1^2$ and Lemma 2.1 is proved.

**3. Maximal examples.** In this section we construct examples that, together with the general bound $\varphi(n, \beta_i) \leq m(n-2)+4$ and its particular improvement given by Theorem 2.2, provide the results displayed in Section 1. We will use $\varepsilon^r \to W$ to denote the $r$-dimensional trivial vector bundle over any space $W$. As mentioned in Section 1, in [**10**] Pergher and Stong constructed, for each $n \geq 1$, maximal involutions $(V^{m(n)}, T_n)$ with the fixed point set having the form $F = F^n \cup \{point\}$.

THEOREM 3.1. *For every $n \geq 4$ even, $\varphi(n, \beta_1) = m(n-2)+2$ and $\varphi(n, \beta_4) = m(n-2)+4$.*

*Proof.* First recall that $\beta_1$ and $\beta_4$ are characterized, respectively, by the lists $(1, 0, 0)$ and $(1, 1, 1)$. Consider the involution $(V^{m(n-2)} \times RP^2, T)$ given by $T(x, y) = (T_{n-2}(x), y)$. The fixed point set of $T$ has the form $(F^{n-2} \times RP^2) \cup RP^2$, and the normal bundle of $RP^2$ in $V^{m(n-2)} \times RP^2$ is $\varepsilon^{m(n-2)} \to RP^2$, which represents $\beta_1$. This gives $\varphi(n, \beta_1) = m(n-2)+2$. Now consider the involution $(V^{m(n-2)} \times RP^2 \times RP^2, T)$, where $T(x, y, z) = (T_{n-2}(x), z, y)$. Again the fixed point set has the form $(F^{n-2} \times RP^2) \cup RP^2$, and in this case the normal bundle of $RP^2$ in $V^{m(n-2)} \times RP^2 \times RP^2$ is $\tau \oplus \varepsilon^{m(n-2)} \to RP^2$, where $\tau$ is the tangent bundle over $RP^2$; this bundle represents $\beta_4$, which gives $\varphi(n, \beta_4) = m(n-2)+4$ (this example was used in [**8**] to show that the general bound $\varphi(n, \beta_4) = m(n-2)+4$ is best possible). □

In order to obtain the next examples, we need the following:

LEMMA 3.2. *Write $n = 2^p q$, where $p \geq 1$, $q$ is odd and $p \leq q$. Then, for each $0 \leq r \leq m(n)$, there exists an involution $S : V^{m(n)} \to V^{m(n)}$ commuting with $T_n$ (thus the isolated fixed point $P$ of $T_n$ is also fixed by $S$) so that the dimension of the vector subspace of the tangent space of $V^{m(n)}$ at $P$ on which the representation of $S$ acts as $-1$ is $m(n)-r$ (equivalently, the dimension of the component of the fixed point set of $S$ containing $P$ is $r$; we say in this case that the representation of $S$ on the tangent space to $V^{m(n)}$ at $P$ has the form $R_+^r \oplus R_-^{m(n)-r}$).*

*Proof.* To construct the maximal involutions $(V^{m(n)}, T_n)$, Stong and Pergher used an inductive procedure on $p \geq 0$ starting at $q \geq 1$. So the idea for constructing $S$ is to insert suitable involutions in the steps of the induction so that in the last step we get the desired $S$. It was known that $m(q) = q+1$ and $(V^{q+1}, T_q) = (RP^{q+1}, T_q)$, where $T_q([x_0, x_1, \ldots, x_{q+1}]) = [x_0, -x_1, \ldots, -x_{q+1}]$ (see [11, Theorem 2.3, page 269]; note that the fixed point set of $T_q$ is $RP^q \cup \{point\}$), and this is the first step of the induction. The next step builds, from a previous involution $(M^m, T)$ with fixed point set of the form $F = F^n \cup \{point\}$ and with $m < 2n+1$, a new involution $(W^{m+2n+1}, \theta)$ with fixed point set of the form $F = F^{2n} \cup \{point\}$, in such a way that if $m = m(n)$, then $m+2n+1 = m(2n)$. Since $m(n) < 2n+1$ for $p < q$ and $m(n) \geq 2n+1$ for $p \geq q$, this construction can be realized only for $0 \leq p < q$, which means that the last $n$ attained is $n = 2^q q$. $W^{m+2n+1}$ is the orbit space

$$W^{m+2n+1} = \frac{S^k \times M^m \times M^m}{K},$$

where $S^k$ is the $k$-dimensional sphere, $k = 2n+1-m$ and $K(x, y, z) = (-x, z, y)$. On $S^k \times M^m \times M^m$ we define the involution $U \times T \times T$, where $U(x_0, x_1, \ldots, x_k) = (x_0, -x_1, \ldots, -x_k)$. This involution commutes with $K$ and then induces an involution on $W^{m+2n+1}$, which is our $\theta$. If $P$ is the isolated fixed point of $T$, $[((1, 0, \ldots, 0), (P, P))]$

is the isolated fixed point of $\theta$ (see [ 10, Section 4, page 83] for the details concerning the computation of the fixed point set of $\theta$. In fact, besides $F^{2n} \cup \{point\}$, $\theta$ has two more $n$-dimensional components whose normal bundles are cobordant; then, up to cobordism, they can be eliminated. These components have no influence in the argument, which involves only the isolated fixed point).

To insert suitable involutions, first define, for $0 \leq r \leq q+1$, the involution $S : RP^{q+1} \to RP^{q+1}$ given by $S([x_0, \ldots, x_{q+1}]) = [x_0, \ldots, x_r, , -x_{r+1}, \ldots, -x_{q+1}]$. Then $S$ commutes with $T_q$ and the isolated fixed point $P = [1, 0, \ldots, 0]$ of $T_q$ belongs to the component of the fixed point set of $S$ given by $RP^r = \{[x_0, \ldots, x_r, 0, \ldots, 0]\}$. This means that the representation of $S$ at $P$ is of the form $R_+^r \oplus R_-^{q+1-r}$. Now consider the involutions $(M^m, T)$ and $(W^{m+2n+1}, \theta)$ as above described, and inductively suppose one has an involution $S : M^m \to M^m$ commuting with $T$ and having representation of the form $R_+^r \oplus R_-^{m-r}$ at the isolated fixed point $P$ of $T$. Denote by $D^r \subset M^m$ the $r$-dimensional component of the fixed point set of $S$ that contains $P$. On $S^k \times M^m \times M^m$ we consider the involution $L \times S \times S$, where $L(x_0, \ldots, x_k) = (x_0, \ldots, x_j, -x_{j+1}, \ldots, -x_k)$ and $0 \leq j \leq k$. This involution commutes with $K$ and then induces an involution $\overline{S}$ on $W^{m+2n+1}$. The component of the fixed point set of $\overline{S}$ that contains the isolated fixed point $[((1, 0, \ldots, 0), (P, P))]$ of $\theta$ is

$$\frac{S^j \times D^r \times D^r}{K},$$

where $S^j \subset S^k$ consists of the points of the form $(x_0, \ldots, x_j, 0, \ldots, 0)$. This component has dimension $j + 2r$, which means that $\overline{S}$ has representation of the form $R_+^{j+2r} \oplus R_-^{m+2n+1-j-2r}$ at the isolated fixed point of $\theta$.

Now note that, by starting either at 0 or at 1, we attain any natural number $r \geq 0$ after an iterated number of steps by either doubling or doubling and adding 1 in each step. To see how to proceed in each step, write $r = 2^{p_1} q_1 = 2^{p_1}((q_1 - 1) + 1) = 2^{p_1}(2^{p_2} q_2 + 1) = 2^{p_1}(2^{p_2}((q_2 - 1) + 1) + 1) = 2^{p_1}(2^{p_2}(2^{p_3} q_3 + 1) + 1) = \cdots$, where $p_i \geq 0$ is even and $q_i$ is odd. To end the proof, we use this principle and our inductive construction of $S$ with $r = 0$ or 1 and $j = 0$ or 1. $\square$

THEOREM 3.3. *For $n \geq 4$ even, where $n - 2 = 2^p q$ with $p \leq q$, and for every $\beta_i$, there are examples of involutions $(M^m, T)$ with fixed data of the form $\eta \to F^n \cup \mu \to F^2$, where $\beta_i = [\mu \to F^2]$ and $m = m(n-2) + 2$. In particular, $\varphi(n, \beta_3) = \varphi(n, \beta_5) = \varphi(n, \beta_7) = m(n-2) + 2$ when $n - 2 = 2^p q$ with $p \leq q$.*

*Proof.* For $0 \leq r \leq m(n-2)$, take an involution $S : V^{m(n-2)} \to V^{m(n-2)}$ commuting with $T_{n-2}$ so that its representation on the tangent space to $V^{m(n-2)}$ at the isolated fixed point of $T_{n-2}$ has the form $R_+^r \oplus R_-^{m(n-2)-r}$. Consider the closed $(m(n-2)+2)$-dimensional manifold given by the orbit space

$$\frac{V^{m(n-2)} \times S^2}{\Theta},$$

where $\Theta$ is the involution $\Theta(x, y) = (S(x), -y)$. On this manifold one has the involution $B([x, y]) = [T_{n-2}(x), y]$, whose fixed point set is

$$\frac{(F^{n-2} \cup \{point\}) \times S^2}{\Theta} = \frac{F^{n-2} \times S^2}{\Theta} \quad \bigcup \quad RP^2$$

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and has in this way the form $F^n \cup F^2$. The normal bundle of $RP^2$ in

$$\frac{V^{m(n-2)} \times S^2}{\Theta}$$

is $(m(n-2)-r)\xi \oplus \varepsilon^r \to RP^2$, where $(m(n-2)-r)\xi \to RP^2$ is the Whitney sum of $m(n-2)-r$ copies of the canonical line bundle $\xi$ over $RP^2$. Since $n \geq 4$, $m(n-2) \geq 5$ and in particular this can be performed for $r = m(n-2)$, $m(n-2)-1$, $m(n-2)-2$ and $m(n-2)-3$. Using the fact that the Stiefel–Whitney class of $(m(n-2)-r)\xi$ is $W = (1+\alpha)^{m(n-2)-r}$, where $\alpha \in H^1(RP^2, Z_2)$ is the generator, we can see that these values give examples $(M^{m(n-2)+2}, T)$ realizing $\beta_1$, $\beta_2$, $\beta_3$ and $\beta_4$, respectively. Now note that if $(M^m, T)$ and $(N^m, S)$ are involutions having fixed set of the form $F^n \cup F^2$, and in such a way that $\mu \mapsto F^2$ represents $\beta_i$ for $(M^m, T)$ and $\beta_j$ for $(N^m, S)$, then $(M^m, T) \cup (N^m, S)$ has still fixed set of the form $F^n \cup F^2$ with $\mu$ representing $\beta_i + \beta_j$; also, the sum of cobordism classes of bundles is compatible with the sum mod 2 of the corresponding characteristic numbers. Then, by summing the obtained examples, we obtain examples for every $\beta_i$, which ends the proof. □

The following theorem solve the question of computing $\varphi(n, \beta_i)$ for every $\beta_i$ and $n \equiv 0 \bmod 4$.

THEOREM 3.4. *For $n \geq 4$ and $n \equiv 0$ mod 4, one has*

$$\varphi(n, \beta_i) = \begin{cases} m(n-2)+2, & \text{if } i = 1, 3, 5 \text{ or } 7, \\ m(n-2)+4, & \text{if } i = 2, 4 \text{ or } 6. \end{cases}$$

*Proof.* Because of Theorem 3.1, for $i = 1$ or 4 there is nothing to prove. Write $n = 4t$, where $t \geq 1$. Since $4t-2$ is of the form $2^p q$ with $p \leq q$, Theorem 3.3 covers the cases $i = 3$, 5 and 7. Therefore, since $\beta_2 + \beta_4 = \beta_6$, it suffices to exhibit a maximal example $(M^{m(n-2)+4}, T)$ for $i = 2$. Write $j = t-1$, that is, $n = 4j+4$ with $j \geq 0$. Note that

$$m(n-2)+4 = m(2(2j+1))+4 = 3(2j+1)+2+4 = 6j+9.$$

Consider the Dold manifold

$$M = P(2j+5, 2j+2) = \frac{S^{2j+5} \times CP^{2j+2}}{\theta}.$$

Here, $CP^{2j+2}$ is the $(2j+2)$-dimensional complex projective space and $\theta$ is the involution $\theta(x, y) = (-x, \overline{y})$, where $\overline{y}$ means complex conjugation. Note that the dimension of $M$ is $6j+9$. On $M$ one has the involution $T : M \to M$ induced by $U \times L$, where

$$U(x_0, x_1, \ldots, x_{2j+5}) = (x_0, x_1, x_2, -x_3, \ldots, -x_{2j+5})$$

and

$$L[z_0, z_1, \ldots, z_{2j+2}] = [z_0, -z_1, -z_2, \ldots, -z_{2j+2}].$$

To find the fixed set, one looks at $(U(x), L[z]) = (x, [z])$ and $(U(x), L[z]) = (-x, [\overline{z}])$. If $(U(x), L[z]) = (x, [z])$, $x = (x_0, x_1, x_2, 0, \ldots, 0) \in S^2$ and $[z] \in CP^0 \cup CP^{2j+1}$, which gives the fixed component

$$\frac{S^2 \times (CP^0 \cup CP^{2j+1})}{\theta} = \frac{S^2}{A} \quad \bigcup \quad \frac{S^2 \times CP^{2j+1}}{A \times C},$$

where $A$ is the antipodal map and $C$ is the complex conjugation; that is, this fixed component is $RP^2 \cup P(2, 2j+1)$, which has the form $RP^2 \cup F^n$. On the other hand, if $(U(x), L[z]) = (-x, [\overline{z}])$, $x = (0, 0, 0, x_3, x_4, \ldots, x_{2j+5}) \in S^{2j+2}$ and $[z] \in RP^{2j+2}$, where $RP^{2j+2} \subset CP^{2j+2}$ consists of the points of the form $[r_0, ir_1, ir_2, \ldots, ir_{2j+2}]$ with $r_i$ real (that is, the points $[z_0, z_1, z_2, \ldots, z_{2j+2}]$ with $z_0 = r_0$ real and $z_p$ pure imaginary for $p > 0$). This gives a fixed component

$$\frac{S^{2j+2} \times RP^{2j+2}}{A \times C}$$

of dimension $4j + 4 = n$. That is, the fixed point set of $(M, T)$ has the form $F^n \cup RP^2$. To find the normal bundle of $RP^2$ in $M$, first denote by $\xi \to P(2j+5, 2j+2)$ the real canonical line bundle coming from $RP^{2j+5}$ and by $\eta \to P(2j+5, 2j+2)$ the complex canonical line bundle coming from $CP^{2j+2}$. If $P(p, t) \subset P(2j+5, 2j+2)$ is a canonically embedded Dold-submanifold of $P(2j+5, 2j+2)$, with $p \leq 2j+5$ and $t \leq 2j+2$, it is known that the normal bundle of $P(p, t)$ in $P(2j+5, 2j+2)$ is $(2j + 5 - p)\xi \oplus (2j + 2 - t)\eta$, where $\xi$ and $\eta$ are restrictions of the previous $\xi$ and $\eta$, and the natural numbers express Whitney sums. For $t = 0$, the total space of $\eta \to P(p, 0)$ is

$$\frac{S^p \times CP^0 \times \mathbb{C}^1}{A \times C \times F} = \frac{S^p \times \mathbb{R}^2}{A \times F},$$

where $\mathbb{R}^2$ ($\cong \mathbb{C}^1$) is the euclidean two-dimensional space and $F(x, y) = (x, -y)$. Then, over $P(p, 0)$, $\eta$ reduces to $\xi \oplus \varepsilon^1$, and the normal bundle of $P(p, 0)$ in $P(2j+5, 2j+2)$ reduces to $(4j + 7 - p)\xi \oplus (2j+2)\varepsilon^1$. Since our $RP^2$ is

$$\frac{S^2 \times CP^0}{A \times C} = P(2, 0),$$

the normal bundle of $RP^2$ in $M$ is then $(4j+5)\xi \oplus (2j+2)\varepsilon^1 \to RP^2$. Since $4j + 5 \equiv 1$ mod 4, this bundle represents $\beta_2$ and the result follows. □

Remark. Although easy, it is illustrative and curious to give a numerical example to compare the bounds cited in Section 1. Take $n = 288$; in this case, the Boardman's bound is 720. For the case $F = F^n \cup F^j$, the (best possible) bounds are 573, 290 and 432 for $j = 0$, 1 and 2, respectively. In this paper we have seen that, for $j = 2$, the bound $m = 432$ can be improved to $m = 430$ if $[\mu \to F^2] = \beta_1, \beta_3, \beta_5$ or $\beta_7$.

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## REFERENCES

**1.** J. M. Boardman, On manifolds with involution, *Bull. Amer. Math. Soc.* **73** (1967), 136–138.

**2.** A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces, I, *Amer. J. Math*. **80** (1958), 458–538.

**3.** P. E. Conner and E. E. Floyd, *Differentiable periodic maps* (Springer-Verlag, Berlin, 1964).

**4.** Suzanne M. Kelton, Involutions fixing $\mathbb{R}P^j \cup F^n$, *Topology Appl*. **142** (2004), 197–203.

**5.** Suzanne M. Kelton, Involutions fixing $\mathbb{R}P^j \cup F^n$, II, *Topology Appl*. **149** (1–3) (2005), 217–226.

**6.** C. Kosniowski and R. E. Stong, Involutions and characteristic numbers, *Topology* **17** (1978), 309–330.

**7.** P. L. Q. Pergher and Fabio G. Figueira, Dimensions of fixed point sets of involutions, *Arch. Math. (Basel)* **87** (3) (2006), 280–288.

**8.** P. L. Q. Pergher and Fabio G. Figueira, Involutions fixing $F^n \cup F^2$, *Topology Appl*. **153** (14) (2006), 2499–2507.

**9.** P. L. Q. Pergher and Fabio G. Figueira, Two commuting involutions fixing $F^n \cup F^{n-1}$, *Geom. Dedicata* **117** (2006), 181–193.

**10.** P. L. Q. Pergher and R. E. Stong, Involutions fixing $\{\text{point}\} \cup F^n$, *Transform. Groups* **6** (2001), 78–85.

**11.** D. C. Royster, Involutions fixing the disjoint union of two projective spaces, *Indiana Univ. Math. J.* **29** (2) (1980), 267–276.