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Boundary Structure of Hyperbolic 3-Manifolds Admitting Annular and Toroidal Fillings at Large Distance

Sangyop Lee and Masakazu Teragaito

Abstract. For a hyperbolic 3-manifold M with a torus boundary component, all but finitely many Dehn fillings yield hyperbolic 3-manifolds. In this paper, we will focus on the situation where M has two exceptional Dehn fillings: an annular filling and a toroidal filling. For such a situation, Gordon gave an upper bound of 5 for the distance between such slopes. Furthermore, the distance 4 is realized only by two specific manifolds, and 5 is realized by a single manifold. These manifolds all have a union of two tori as their boundaries. Also, there is a manifold with three tori as its boundary which realizes the distance 3. We show that if the distance is 3 then the boundary of the manifold consists of at most three tori.

1 Introduction

Let M be a hyperbolic 3-manifold with a torus boundary component T_0 in the sense that M with its boundary tori removed has a complete hyperbolic structure with totally geodesic boundary. For a slope γ on T_0, $M(\gamma)$ denotes the manifold obtained by γ-Dehn filling on M. That is, $M(\gamma) = M \cup V_\gamma$, where V_γ is a solid torus glued to M along T_0 in such a way that γ bounds a disk in V_γ. A 3-manifold is said to be *annular* (resp. *toroidal*) if it contains an essential annulus (resp. torus). Suppose that $M(\alpha)$ is annular and $M(\beta)$ is toroidal for slopes α and β on T_0. Gordon [2] showed that $\Delta(\alpha, \beta) \leq 5$, where $\Delta(\alpha, \beta)$ denotes the *distance* between two slopes, which is their minimal geometric intersection number. Furthermore, Gordon and Wu [6] showed that the distance 5 is realized by a single manifold and the distance 4 is realized by two specific manifolds. These manifolds are the exteriors of the Whitehead sister (or $(-2, 3, 8)$-pretzel) link, the Whitehead link and the 2-bridge link corresponding to $3/10$, using Conway's notation, in the 3-sphere S^3. Following Gordon [3], let us define

$$\Delta^k(A, T) = \max\{\Delta(\alpha, \beta) : \text{there is a hyperbolic 3-manifold } M \text{ such that } \partial M \text{ is a disjoint union of } k \text{ tori, and } \alpha, \beta \text{ are slopes on some component of } \partial M \text{ such that } M(\alpha) \text{ is annular and } M(\beta) \text{ is toroidal}\}$$

for $k \geq 2$. (For other types $X, Y \in \{S, D, A, T\}$, $\Delta^k(X, Y)$ is defined similarly, but we do not need it.) Thus $\Delta^2(A, T) = 5$. Also, there are infinitely many hyperbolic manifolds realizing the distance three [6]. Among them, there is a hyperbolic 3-manifold,

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called the *magic manifold*, which is the exterior of a certain 3-component link in S^3. Hence $\Delta^3(A,T) = 3$. Gordon [3] gave an example showing $\Delta^k(A,T) \geq 2$ for any $k \geq 4$. Thus $\Delta^k(A,T) = 2$ or 3 for $k \geq 4$. The purpose of this paper is to determine this value.

Theorem 1.1 *Let M be a hyperbolic 3-manifold with a torus boundary component T_0 and suppose that there are two slopes α, β on T_0 such that $M(\alpha)$ is annular and $M(\beta)$ is toroidal. If $\Delta(\alpha,\beta) = 3$, then ∂M is a union of at most three tori. In particular, $\Delta^k(A,T) = 2$ for any $k \geq 4$.*

This gives a partial answer to [3, Question 5.3].

In Section 2, we prepare the basic facts about labelled graphs. In particular, the key is Lemma 2.4 which claims that neither graph contains both a black Scharlemann cycle and a white Scharlemann cycle. Section 3 is devoted to a special case where one graph has a single vertex, and Section 4 deals with the case where the graph on the annulus has two vertices. Section 5 deals with the generic case. To prove Theorem 1.1, we need to consider the situation that $M(\beta)$ contains a Klein bottle. This case is treated in Sections 6 and 7.

2 Preliminaries

An annulus or torus is *essential* if it is incompressible, boundary-incompressible and is not boundary-parallel. For two slopes α and β on T_0, we suppose that $M(\alpha)$ is annular and $M(\beta)$ is toroidal. That is, $M(\alpha)$ (resp. $M(\beta)$) contains an essential annulus (resp. torus).

To prove Theorem 1.1, we assume, by way of contradiction, from now on, that $\Delta(\alpha,\beta) = 3$ and ∂M is not a union of at most three tori.

Lemma 2.1 *$M(\alpha)$ and $M(\beta)$ are irreducible and boundary-irreducible.*

Proof Since M is large in the sense of [14], $M(\alpha)$ and $M(\beta)$ are irreducible by [14, Theorems 4.1 and 5.1]. Boundary-irreducibility follows from [5, 7]. ∎

Let $\widehat{S}$ be an essential annulus in $M(\alpha)$. For a core K_α of the attached solid torus V_α, we can assume that $\widehat{S}$ meets K_α transversely. Then $\widehat{S} \cap V_\alpha$ is a disjoint union of meridian disks of V_α, $u_1, u_2, \ldots, u_s$, numbered successively along V_α, and s can be chosen to be minimal among all essential annuli. Similarly, we consider an essential torus $\widehat{T}$ in $M(\beta)$, meeting a core K_β of V_β transversely. Then $\widehat{T} \cap V_\beta$ is a union of meridian disks $v_1, v_2, \ldots, v_t$, and t is chosen to be minimal. Let $S = \widehat{S} \cap M$ and $T = \widehat{T} \cap M$. We can assume that no circle component in $S \cap T$ bounds a disk in S or T, since both surfaces are incompressible.

In the usual way [1,2,6], the arc components of $S \cap T$ define labelled graphs G_S on $\widehat{S}$ and G_T on $\widehat{T}$. The vertices of G_S (resp. G_T) are $u_1, u_2, \ldots, u_s$ (resp. $v_1, v_2, \ldots, v_t$). For an edge of G_S, if its endpoint lies in $\partial u_i \cap \partial v_j$, then the point is labelled j at u_i. Thus the sequence of labels $1, 2, \ldots, t$ is repeated three times around each u_i, and so u_i has degree $3t$. Similarly, the edges of G_T are labeled, and the sequence $1, 2, \ldots, s$

appears three times around v_j. An edge with label i at one of its endpoints is called an *i-edge*. Also, an edge with labels i and j is called a *$\{i, j\}$-edge*. An edge is said to be *level* if its endpoints have the same label. Notice that there is a one-to-one correspondence between the edges of G_S and G_T, and that neither graph contains a *trivial loop*, which bounds a 1-sided disk face. Throughout the paper, two graphs on a surface are considered to be equivalent if there is a homeomorphism of the surface sending one graph to the other.

Each vertex of G_S is given a sign, according to the sign of the intersection point of K_α with $\widehat{S}$ with respect to some chosen orientations of M, $\widehat{S}$ and K_α. Similarly, we give a sign to each vertex of G_T. Two vertices are *parallel* if they have the same sign, otherwise they are *antiparallel*. An edge is *positive* if it connects parallel vertices. Otherwise, it is *negative*. In particular, a loop is positive. A point at a vertex is called a *positive edge endpoint* if there is a positive edge incident to it. Otherwise, it is a *negative edge endpoint*.

For a graph $G = G_S$ or G_T, let G^+ denote the subgraph consisting of all vertices and all positive edges of G. Also, let G^+_x be the subgraph of G^+ consisting of all vertices and all x-edges of G^+ for a label x. A disk face of G^+_x is called an *x-face*. The *reduced graph* $\overline{G}$ of G is obtained from G by amalgamating each family of mutually parallel edges into a single edge.

A cycle σ consisting of positive edges is a *Scharlemann cycle* if it bounds a disk face of the graph, and all the edges in σ have the same pair of labels $\{i, i+1\}$ at their endpoints, called the *label pair* of σ. The *length* of σ is the number of edges in σ. In particular, a Scharlemann cycle of length two is called an *S-cycle*. If σ is surrounded by a cycle τ, that is, each edge of τ is immediately parallel to an edge of σ, then τ is called an *extended Scharlemann cycle* (see [4]).

Lemma 2.2

(i) *There are no two edges which are parallel in both graphs.*

(ii) The parity rule: *An edge is positive in one graph if and only if it is negative in the other.*

(iii) *The edges of a Scharlemann cycle in G_S (resp. G_T) do not lie in a disk in $\widehat{T}$ (resp. $\widehat{S}$).*

(iv) *If G_S (resp. G_T) contains a Scharlemann cycle, then $\widehat{T}$ (resp. $\widehat{S}$) is separating, and so t (resp. s) is even.*

(v) *If $t > 2$ (resp. $s > 2$), then G_S (resp. G_T) cannot contain an extended Scharlemann cycle.*

Proof (i) This is [2, Lemma 2.1]. See also [6, Lemma 2.2]. (ii) can be found in [1, p. 279]. (iii) and (iv) are [6, Lemma 2.2]. (v) For G_S, this is [4, Theorem 3.2]. For G_T, we refer to [11]. We remark that only extended S-cycles are considered in [6, 14]. ∎

Theorem 2.3 *$M(\beta)$ does not contain a Klein bottle meeting a core K_β of V_β in at most $t/2$ points.*

Theorem 2.3 will be proved in Sections 6 and 7.

If G_S contains a Scharlemann cycle, $M(\beta)$ is split into two pieces $\mathcal{B}$ and $\mathcal{W}$ along $\widehat{T}$. We call them the *black side* and the *white side* of $\widehat{T}$, respectively. Also, a disk face of G_S is said to be *black* or *white*, according as it lies in $\mathcal{B}$ or $\mathcal{W}$. In particular, a Scharlemann cycle whose disk face is black (white) is called a *black (white) Scharlemann cycle*. This is similar for G_T.

For the remainder of the section, let $H_{i,i+1}$ be that part of V_β between v_i and v_{i+1}.

Lemma 2.4 *Neither graph contains a black Scharlemann cycle and a white Scharlemann cycle simultaneously.*

Proof Assume that G_S contains a black Scharlemann cycle σ_1 and a white Scharlemann cycle σ_2. Let D_i be the disk face bounded by σ_i, and $\{k_i, k_i+1\}$ be the label pair of σ_i. Let $X = N(\widehat{T} \cup H_{k_1,k_1+1} \cup H_{k_2,k_2+1} \cup D_1 \cup D_2)$. Then ∂X consists of two tori T_1 and T_2, each of which intersects K_β fewer than t times. By the minimality of $\widehat{T}$, each T_i is boundary-parallel or bounds a solid torus in $M(\beta)$. Thus ∂M consists of at most three tori, contradicting our assumption.

A similar construction works for G_T. By using the disk faces bounded by a black Scharlemann cycle and a white Scharlemann cycle in G_T, we obtain two annuli S_1 and S_2, each of which intersects K_α fewer than s times. By the minimality of $\widehat{S}$, each S_i is boundary parallel. Hence ∂M is a union of two tori, a contradiction. ■

Lemma 2.5 *G_S satisfies the following.*

(i) *There are no two S-cycles with disjoint label pairs.*
(ii) *Any family of mutually parallel positive edges in G_S contains at most $t/2+1$ edges. If $\widehat{T}$ is non-separating, then it contains at most $t/2$ edges.*
(iii) *Either any family of mutually parallel negative edges in G_S contains at most t edges, or all vertices of G_T are parallel.*

Proof (i) Let σ_1 and σ_2 be S-cycles with disjoint label pairs. Let $\{k_i, k_i+1\}$ be the label pair of σ_i, and let D_i be the face bounded by σ_i, $i = 1, 2$. Shrinking H_{k_i,k_i+1} to its core in $H_{k_i,k_i+1} \cup D_i$ gives a Möbius band B_i whose boundary is essential on $\widehat{T}$, by Lemma 2.2(iii). By ∂B_1 and ∂B_2, $\widehat{T}$ is split into two annuli A_1 and A_2. If A_i contains a_i vertices in its interior, then the Klein bottle $F_i = B_1 \cup A_i \cup B_2$ meets K_β in $a_i + 2$ points. The torus $\partial N(F_i)$ is incompressible by the irreducibility of $M(\beta)$. If it is boundary parallel in $M(\beta)$, then $M(\beta) = N(F_i)$ is not toroidal. Hence this torus is essential, and so $2(a_i+2) \geq t$, giving $a_i \geq t/2 - 2$. Since $a_1 + a_2 = t - 4$, $a_1 = a_2 = t/2 - 2$. But this contradicts Theorem 2.3.

(ii) If $t > 2$, then such a family contains at most $t/2+2$ edges by [13, Lemma 1.4], and moreover, if it contains $t/2+2$ edges, then it contains two S-cycles with disjoint label pairs, which contradicts (i). If $t = 2$, then three parallel edges contain a black S-cycle and a white S-cycle, contradicting Lemma 2.4. When $t = 1$, G_T contains only positive edges, and so G_S has no positive edges by the parity rule.

If a family of parallel positive edges contains more than $t/2$ edges, then the family contains an S-cycle. The second claim follows immediately from Lemma 2.2(iv).

(iii) See [6, Lemma 2.3] for $t > 2$. Assume $t = 2$ and that the two vertices of G_T are antiparallel. Suppose that G_S has three mutually parallel negative edges e_1, e_2, e_3, numbered successively. Then all are level by the parity rule. Hence we can assume that e_1 and e_3 have label 1. Since two loops at each vertex of G_T are parallel (see [2, Lemma 5.2]), e_1 and e_3 correspond to parallel loops at v_1 in G_T. This contradicts Lemma 2.2(i). ■

Lemma 2.6 *G_T satisfies the following.*

(i) *If $s > 2$, then any family of mutually parallel positive edges in G_T contains at most $s/2 + 1$ edges. If $\widehat{S}$ is non-separating, then it contains at most $s/2$ edges.*
(ii) *Any family of mutually parallel negative edges in G_T contains at most s edges.*
(iii) *All Scharlemann cycles in G_T have the same label pair.*

Proof These are [6, Lemma 2.5]. ■

Lemma 2.7 *G_S satisfies the following.*

(i) *At most two labels can be labels of S-cycles.*
(ii) *At most four labels can be labels of Scharlemann cycles.*

Proof (i) If there are three labels of S-cycles, then there are two S-cycles with disjoint label pairs and with the same color by Lemma 2.4. This is impossible, by Lemma 2.5(i).

(ii) If not, G_S has three Scharlemann cycles $\sigma_1, \sigma_2, \sigma_3$ with mutually disjoint label pairs and with the same color Lemma 2.4. Let D_i be the face bounded by σ_i, and let $\{k_i, k_i+1\}$ be the label pair of σ_i. We can assume that $D_i \subset \mathcal{B}$. On $\widehat{T}$, there are mutually disjoint annuli A_i which contain the edges of σ_i, respectively. Define $M_i = N(A_i \cup H_{k_i,k_i+1} \cup D_i) \subset \mathcal{B}$. Let $B_i = \mathrm{cl}(\partial M_i - A_i)$. Then a new torus $T_i = (\widehat{T} - A_i) \cup B_i$ meets K_β fewer than t times. Hence T_i is compressible or boundary parallel. If one of T_i is compressible, the argument in the proof of [4, Theorem 3.5] without any change gives a contradiction. Thus any T_i is boundary parallel. Let $Z_1 = \mathrm{cl}(\mathcal{B} - M_1)$. Then $Z_1 = T^2 \times I$. Since $M_2 \subset Z_1$, M_2 is a solid torus, and moreover, B_2 is parallel to A_2 through M_2. This contradicts [4, Claim 3.6]. ■

If G_S contains a Scharlemann cycle with label i, then i is called an *S-label*. Otherwise, i is called a non-S-label.

Lemma 2.8 *Let $t \geq 3$. Any x-face in G_S has at least four sides for a non-S-label x.*

Proof Assume not. By Lemmas 2.2(v) and 2.5(ii), G_S cannot contain a two-sided x-face. Let D be a 3-sided x-face in G_S. By [8, Proposition 5.1], D contains a Scharlemann cycle. Since G_S cannot contain an extended Scharlemann cycle by Lemma 2.2(v), D contains an S-cycle. By using Lemma 2.7(1), the proof of [4, Lemma 5.1] shows that D contains an S-cycle with face f, and the bigon g_1 and the 3-gon g_2 adjacent to f have only two kinds of corners. See [4, Figure 5.4]. For convenience, we assume that f has two $(1,2)$-corners, and g_i has $(t,1)$- and $(2,3)$-corners. Let $A_{i,i+1}$

be the annulus in ∂V_β between v_i and v_{i+1}. Then $(\widehat{T} - \mathrm{Int}(v_t \cup v_1 \cup v_2 \cup v_3)) \cup (A_{t,1} \cup A_{2,3})$ is a genus three closed surface, on which ∂g_1 and ∂g_2 are homologically independent. (This means that the genus three closed surface will be compressed to a torus along g_1 and g_2.) Hence $N(\widehat{T} \cup H \cup f \cup g_1 \cup g_2)$ has two torus boundary components, where H is the part of V_β between v_t and v_3, containing v_1. Since each torus meets K_β fewer than t times, they are inessential in $M(\beta)$. Then M is bounded by at most three tori as in the proof of Lemma 2.4, a contradiction. ■

3 The Case Where One Graph Has a Single Vertex

In this section, we treat the case where $s = 1$ or $t = 1$.

Lemma 3.1 $s \neq 1$.

Proof Assume $s = 1$. Since the vertex u_1 of G_S has degree $3t$, t must be even. Also, all edges of G_S are positive and parallel. If $t > 2$, then $3t/2 \leq t/2 + 1$, giving $t \leq 1$, a contradiction. Hence $t = 2$. But then G_S contains a black S-cycle and a white S-cycle, which contradicts Lemma 2.4. ■

Lemma 3.2 *If $t = 1$, then $s = 2$.*

Proof Assume $t = 1$ and $s \geq 3$. There are $3s/2$ edges (so s is even) in G_T, which are divided into at most three families of mutually parallel edges (see [2, Lemma 5.1]). Since each family contains at most $s/2+1$ edges by Lemma 2.6(1), G_T has at least two families. If there are only two families, then $4(s/2+1) \geq 3s$ gives $s \leq 4$. Hence $s = 4$. Then G_T consists of two families of three mutually parallel edges. By examining the labels, this contradicts the parity rule. Therefore, G_T contains three families.

We write $G_T = H(q_1, q_2, q_3)$ when each family contains q_1, q_2, q_3 edges, respectively. Note that $H(q_1, q_2, q_3)$ is invariant under any permutation of the q_i's. If $q_i \leq s/2$ for each i, then $6 \cdot s/2 \geq 2(q_1 + q_2 + q_3) = 3s$ gives $q_1 = q_2 = q_3 = s/2$, and so $G_T = H(s/2, s/2, s/2)$. It is easy to see that G_T contains an extended Scharlemann cycle of length three, which is impossible by Lemma 2.2(v). Hence we may assume $q_1 = s/2 + 1$. Let Q_i denote the family of parallel edges containing q_i edges. Then Q_1 contains an S-cycle at one end (see [13, Lemma 1.4]). By examining the labels, $q_2 + q_3 \equiv 0 \pmod{s}$ and $q_2 + q_3 \equiv s - 2 \pmod{s}$. This implies $s = 2$, a contradiction. ■

Lemma 3.3 $t \neq 1$.

Proof Assume $t = 1$. By Lemma 3.2, $s = 2$. Then G_T is $H(3, 0, 0)$, $H(2, 1, 0)$, or $H(1, 1, 1)$. If G_T is $H(3, 0, 0)$ or $H(1, 1, 1)$, then G_T contains a black Scharlemann cycle and a white Scharlemann cycle, contradicting Lemma 2.4. Clearly, $H(2, 1, 0)$ contradicts the parity rule. ■

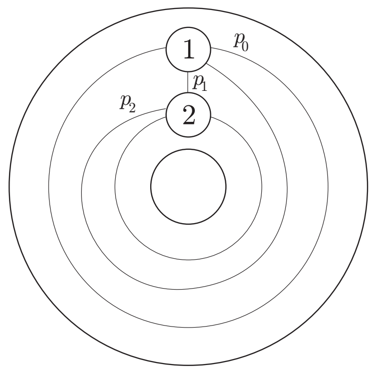

Figure 1

4 The Case $s = 2$

In this section, we consider the case where $s = 2$ and $t \geq 2$. Then the reduced graph $\overline{G}_S$ of G_S is a subgraph of the graph shown in Figure 1. Notice that u_1 and u_2 are incident to the same number of loops in G_S. We write $G_S = G(p_0, p_1, p_2)$ when u_i is incident to p_0 loops, and the other two families of parallel edges contain p_1 and p_2 edges, respectively. Clearly, $G(p_0, p_1, p_2)$ is equivalent to $G(p_0, p_2, p_1)$. We divide the argument into two cases.

4.1 When the Two Vertices of G_S Are Parallel

Lemma 4.1 *$t > 2$.*

Proof Assume $t = 2$. Then $p_i \leq 2$ for any i by Lemma 2.5(ii), and so G_S is $G(2, 2, 0)$, $G(2, 1, 1)$, or $G(1, 2, 2)$. In any case, G_S contains a black Scharlemann cycle and a white Scharlemann cycle, since any disk face is a Scharlemann cycle. This contradicts Lemma 2.4. ■

Lemma 4.2 *$\widehat{T}$ is separating and $t = 4$.*

Proof Since $3t = 2p_0 + p_1 + p_2$, $p_i > t/2$ for some i. Thus G_S contains an S-cycle, and so $\widehat{T}$ is separating and t is even by Lemma 2.2(iv). Hence we have $t \geq 4$. Notice that $p_i \leq t/2 + 1$ for any i by Lemma 2.5(ii). Hence $3t \leq 4(t/2 + 1) = 2t + 4$, giving $t \leq 4$. ■

Proposition 4.3 *The two vertices of G_S cannot be parallel.*

Proof Since $p_i \leq 3$ for any i by Lemma 2.5(ii), $G_S = G(3, 3, 3)$. Once we fix the labeling around u_1, by the parity rule there are only two possibilities for the labeling around u_2. In any case, G_S contains a black Scharlemann cycle and a white Scharlemann cycle, a contradiction. ■

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4.2 When the Two Vertices of G_S Are Antiparallel

Lemma 4.4 $p_0 \neq 0$.

Proof If $p_0 = 0$, then all edges of G_S connect u_1 with u_2. Hence all edges of G_T are positive by the parity rule. Notice that any disk face of G_T is a Scharlemann cycle. Since G_T has $3t$ edges, it contains at least $2t$ disk faces. If these disk faces have the same color, then G_T has at least $4t$ edges, a contradiction. Thus G_T contains a black Scharlemann cycle and a white Scharlemann cycle, contradicting Lemma 2.4. ∎

Lemma 4.5 *G_S equals $G(t/2, t, t)$, $G(t/2+1, t, t-2)$, or $G(t/2+1, t-1, t-1)$.*

Proof Since $p_0 \neq 0$, G_S contains a positive edge, and not all the vertices of G_T are parallel. This implies $p_i \leq t$ for $i = 1, 2$ by Lemma 2.5(iii). By $3t = 2p_0 + p_1 + p_2 \leq 2p_0 + 2t$, $p_0 \geq t/2$. If $p_0 = t/2$, then $G_S = G(t/2, t, t)$. If $p_0 > t/2$, then G_S contains an S-cycle, and hence $\widehat{T}$ is separating and t is even. Thus $p_0 = t/2 + 1$, and then the conclusion follows immediately. ∎

Lemma 4.6 *G_T cannot contain an S-cycle.*

Proof Let σ be an S-cycle in G_T whose disk face is f. The edges of σ form an essential cycle in $\widehat{S}$ by Lemma 2.2(iii). Let H be the part of V_α between u_1 and u_2 meeting ∂f. Then shrinking H into its core in $H \cup f$ gives a Möbius band B' whose boundary is an essential loop on $\widehat{S}$. The union of B' and an annulus in $\widehat{S}$ between $\partial B'$ and ∂S gives a Möbius band $\widehat{B}$ properly embedded in $M(\alpha)$ which meets K_α in one point. Let $X = N(\widehat{B})$ and let $W = M(\alpha) - \operatorname{Int} X$. Then the frontier $\widehat{Q}$ of X is an incompressible annulus. If $\widehat{Q}$ is boundary parallel, then $M(\alpha)$ has a single torus as boundary, a contradiction. Hence $\widehat{Q}$ is essential. Let $Q = \widehat{Q} \cap M$, and let $A = \partial V_\alpha \cap W$. Then $F = Q \cup A$ is a twice-punctured torus.

Let $B = \widehat{B} \cap M$. If B is compressible in M, then let δ be a compressing disk for B. Since $\partial\delta$ is orientation-preserving on B, it bounds a disk in $\widehat{B}$ or is parallel to $\partial\widehat{B}$. The former implies that M contains a properly embedded Möbius band, contradicting the hyperbolicity of M. The latter means that $M(\alpha)$ contains a projective plane, and so $M(\alpha)$ is reducible, contradicting Lemma 2.1. Hence B is incompressible. Also, if B is boundary compressible, then K_α can be isotoped to the core of $\widehat{B}$ by using a boundary compressing disk. Then M contains an essential annulus. Hence B is boundary incompressible.

We construct another graph pair $\{G_B, G_T^B\}$ from B and T in the usual way. There is no trivial loop in each graph. Note that G_B has a single vertex, and G_T^B consists of t vertices of degree three and $3t/2$ edges. In fact, the double cover of G_B is a subgraph of the graph shown in Figure 1. By an Euler characteristic calculation, G_T^B contains a disk face D'. Let $D = D' \cap W$. Notice that ∂D is essential on F. For ∂D runs on Q and A alternately, and $\partial D \cap Q$ consists of arcs as shown in Figure 1. Surgering F along D gives either an annulus or a disjoint union of an annulus and a torus, according as ∂D is non-separating or separating on F. In any case, the resulting surface is disjoint from K_α. Hence the annulus component is boundary parallel, and the torus component,

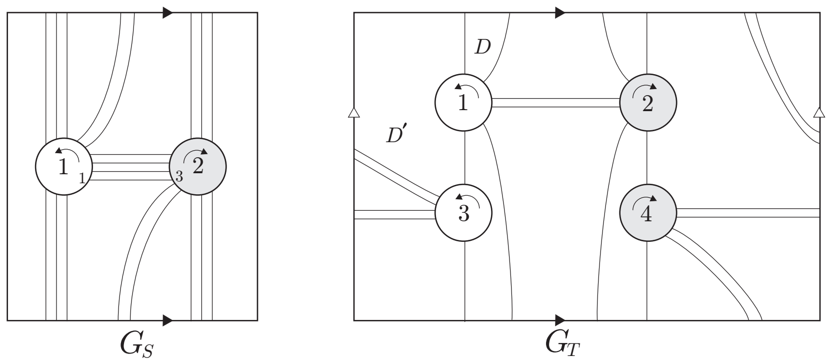


Figure 2

if it exists, is inessential. Thus $M(\alpha)$ is bounded by at most two tori, a contradiction. ∎

Lemma 4.7 $t = 2$.

Proof Assume $t > 2$. By Lemma 4.5, there are three possibilities for G_S by Lemma 4.5.

If $G_S = G(t/2, t, t)$, then G_T has $2t$ positive edges by the parity rule. Hence G_T^+ has at least t disk faces. Notice that such a disk face is also a face of G_T, and so it is bounded by a Scharlemann cycle. Hence we may assume that such disk faces are all black by Lemma 2.4. Also, such a disk face has at least three sides by Lemma 4.6. Thus there are at least $3t$ positive edges, a contradiction.

If $G_S = G(t/2+1, t, t-2)$, then the same argument yields a contradiction, unless $t = 4$. (Notice that $p_0 = t/2 + 1$ implies that $\widehat{T}$ is separating and t is even, by Lemma 2.2(iv).) Suppose $t = 4$ and $G_S = G(3, 4, 2)$. Let Q be the family of 4 negative edges in G_S, and let σ be the associated permutation to Q. That is, each edge of Q has label i at u_1 and $\sigma(i)$ at u_2. If σ is the identity, then G_S contains two S-cycles with disjoint label pairs, which contradicts Lemma 2.5. Hence $\sigma = (13)(24)$. In this case, G_T is uniquely determined. First, the edges of the two S-cycles with label pair $\{3, 4\}$ form essential cycles. The edges of Q form two essential cycles by Lemma 2.2(i). By examining labels, the two edges between v_1 and v_2 turn out to be parallel. See Figure 2. Then G_T has a Scharlemann cycle of length three with face D. Thus $M(\alpha)$ is split into two pieces $\mathcal{B}$ and $\mathcal{W}$ along $\widehat{S}$. We may assume that $D \subset \mathcal{B}$. Let $H = V_\alpha \cap \mathcal{B}$. Let $X = N(\widehat{S} \cup H \cup D) \subset \mathcal{B}$. By the minimality of $\widehat{S}$, the annulus $\mathrm{cl}(\partial X - \widehat{S})$ is boundary parallel. Thus $\partial\mathcal{B}$ is a torus. Let D' be the white face as shown in Figure 2. Similarly, we can see that $\partial\mathcal{W}$ is a torus by using D'. Thus $M(\alpha)$ is bounded by a single torus, a contradiction.

If $G_S = G(t/2+1, t-1, t-1)$, then the two families of loops at u_1 and u_2 contain S-cycles. Hence t is even. By examining labels, such an S-cycle is located at one

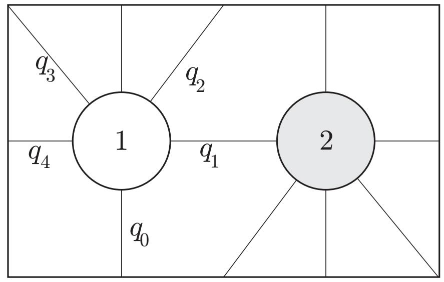


Figure 3

end of each family. Then it is obvious that these two S-cycles have distinct colors, contradicting Lemma 2.4. ■

By Lemma 4.7, G_T has only two vertices. The reduced graph $\overline{G}_T$ is a subgraph of the graph shown in Figure 3 (see [2, Lemma 5.2]). We say $G_T = H'(q_0, q_1, q_2, q_3, q_4)$, where q_i denotes the number of edges in the corresponding family of parallel edges. Note that

$$H'(q_0, q_1, q_2, q_3, q_4) \cong H'(q_0, q_3, q_4, q_1, q_2) \cong H'(q_0, q_4, q_3, q_2, q_1)$$
$$\cong H'(q_0, q_2, q_1, q_4, q_3).$$

Proposition 4.8 *The two vertices of G_S cannot be antiparallel.*

Proof By Lemma 4.5, G_S is $G(1, 2, 2)$, $G(2, 1, 1)$, or $G(2, 2, 0)$.

If $G_S = G(1, 2, 2)$, then G_T is $H'(2, 1, 1, 0, 0)$ or $H'(2, 2, 0, 0, 0)$. Then G_T contains an S-cycle, contradicting Lemma 4.6. If $G_S = G(2, 1, 1)$, then G_S contains a black Scharlemann cycle and a white Scharlemann cycle, contradicting Lemma 2.4.

Suppose $G_S = G(2, 2, 0)$. Then G_S contains two S-cycles ρ_1 and ρ_2 of the same color. Let f_i be its face for $i = 1, 2$, and let A be the annulus part of ∂V_β between v_1 and v_2, meeting f_i. Notice that $q_0 = 1$ and $(q_1 + q_2, q_3 + q_4)$ is $(3, 1)$, $(2, 2)$, or $(4, 0)$, up to equivalence. By the parity rule, $(3, 1)$ is impossible. Thus G_T is $H'(1, 1, 1, 2, 0)$, $H'(1, 2, 0, 2, 0)$, $H'(1, 1, 1, 1, 1)$, or $H'(1, 2, 2, 0, 0)$.

First, $H'(1, 1, 1, 2, 0)$ contradicts the parity rule. If $G_T = H'(1, 2, 0, 2, 0)$, ∂f_1 and ∂f_2 cannot be located on $T \cup A$ simultaneously. Assume $G_T = H'(1, 1, 1, 1, 1)$. Then there are two disjoint rectangles R_1 and R_2 in A split by $\partial f_1 \cup \partial f_2$ such that $f_i \cup R_i$ gives a Möbius band B_i. Thus we have two Möbius bands B_1 and B_2 whose boundaries are disjoint on $\widehat{T}$. Hence $M(\beta)$ contains a Klein bottle as a union of B_1, B_2 and an annulus on $\widehat{T}$, meeting K_β once. This contradicts Theorem 2.3. Finally, assume $G_T = H'(1, 2, 2, 0, 0)$. Then G_T contains two 3-gons and two bigons. Let f be any one of the 3-gons and g any one of the bigons. Let A' (resp. A'') be the part of ∂V_α between u_1 and u_2 meeting ∂f (resp. ∂g). Then ∂f is a non-separating curve on the surface $S \cup A'$, so surgering $S \cup A'$ along f gives rise to a boundary parallel annulus in $M(\alpha)$. Thus $\widehat{S}$ is separating in $M(\alpha)$. On the other hand, surgering $S \cup A''$ along g gives rise to a surface disjoint from K_α, which is an annulus or a disjoint union of an annulus and a torus, according as ∂g is non-separating or separating on $S \cup A''$. As in the proof of Lemma 4.6, $M(\alpha)$ is bounded by at most two tori, a contradiction. ■

5 The Generic Case

Finally, we consider the case where $s \geq 3$ and $t \geq 2$. Since all Scharlemann cycles of G_T have the same label pair by Lemma 2.6(iii), we can assume that $\{1, 2\}$ is the label pair, if it exists. Then these labels are S-labels of G_T, and the vertices u_1 and u_2 are referred to as the *S-vertices* of G_S.

Lemma 5.1 *G_T does not contain an x-face for a non-S-label x.*

Proof This is Theorem 4.5 of [11]. ■

Lemma 5.2 *Any vertex of G_S that is not an S-vertex has at least $2t$ positive edge endpoints.*

Proof Assume that u_i is not an S-vertex. If it has at least $t+1$ negative edge endpoints, then G_T has at least $t+1$ positive i-edges. Let Γ_i be the subgraph of G_T consisting of all vertices and all positive i-edges of G_T. Then an Euler characteristic calculation shows that Γ_i has a disk face, which is an i-face. This contradicts Lemma 5.1. ■

Lemma 5.3 *An S-vertex of G_S, if it exists, has at least t positive edge endpoints.*

Proof Let u_1 be an S-vertex. Suppose that u_1 has k negative edge endpoints. Then G_T has k positive 1-edges. Hence G_T has at least $k - t$ 1-faces. Recall that each 1-face contains a Scharlemann cycle [8]. Thus there are at least $k - t$ Scharlemann cycles with label pair $\{1, 2\}$. Then there are at least $2(k-t)$ positive 1-edges, since all Scharlemann cycles have the same color. We have $2(k-t) \leq k$, and so $k \leq 2t$. Hence u_1 has at least t positive edge endpoints. ■

Let us consider G_S^+, which consists of all vertices and all positive edges of G_S. Let Λ be a component of G_S^+. If there is a disk D in $\widehat{S}$ such that $\Lambda \subset \operatorname{Int} D$, then Λ is said to have a *disk support*. Otherwise, there is an annulus A in $\widehat{S}$, which is called an *annulus support*, such that $\Lambda \subset \operatorname{Int} A$. Clearly, the core of A is parallel to the core of $\widehat{S}$. Furthermore, if Λ has a support F, which is a disk or an annulus, such that $F \cap G_S^+ = \Lambda$, then Λ is called an *extremal component* of G_S^+. Clearly, if there is no component of G_S^+ with a disk support, then any component of G_S^+ is an extremal one with an annulus support.

Suppose that Λ is an extremal component with support F. A vertex u is a *cut vertex* if $\Lambda - u$ has at least two components. We remark that Λ may have loops. Also,

u is called an *interior vertex* if there is no arc ξ in F connecting u to ∂F such that $\xi \cap \Lambda = u$. Otherwise, u is called a *boundary vertex*. Furthermore, an *interior edge* is an edge which cannot admit an arc ξ connecting a middle point x of the edge to ∂F such that $\xi \cap \Lambda = x$. The others are *boundary edges*. When F is an annulus, a vertex u is called a *pinched vertex* if there is a spanning arc ξ of F such that $\xi \cap \Lambda = u$, and a *pinched edge* is defined similarly. In particular, both endpoints of a pinched edge are pinched vertices. Finally, u is said to be *good* if all positive edge endpoints at u are successive. Thus, if u is neither a cut vertex nor a pinched vertex, then it is good.

A subgraph B of Λ is called a *disk block* of G_S^+ if B contains at most one cut vertex of Λ and there is a disk D in $\widehat{S}$ such that $D \cap G_S^+ = B$ and $\partial D \cap B$ is either empty or a single vertex. We remark that a disk block is connected and that a disk block cannot contain a loop which is essential in $\widehat{S}$, but it may contain a loop which is inessential in $\widehat{S}$. If B has an S-vertex u, then u must appear as a boundary vertex of B, because the edges of a Scharlemann cycle in G_T do not lie in a disk in $\widehat{S}$ by Lemma 2.2(iii).

5.1 The Case $t = 2$

To eliminate the case where $t = 2$, we prove three lemmas. Recall that any non-S-vertex has at least four positive edge endpoints by Lemma 5.2, while any S-vertex has at least two positive edge endpoints by Lemma 5.3.

Lemma 5.4 *Any component of G_S^+ has an annulus support, and hence is extremal.*

Proof If G_S^+ has a component with a disk support, then there is an extremal component Λ with a disk support. By Lemmas 5.2 and 5.3, it contains at least two vertices. One of the vertices is good and has at least four successive positive edge endpoints by Lemma 5.2. Hence Λ has a black face and a white face, which contradicts Lemma 2.4, because any disk face of G_S^+ is bounded by a Scharlemann cycle.

Therefore we have shown that any component of G_S^+ has an annulus support. Also, this implies that any component is extremal. ■

Lemma 5.5 *G_S^+ has at most two disk blocks, each of which consists of two vertices and a pair of parallel edges. In particular, a non-cut vertex is an S-vertex.*

Proof Let B be a disk block. If B has an interior edge, then there is a black face and a white face, contradicting Lemma 2.4. Hence B has no interior edge. Thus B is either a single edge or a cycle. However, the former is impossible by Lemmas 5.2 and 5.3. Hence B is a cycle. If the length of B is more than two, then there is a non-cut vertex, which is not an S-vertex, contradicting Lemma 5.2. Hence B is length two, and Lemma 5.2 implies that a non-cut vertex must be an S-vertex.

Since G_S has at most two S-vertices, there are at most two disk blocks. ■

Lemma 5.6 *Any component of G_S^+ containing a non-S-vertex is a cycle of bigons.*

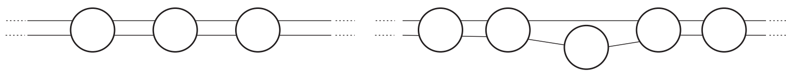


Figure 4

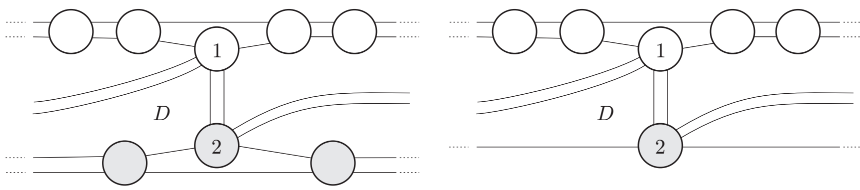


Figure 5

Proof Let Λ be a component containing a non-S-vertex u. Recall that every face of Λ is a disk bounded by a Scharlemann cycle. Hence Λ has no interior vertex.

First, assume that Λ has no cut vertex. Recall that any non-S-vertex has at least four positive edge endpoints. Also, Λ has at most one S-vertex. If a non-S-vertex is not pinched, then Λ has a black face and a white face. Hence any non-S-vertex is pinched, and has degree 4. Thus Λ is either a cycle of bigons, or a cycle of bigons with one bivalent vertex added, which is an S-vertex. See Figure 4.

Suppose that Λ contains a bivalent S-vertex u_1, say. Then the configuration of G_S near u_1 looks like Figure 5. Notice that u_1 has four negative edges, so G_T has at least two 1-faces, which must be bigons bounded by S-cycles. Thus G_T contains two S-cycles. Let D be the disk face as shown there.

Since G_S has an S-cycle, $\widehat{T}$ is separating in $M(\beta)$, and so ∂V_β is divided into two annuli A_1, A_2, where A_1 meets ∂D. Let $T_1 = T \cup A_1$ and $T_2 = T \cup A_2$. Since ∂D is non-separating on T_1, surgering T_1 along D gives a torus disjoint from K_β. On the other hand, surgering T_2 along the face bounded by an S-cycle whose color is distinct from that of D, also gives a torus disjoint from K_β. Thus $M(\beta)$ is bounded by at most two tori, a contradiction. Hence we can conclude that any component of G_S^+ containing a non-S-vertex is a cycle of bigons, possibly of length one.

Next, assume that Λ has a cut vertex. By Lemma 5.5, there are only two possibilities for Λ as shown in Figure 6. However, we can still choose a disk face D as in Figure 5. Thus a similar argument leads to a contradiction. ∎

Proposition 5.7 $t \neq 2$.

Proof By Lemma 5.6, any component of G_S^+ containing a non-S-vertex is a cycle of bigons. All bigons have the same color by Lemma 2.4, and hence any non-S-vertex is incident to exactly two adjacent negative edges. This implies that each non-S-label appears once at each vertex of G_T among positive edge endpoints.

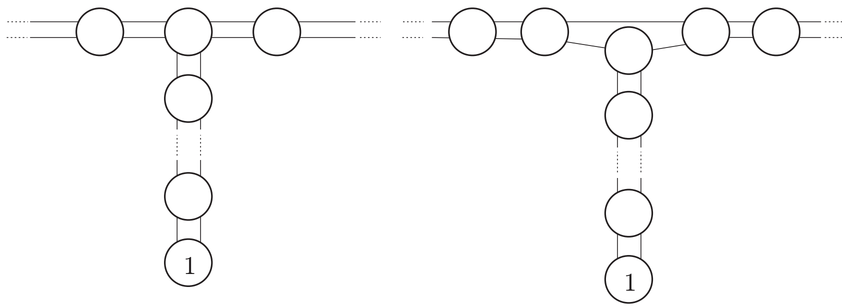


Figure 6

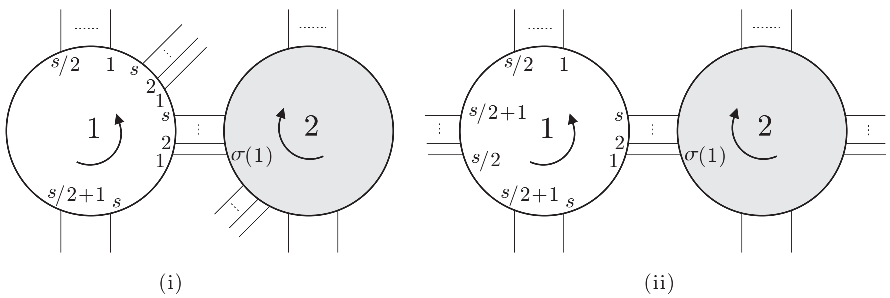


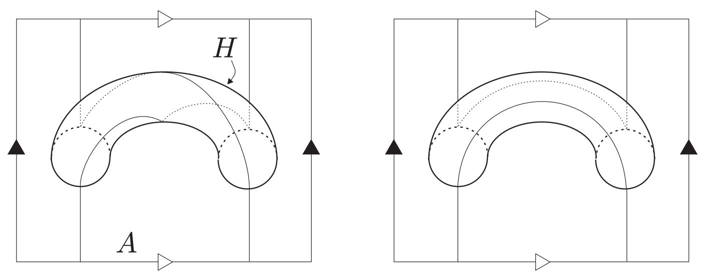
Figure 7

Suppose that G_T has no Scharlemann cycle. Then every label appears once at each vertex among positive edge endpoints. Also, the two edges of any of the bigons in G_S belong to the same pair of families of mutually parallel negative edges in G_T by [5, Lemma 5.2]. (Otherwise, $M(\beta)$ would contain a Klein bottle meeting K_β once.) Hence G_T has only two families of s mutually parallel negative edges. Thus G_T is either $H'(s/2, s, s, 0, 0)$ or $H'(s/2, s, 0, s, 0)$.

If G_T has a Scharlemann cycle, then each vertex of G_T has at least $s + 2$ positive edge endpoints, and so just $s/2 + 1$ loops by Lemma 2.6(i), two of which form an S-cycle. Then we see that G_T is $H'(s/2 + 1, s, s - 2, 0, 0)$ or $H'(s/2 + 1, s - 2, 0, s, 0)$.

We consider these four cases.

Case (A): $G_T = H'(s/2, s, s, 0, 0)$. We can assume that the labels in G_T are as in Figure 7(i). Let Q_1 and Q_2 be the families of mutually parallel negative edges with $q_1(= s)$ and $q_2(= s)$ edges, respectively. Let σ be the associated permutation to Q_1 such that an edge of Q_1 has label x at v_1 and label $\sigma(x)$ at v_2. Clearly, Q_2 also has the same associated permutation σ. Since the edges of Q_1 and Q_2 form cycles of bigons in G_S, σ^2 is the identity. Therefore $\sigma(x) = x$ or $\sigma(x) = x + s/2$.



(i) (ii)

Figure 8

Assume that σ is the identity. Then G_S consists of $s/2$ copies of a graph isomorphic to $G(2,2,0)$ or $G(2,1,1)$. Let D be a 3-gon in G_T. Notice that D is one-cornered. Using D, one can see that $\widehat{S}$ is separating in $M(\alpha)$ and the side of $\widehat{S}$ containing D is bounded by a torus. Also, take a bigon D' among the edges of Q_1, lying on the opposite side. Moreover, we can choose D' so that its edges bound an annulus in $\widehat{S}$ disjoint from the vertices of G_S, as an innermost one. Let D' be bounded by an x-edge and an $(x+1)$-edge, and let A be the annulus in ∂V_α between u_x and u_{x+1}. Then surgering $(\widehat{S} - \mathrm{Int}(u_x \cup u_{x+1})) \cup A$ along D' gives either an annulus, or a disjoint union of an annulus and a torus. In any case, the annulus component meets K_α fewer times than $\widehat{S}$, and the torus component is disjoint from K_α. Hence the annulus component is boundary parallel and the torus component is inessential. Thus $M(\alpha)$ is bounded by at most two tori.

Next, assume that $\sigma(x) = x + s/2$. Then we see that the two $\{1, s\}$-loops in G_T bound a bigon face E in G_S. But ∂E runs like Figure 8(i), and so $M(\beta)$ contains a Klein bottle meeting K_β once, obtained from $E \cup H \cup A$ by shrinking H radially into its core, where H is the 1-handle part of V_β meeting E and A is the annular region on $\widehat{T}$ between the two $\{1, s\}$-loops. This contradicts Theorem 2.3.

Case (B): $G_T = H'(s/2, s, 0, s, 0)$. We can assume that the labels in G_T are as in Figure 7(ii). Similarly, we can see that two families Q_1 and Q_2 of mutually parallel negative edges induce the same permutation σ, and σ^2 is the identity.

If σ is the identity, then take the two $\{1, s\}$-loops in G_T. They bound a bigon E in G_S, and ∂E runs like Figure 8(ii). But consider any S-cycle in G_S. It has one edge in each of Q_1 and Q_2, but we cannot connect them on ∂V_β.

When $\sigma(x) = x + s/2$, the same argument as in Case (A) gives a contradiction.

Case (C): $G_T = H'(s/2+1, s, s-2, 0, 0)$. The labels in G_T can be assumed to be as in Figure 9(i). But this implies that the component of G_S^+ containing u_3 is not a cycle of bigons, contradicting Lemma 5.6.

Case (D): $G_T = H'(s/2+1, s-2, 0, s, 0)$. Then the labels in G_T can be assumed to be as in Figure 9(ii). The two $\{3, s\}$-loops in G_T bound a bigon in G_S. Then the same argument as in Case (B) leads to a contradiction. ∎

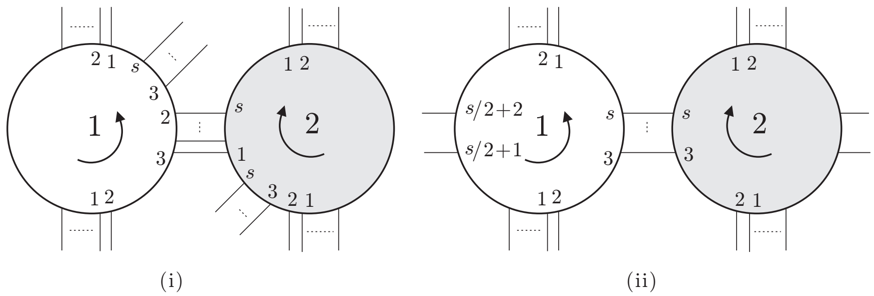


Figure 9

5.2 The Cases $t \geq 5$ and $t = 3$

In this subsection, we eliminate the two cases where $t \geq 5$ and $t = 3$. When $t = 3$, G_S has no Scharlemann cycle by Lemma 2.2(iv), and so no S-label. If $t \geq 5$, then G_S has a non-S-label by Lemma 2.7(ii).

Lemma 5.8 *G_S^+ has no disk block.*

Proof Let B be a disk block of G_S^+. It has at most one cut vertex of G_S^+ and at most one S-vertex among boundary vertices. Let V_i, V_b, V_c, V_s be the number of interior, boundary, cut and S-vertices, respectively. Here a cut vertex means a cut vertex of G_S^+. Then $V_c, V_s \leq 1$. Possibly, an S-vertex is a cut vertex. In this case, we set $V_s = 0$ and $V_c = 1$.

Let x be a non-S-label. Any interior vertex is incident to three positive x-edges, any boundary vertex, except a cut vertex and an S-vertex, is incident to at least two such edges by Lemma 5.2, and an S-vertex is incident to at least one such edge by Lemma 5.3. Consider the subgraph B^x of B consisting of all vertices and all x-edges of B. We remark that B^x may be disconnected, and may have many cut vertices. Let V, E, F be the number of vertices, edges, disk faces of B^x, respectively, as a graph in a disk. Then $V = V_i + V_b$ and $F \geq 1 - V + E$. By counting x-edges, we have

$$E \geq 3V_i + 2(V_b - V_c - V_s) + V_s = 3V - V_b - 2V_c - V_s. \tag{5.1}$$

Since each disk face of B^x has at least four sides by Lemma 2.8,

$$2E \geq 4F + V_b' \geq 4(1 - V + E) + V_b, \tag{5.2}$$

where V_b' is the number of boundary vertices of B^x. (Notice $V_b' \geq V_b$.) These give $3V - V_b - 2V_c - V_s \leq 2V - 2 - V_b/2$. Equivalently, $V - V_b/2 + 2 \leq 2V_c + V_s$. Hence $V_c = V_s = 1$ and $V = V_b = 2$. This implies that B is a family of at least t parallel positive edges joining two vertices, which contradicts Lemma 2.5(ii). ■

Lemma 5.9 *Any component of G_S^+ has an annulus support, and is extremal.*

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Proof If G_S^+ has a component with a disk support, then there is an extremal one Λ with a disk support. Hence Λ contains a disk block, contradicting Lemma 5.8. ■

Proposition 5.10 $t = 4$.

Proof Choose an outermost component Λ of G_S^+. There is an annulus A in $\widehat{S}$ such that $\Lambda \subset \operatorname{Int} A$, $A \cap G_S^+ = \Lambda$ and A contains one component of $\partial\widehat{S}$. After capping off that component of $\partial\widehat{S}$ with a disk, we regard Λ as lying in a disk. From this viewpoint, we consider its interior and boundary vertices. Let V_i, V_b, V_s be the number of interior, boundary, and S-vertices of Λ, respectively. We remark that Λ has a disk face f containing the disk capped off in its interior, where f may be a monogon. Also, Λ may have an S-vertex, and a cut vertex (of Λ) among its boundary vertices. But any boundary vertex is good by Lemma 5.8.

Let x be a non-S-label. Consider the subgraph Λ^x of Λ consisting of all vertices and all x-edges of Λ, as a graph in a disk. We remark that Λ^x may be disconnected. Let V, E, F be the number of vertices, edges, disk faces of Λ^x. Then $F \geq 1 - V + E$ and $V = V_i + V_b$. Each interior vertex of Λ is incident to three positive x-edges, each boundary vertex is incident to at least two such edges, and an S-vertex is incident to at least one such edge. Hence we have

$$E \geq 3V_i + 2(V_b - V_s) + V_s = 3V - V_b - V_s. \tag{5.3}$$

Also, since each disk face of Λ^x, possibly except one, has at least four sides,

$$2E \geq 4(F - 1) + 1 + V_b' \geq 4(E - V) + 1 + V_b, \tag{5.4}$$

where V_b' is the number of boundary vertices of Λ^x itself. These give $3V - V_b - V_s \leq E \leq 2V - V_b/2 - 1/2$, equivalently $V_i + V_b/2 \leq V_s - 1/2$. Then $V_s = 1$, $V_i = 0$ and $V_b = 1$. This means that Λ is an S-vertex with at least $t/2$ parallel loops.

Choose another outermost component Λ' of G_S^+ near the other component of $\partial\widehat{S}$. The same argument shows that Λ' consists of an S-vertex and at least $t/2$ parallel loops. Since the two S-vertices are connected with the edges of Scharlemann cycles, G_S^+ cannot have other components than Λ and Λ'. But this means $s = 2$, a contradiction. ■

5.3 The Case $t = 4$

Again, we can show that G_S^+ has no disk block as in Lemma 5.8, but it needs another argument.

Lemma 5.11 *G_S^+ has no disk block.*

Proof Let B be a disk block. We use the same notation as in the proof of Lemma 5.8. By Lemma 2.7(i), we can choose a label x which is not a label of an S-cycle. Then (5.1) holds. Since each disk face of B^x has at least three sides, (5.2) changes to $2E \geq 3F + V_b' \geq 3(1 - V + E) + V_b$. These give $3V - V_b - 2V_c - V_s \leq E \leq 3V - 3 - V_b$,

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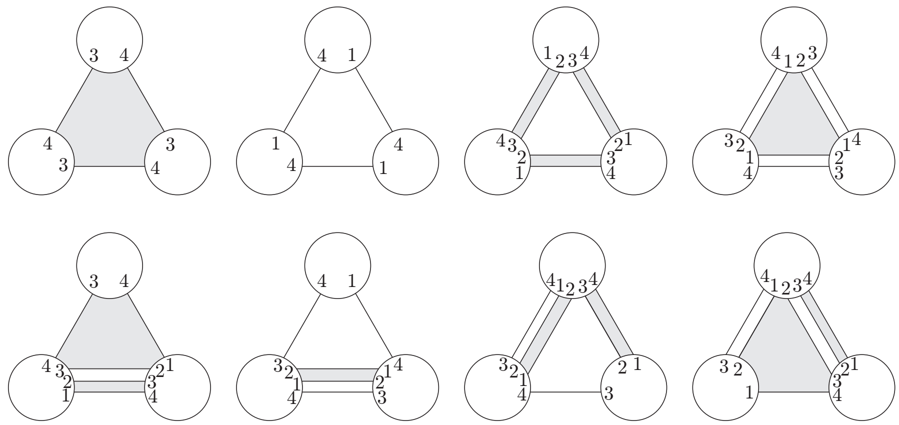

Figure 10

equivalently, $2V_c + V_s \geq 3$. Hence $V_c = V_s = 1$, and all inequalities above are equalities. So, $E = 3V - 3 - V_b$, $F = 2V - V_b - 2 = 2V_i + V_b - 2 \geq 0$ and each disk face of B^x is 3-sided.

If $F = 0$, then $V = V_b = 2$ and hence B is a family of at least t mutually parallel edges, contradicting Lemma 2.5(ii). Thus $F > 0$.

We may assume that $x = 4$ without loss of generality. Figure 10 lists all possible 3-sided faces of B^x, where all edges of G_S are indicated.

Extended Scharlemann cycles are impossible. The last four configurations can be eliminated in the same way. For example, the last configuration contains a white S-cycle and two two-cornered black faces, a bigon and a 3-gon adjacent to the S-cycle. These black faces are homologically independent. Hence $M(\beta)$ is bounded by at most two tori. Thus only the first and second configurations are possible, and they cannot occur simultaneously by Lemma 2.4. Hence we may assume that all faces of B^x are bounded by black Scharlemann cycles with label pair $\{3, 4\}$. Of course, this is impossible if $F > 1$. But if $F = 1$, then $V_i = V_b = 1$ or $V = V_b = 3$. In the former, B^x has a vertex of degree one, which contradicts Lemma 2.5(ii). In the latter, B^x is a cycle of length three, and so the vertex other than the cut vertex and the S-vertex is incident to at least t parallel positive edges in B, again contradicting Lemma 2.5(ii). ■

Hence again Lemma 5.9 holds.

Proposition 5.12 $t \neq 4$.

Proof Assume $t = 4$. We use the same notation as in the proof of Proposition 5.10. Let x be a label of G_S which is not a label of an S-cycle. Then we have (5.3). Since each disk face of Λ^x, possibly except one, has at least three sides, (5.4) changes to $2E \geq 3(F-1)+1+V'_b \geq 3(E-V)+1+V_b$, where V'_b denotes the number of boundary vertices of Λ^x. These give $3V - V_b - V_s \leq E \leq 3V - V_b - 1$. Hence $V_s = 1$ and all

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inequalities above are equalities, and then $E = 3V - V_b - 1$, $F = 2V - V_b = 2V_i + V_b$, and each disk face of Λ^x is three sided.

If $F = 2$, then $V = V_b = 2$ and $E = 3$. Then Λ has two vertices, one of which is a pinched vertex and the other is an S-vertex. By examining the labels around the vertices, we can see that Λ contains two S-cycles with disjoint label pairs. This contradicts Lemma 2.5(i). If $F > 2$, then the same argument as in the proof of Lemma 5.11 is applicable. Thus $F = 1$. Then $V = V_b = 1$ and so Λ consists of an S-vertex and parallel loops.

Similarly, another outermost component of G_S^+ near the other component of $\partial\widehat{S}$ consists of an S-vertex with parallel loops. Then $s = 2$ as in the proof of Proposition 5.10, a contradiction. ∎

6 Klein Bottle

In the remainder of the paper, we prove Theorem 2.3. Suppose that $M(\beta)$ contains a Klein bottle $\widehat{P}$ which meets K_β in p ($\leq t/2$) points, and that p is minimal among all Klein bottles in $M(\beta)$. Then $\widehat{P}$ meets V_β in a disjoint union of meridian disks $w_1, w_2, \ldots, w_p$ numbered successively along V_β. Let $P = \widehat{P} \cap M$, and let N be a thin neighborhood of $\widehat{P}$.

Lemma 6.1 *P is incompressible and boundary incompressible.*

Proof See [11, Lemma 2.1]. ∎

Thus we can assume that no circle component of $S \cap P$ bounds a disk in S or P. From the arc components of $S \cap P$, we have a graph pair in the usual way. By abuse of notation, we denote the pair by $\{G_S, G_P\}$ in the rest of paper. Since P is non-orientable, we cannot give a sign to a vertex of G_P. However, there is a way to give a sign to an edge of G_P (see [10]). Then the parity rule survives without any change. We remark that a positive edge of G_S can be a level edge. It corresponds to an orientation-reversing loop on $\widehat{P}$. Also, there are no two edges which are parallel in both graphs [2, Lemma 2.1].

If $p > 2$, a triple $\{e_1, e_2, e_3\}$ of mutually parallel positive edges in G_S is called a *generalized S-cycle* if e_2 is a level edge with label i, and e_1 and e_3 have label pair $\{i-1, i+1\}$ at their endpoints.

Lemma 6.2 *If $p \geq 2$, then G_S satisfies the following.*

(i) *There is no Scharlemann cycle.*
(ii) *If $p \geq 3$, then there is no generalized S-cycle.*
(iii) *At most two labels can be labels of positive level edges.*
(iv) *Any family of parallel positive edges contains at most $p/2 + 1$ edges.*
(v) *Any family of parallel negative edges contains at most p edges.*

Proof (i) See [12, Lemma 3.2]. (The argument works for a Scharlemann cycle with any length.) (ii) is [12, Lemma 3.3]. (iii) follows from the facts that a positive level edge in G_S corresponds to an orientation-reversing loop in $\widehat{P}$ and that a Klein bottle contains at most two disjoint Möbius bands.

(iv) Let Q be a family of mutually parallel positive edges in G_S. Let $|Q|$ denote the number of edges in Q. Suppose $|Q| > p/2 + 1$.

Assume $p = 2$. If an edge in Q is level, then all edges are level. Since any two level edges with the same label are parallel in G_P, there would be two edges which are parallel in both graphs, a contradiction. If no edge in Q is level, then Q contains an S-cycle, contradicting (i).

Assume $p > 2$. Then Q would contain an S-cycle or a generalized S-cycle, a contradiction.

(v) Let $e_1, e_2, \ldots, e_p, e_1'$ be mutually parallel negative edges in G_S, numbered successively. We may assume that e_i has label i at one vertex for $i = 1, 2, \ldots, p$, so e_1' has label 1 at the same vertex. If e_i has label $\sigma(i)$ at the other end, we have the associated permutation σ. According to the orbits of σ, the edges e_i form essential orientation-preserving cycles on $\widehat{P}$ by [2, Lemma 2.3]. Let L be the cycle through vertex w_1. Then e_1' is not parallel to e_1. However, then a new cycle $(L - e_1) \cup e_1'$ is inessential on $\widehat{P}$, a contradiction. (This is essentially the same as the proof of [2, Lemma 4.2].) ■

Lemma 6.3 *Let $p \geq 3$. If x is not a label of a positive level edge in G_S, then any x-face in G_S has at least four sides.*

Proof First, there is no two-sided x-face, since such a face contains an S-cycle or a generalized S-cycle. Let D be a three-sidedd x-face, and let $\Gamma = G_S \cap D$. If Γ does not contain a level edge, then there is a Scharlemann cycle by [8], contradicting Lemma 6.2(i). Hence Γ contains a level edge. Notice that the faces of Γ consist of a single 3-gon f and bigons. Since Γ cannot contain a generalized S-cycle, any level edge appears in the 3-gon f. There are two cases.

Case (1): Only one label is a label of positive level edges in Γ.

Then, in fact, Γ contains only one level edge e. We may assume that it has label 1. Clearly, the bigon g adjacent to e has two corners $(1, 2)$ and $(p, 1)$. Moreover, the 3-gon f is also two-corned. That is, it has only $(1, 2)$-corners and $(p, 1)$-corners [9, Claim 3.7] (or see [11]).

Let H be the part of V_β between w_p and w_1, containing w_2. Let

$$X = N(\widehat{P} \cup H \cup f \cup g).$$

Then ∂X is a torus intersecting K_β fewer than t times. Hence it is boundary parallel in $M(\beta)$ or compressible. Thus $M(\beta)$ is bounded by at most one torus, a contradiction.

Case (2): Two labels are labels of positive level edges in Γ.

We may assume that the 3-gon f contains a level edge e_1 with label 1 and a level edge e_2 with label 2. Let g_i be the bigon adjacent to f, sharing e_i for $i = 1, 2$. Let H be the part of V_β between w_p and w_3, containing w_1. Construct $N(\widehat{P} \cup H \cup f \cup g_1 \cup g_2)$ as above. Then a similar argument to Case (1) implies a contradiction. ■

Lemma 6.4 $s \neq 1$.

Proof Assume $s = 1$. Notice that p is even, since the vertex of G_S has degree $3p$. There are $3p/2$ parallel loops in G_S, but this contradicts Lemma 6.2(iv), because $3p/2 > p/2 + 1$. ∎

Lemma 6.5 $p \neq 1$.

Proof Assume $p = 1$. By an Euler characteristic calculation, G_S has a disk face D. Let $X = N \cup V_\beta$. Then ∂X is a genus 2 closed surface disjoint from K_β. Let $D' = D - \operatorname{Int} X$. Surger ∂X along D'. The resulting surface is either a torus or a disjoint union of two tori, according as $\partial D'$ is non-separating or separating on ∂X. Thus $M(\beta)$ is bounded by at most two tori, a contradiction. ∎

Lemma 6.6 $s \geq 3$.

Proof By Lemmas 6.4 and 6.5, $s \geq 2$ and $p \geq 2$. Suppose $s = 2$. Then we can use the same notation $G_S = G(p_0, p_1, p_2)$ as in Section 4.

First assume $p = 2$. Since G_S cannot contain an S-cycle, $p_0 \leq 1$. By Lemma 6.2(iv) and (v), $p_i \leq 2$ for $i = 1, 2$. Thus $G_S = G(1, 2, 2)$, and there are two bigons and two 3-gons. Take a bigon D_1 and a 3-gon D_2. Let $X = N \cup V_\beta$, and $D_i' = D_i - \operatorname{Int} X$. Then ∂X is a genus 3 closed surface, on which $\partial D_1'$ and $\partial D_2'$ are homologically independent. Thus $M(\beta)$ is bounded by at most one torus, a contradiction.

Next, assume $p \geq 3$. By Lemma 6.2(iv), $p_0 \leq p/2 + 1$, but if the equality holds, there is an S-cycle. Hence $p_0 \leq (p + 1)/2$.

If the two vertices of G_S are parallel, then $p_i \leq p/2 + 1$ for $i = 1, 2$. So

$$3p \leq 2 \cdot (p + 1)/2 + 2(p/2 + 1) = 2p + 3,$$

giving $p \leq 3$. When $p = 3$, we have $p_i \leq 2$, giving

$$3p \leq 2 \cdot (p + 1)/2 + 2 \cdot (p + 1)/2 = 2p + 2.$$

This is a contradiction.

Therefore the two vertices of G_S are antiparallel. By Lemma 6.2(v), $p_i \leq p$ for $i = 1, 2$. So $3p = 2p_0 + p_1 + p_2 \leq 2p_0 + 2p$, giving $p_0 \geq p/2$. Hence $p_0 = p/2$ if p is even, and $p_0 = (p + 1)/2$ if p is odd. This implies that $G_S = G(p/2, p, p)$ if p is even, and $G_S = G((p+1)/2, p, p-1)$ if p is odd. For both cases, the same argument as the case $G_S = G(t/2, t, t)$ in the proof of Lemma 4.7 works. ∎

Lemma 6.7 *Any vertex of G_S, except S-vertices, has at least $2p$ positive edge endpoints. An S-vertex, if it exists, has at least p positive edge endpoints.*

Proof Lemma 5.1 holds again. Hence the proofs of Lemmas 5.2 and 5.3 work. ∎

Proposition 6.8 *$p \geq 3$ is impossible.*

Proof Using Lemma 6.3, instead of Lemma 2.8, the proof of Lemma 5.8 works. Hence G_S^+ does not contain a disk block. Then the proofs of Lemma 5.9 and Proposition 5.10 are applicable. ■

7 A Special Case: $p = 2$

Finally, we eliminate the situation where $s \geq 3$ and $p = 2$. Recall that any vertex of G_S, except S-vertices, has at least four positive edge endpoints, and that any S-vertex, if it exists, has at least two positive edge endpoints by Lemma 6.7.

Let $W = \mathrm{cl}(M(\beta) - N)$. We say that N is the black region, and W is the white region. Let $T = \partial N - \operatorname{Int} V_\beta$. As usual, S and T give a labeled graph pair $\{G'_S, G_T\}$. In fact, G_T is a double cover of G_P. The disk faces of G'_S are divided into black and white faces as usual. Thus any black bigon of G'_S corresponds to an edge of G_S.

Consider the genus three surface $R = \partial(N \cup V_\beta)$, which is disjoint from K_β.

Lemma 7.1 *For any two white disk faces of G'^+_S, their boundaries are parallel in R. In particular, all white disk faces of G_S^+ have the same number of sides, and G_S^+ cannot contain two adjacent 3-gons.*

Proof Suppose that G'^+_S contains two white disk faces whose boundaries are not parallel in R. Surgering R along them gives a torus or a disjoint union of two tori. Since the surface is disjoint from K_β, $M(\beta)$ is bounded by at most two tori, a contradiction.

Let f and g be adjacent 3-gons in G_S^+. Consider the two white faces f' and g' of G'^+_S corresponding to f and g, respectively. Then $\partial f'$ and $\partial g'$ are not parallel on R. ■

Lemma 7.2 *At any vertex of G_S, there are no consecutive pairs of parallel positive edges.*

Proof Otherwise, there are two consecutive bigons. However, it is easy to see that the corresponding white bigons have non-parallel boundaries on R. This contradicts Lemma 7.1. (See also [10, Lemma 6.3].) ■

We divide the argument into two cases.

Case (A): G_S^+ contains a bigon. Then all disk faces of G_S^+ are bigons by Lemma 7.1.

Lemma 7.3 *G_S^+ has at most two disk blocks. Any disk block consists of two vertices, one of which is an S-vertex, and a pair of parallel edges.*

Proof Let B be a disk block. Since all faces of B are disks, they are bigons. Thus B has only two vertices. By Lemma 7.2, one vertex is an S-vertex. Also, the other is a cut vertex of G_S^+.

Since G_S has at most two S-vertices, there are at most two disk blocks in G_S^+. ■

Lemma 7.4 *Any component of G_S^+ has an annulus support, and is extremal.*

Proof If there is a component with a disk support, then there is an extremal one, say Λ, with a disk support. Notice that all faces of Λ are disks, and hence bigons. Thus there are two consecutive bigons at a non-S-vertex, which is not a cut vertex of Λ. This contradicts Lemma 7.2. ■

Lemma 7.5 *Let Λ be an outermost component of G_S^+. Then Λ consists of two vertices together with a loop at one vertex and a pair of parallel level edges connecting the two vertices. Moreover, one vertex is an S-vertex.*

Proof By Lemma 7.2, Λ has no interior vertex. If Λ has no disk block, then it is a cycle of bigons, contradicting Lemma 7.2. (If the cycle is length one, then there is an S-cycle.) Also, any boundary vertex is incident to a disk block. Since there is only one disk block incident to Λ, we have the conclusion. ■

Lemma 7.6 *Case (A) is impossible.*

Proof By Lemmas 7.4 and 7.5, G_S^+ has two components Λ_1 and Λ_2, each of which satisfies the conclusion of Lemma 7.5. We may assume that Λ_i contains an S-vertex u_i for $i = 1, 2$. Notice that u_1 and u_2 are joined by the edges of a Scharlemann cycle in G_T, which do not lie on a disk in $\widehat{S}$ by Lemma 2.2(iii). Hence G_S^+ consists of Λ_1 and Λ_2, so $s = 4$. Since u_1 is incident to four negative edges, G_P contains at least two 1-faces by an Euler characteristic calculation. Each 1-face contains a Scharlemann cycle. Thus G_P has at least two Scharlemann cycles, so the four negative edges at u_1 are the edges of Scharlemann cycles in G_P. This is similar for u_2. Then the non-S-vertex of Λ_1 cannot be incident to a negative edge, a contradiction. ■

Case (B): Any disk face of G_S^+ has at least three sides.

Lemma 7.7 *G_S^+ has no disk block.*

Proof Let B be a disk block. It has at most one cut vertex and at most one S-vertex among boundary vertices. Let V, E, F be the number of vertices, edges, faces of B, respectively. Let V_i, V_b, V_c, V_s be the number of interior, boundary, cut, and S-vertices of B, respectively. Then $V = V_i + V_b$ and $V_c, V_s \leq 1$. (If an S-vertex is a cut vertex, then set $V_c = 1$ and $V_s = 0$.)

Any interior vertex has degree six, any boundary vertex, except a cut vertex and an S-vertex, has degree at least four, and a cut vertex or an S-vertex has degree at least two. By counting degrees,

$$2E \geq 6V_i + 4(V_b - V_c - V_s) + 2V_c + 2V_s = 6V - 2V_b - 2V_c - 2V_s.$$

Since each face of B has at least three sides, $2E \geq 3F + V_b = 3(1 - V + E) + V_b$. Then $3V - V_b - V_c - V_s \leq 3V - 3 - V_b$, and hence $V_c + V_s \geq 3$, a contradiction. ■

Lemma 7.8 *Case (B) is impossible.*

Proof By Lemma 7.7, any component of G_S^+ has an annulus support, and is extremal. Let Λ be an outermost component. After capping off the component of $\partial\widehat{S}$ near Λ, we regard Λ as lying in a disk. From this view point, we consider its interior vertices and boundary vertices. Let V, E, F be the number of vertices, edges, and disk faces of Λ, respectively. Let V_i, V_b, V_s be the number of interior, boundary, and S-vertices of Λ. Here Λ may have a monogon, which includes the disk capped off. As before, $2E \geq 6V_i + 4(V_b - V_s) + 2V_s = 6V - 2V_b - 2V_s$. Since each disk face of Λ, except at most one, has at least three sides, $2E \geq 3(F - 1) + 1 + V_b = 3E - 3V + 1 + V_b$. Then $3V - V_b - V_s \leq 3V - V_b - 1$, equivalently, $V_s \geq 1$. Thus $V_s = 1$ and all inequalities above are equalities. So, each disk face of Λ, except one monogon, has three sides. Since Λ has an S-vertex, $\widehat{S}$ is separating in $M(\alpha)$ and G_S^+ has exactly two components, Λ and Λ', where Λ' is another outermost component.

If $F = 1 - V + E = 2V - V_b > 2$, then Λ contains two adjacent 3-gons, contradicting Lemma 7.1. If $F = 1$, then $V = V_b = V_s = 1$ and $E = 1$. Hence Λ is an S-vertex with a loop. Similarly, Λ' has the same form. But this means $s = 2$, a contradiction. If $F = 2$, then $V = V_b = 2$ and $E = 3$. Then Λ consists of one pinched vertex and one bivalent S-vertex. Again, Λ' has the same form. Since u_1 is incident to four negative edges, G_P contains at least two Scharlemann cycles as in the proof of Lemma 7.6. Then any pinched vertex cannot be incident to a negative edge, a contradiction. ■

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Department of Mathematical Sciences
Seoul National University
San 56-1, Shinrim-dong, Kwanak-gu
Seoul 151-747
Korea
e-mail: sangyop@snu.ac.kr

Department of Mathematics and Mathematics Education
Hiroshima University
1-1-1 Kagamiyama
Higashi-hiroshima
Japan 739-8524
e-mail: teragai@hiroshima-u.ac.jp