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GEOMETRIC INVARIANT THEORY FOR HOLOMORPHIC FOLIATIONS ON $\mathbb{CP}^2$ OF DEGREE 2

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Abstract. Let $\mathcal{F}_2$ be the space of the holomorphic foliations on $\mathbb{CP}^2$ of degree 2. In this paper we study the linear action $PGL(3, \mathbb{C}) \times \mathcal{F}_2 \to \mathcal{F}_2$ given by $gX = DgX \circ (g^{-1})$ in the sense of the Geometric Invariant Theory. We obtain a characterisation of unstable and stable foliations according to properties of singular points and existence of invariant lines. We also prove that if X is an unstable foliation of degree 2, then X is transversal with respect to a rational fibration. Finally we prove that the geometric quotient of non-degenerate foliations without invariant lines is the moduli space of polarised del Pezzo surfaces of degree 2.

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1. Introduction. In this paper we study the properties of holomorphic foliations through the Geometric Invariant Theory (GIT), which was mainly developed by Hilbert and Mumford (see [**6**]).

The Geometric Invariant Theory tells us that it is possible to study the action of a reductive group G on a projective variety V by stratifying the points of variety in two categories: unstable points and semistable points. By restricting the action of G to the semistable points we obtain what is called a good quotient. The set of semistable points contains the open set of stable points and the restriction of the action to the stable points gives us a geometric quotient.

In most of the cases the variety V consists of certain geometric objects, such as algebraic curves or hypersurfaces. Furthermore, the usual action of G on V is such that objects are in the same orbit if and only if they are isomorphic.

The unstable points form a Zariski closed set in V and are in some sense degenerate objects. For example, if we consider the natural action of $PGL(3, \mathbb{C})$ on $\mathbb{CP}^9$, where $\mathbb{CP}^9$ is the space of plane curves of degree 3, then a cubic plane curve is unstable if and only if it has a triple point, or a cusp, or two components tangent at a point. Another example is the action of $PGL(2, \mathbb{C})$ in the space of binary forms of degree d. In this case a binary form of degree d is semistable if and only if it has no root of multiplicity greater that $\frac{d}{2}$ (see [**13**]).

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When V is the space of holomorphic foliations of degree d on the complex projective space $\mathbb{CP}^n$ and the group of automorphisms of $\mathbb{CP}^n$ acts by change of coordinates, Gómez-Mont and Kempf have proved in [**7**] that a foliation with only non-degenerate singularities is stable and that the distribution of its singular set is also stable. In the particular case of the projective plane $\mathbb{CP}^2$ that we have, an unstable foliation has degenerate singularities and, in some cases, it has an invariant line (see [**1**]).

In this work we analyse the properties of stable and unstable holomorphic foliations on $\mathbb{CP}^2$ of degree 2. In the following two sections we will make a summary of the Geometric Invariant Theory and the Theory of Holomorphic Foliations on $\mathbb{CP}^2$. In Section 4 we obtain the generators of unstable foliations, and give a characterisation of unstable and stable foliations according to the multiplicity and the Milnor number of its singular points and the existence of invariant lines.

In the next section we show that a generic unstable foliation of degree 2 is a Riccati foliation, i.e., there exists a rational fibration on $\mathbb{CP}^1$ whose generic fibre is transverse to the leaves of the foliation. In the last section we will prove that the geometric quotient of non-degenerate holomorphic foliations on $\mathbb{CP}^2$ of degree 2 without invariant lines is the coarse moduli space of the polarised del Pezzo surfaces of degree 2.

2. Geometric Invariant Theory (see [13]). The following is a summary of the Geometric Invariant Theory, which will be required for the sequel. All the definitions and results can be found in [**13**].

Let V be a projective variety in $\mathbb{CP}^n$, and consider a reductive group G acting linearly on V.

Definition 1. Let $x \in V \subset \mathbb{CP}^n$, and consider $\overline{x} \in \mathbb{C}^{n+1}$ such that $\overline{x} \in x$. Denote by $O(\overline{x})$ the orbit of $\overline{x}$ in the affine cone of V, then
(i) x is unstable if $0 \in \overline{O(\overline{x})}$.
(ii) x is semistable if $0 \notin \overline{O(\overline{x})}$. The set of semistable points will be denoted by V^{ss}.
(iii) x is stable if it is semistable, the orbit of x, $O(x)$, is closed in V^{ss} and $\dim O(x) = \dim G$. The set of stable points will be denoted by V^s.

The main result in the Geometric Invariant Theory is the following:

Theorem 1. (i) *There exists a good quotient (Y, ϕ) of V^{ss} by G, where Y is projective, and*
(ii) *there exists an open set $Y^s \subset Y$ such that $\phi^{-1}(Y^s) = V^s$ and (Y^s, ϕ) is a geometric quotient of V^s by G.*

Now we describe the Hilbert-Mumford criterion for finding the unstable points for a linear action. Let $\lambda : \mathbb{C}^* \to G$ be a 1-parameter subgroup (1-PS). Then we have a morphism, which we also denote by λ:

$$\mathbb{C}^* \to GL(n+1, \mathbb{C}),$$
$$t \mapsto \lambda(t) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}, v \mapsto \lambda(t)v,$$

and we know that this is a diagonal representation of $\mathbb{C}^*$. Therefore there exists a basis $\{v_0, \dots, v_n\}$ of $\mathbb{C}^{n+1}$ such that, for all $t \in \mathbb{C}^*$, $\lambda(t)v_i = t^{r_i} v_i$, where $r_i \in \mathbb{Z}$. This integer r_i is called the weight of v_i with respect to the action of λ on $\mathbb{C}^{n+1}$.

DEFINITION 2. Let $x \in V$ and let λ be a 1-PS. If $\bar{x} \in x$ and $\bar{x} = \sum_{i=0}^{n} a_i v_i$, then $\lambda(t)\bar{x} = \sum_{i=0}^{n} t^{r_i} a_i v_i$. We define the following function:

$$\mu(x, \lambda) := min\{r_i : a_i \neq 0\}.$$

The numerical criterion can now be stated.

THEOREM 2. (i) *x is stable if and only if $\mu(x, \lambda) < 0$ for every 1-PS, λ, of G,*
(ii) *x is unstable if and only if there exists a 1-PS, λ, of G such that $\mu(x, \lambda) > 0$.*

DEFINITION 3. We will say that a point $x \in V$ is λ-unstable or unstable with respect to λ if $\mu(x, \lambda) > 0$.

The following is a useful tool for the method of 1-PS when $G = SL(n, \mathbb{C})$. We formulate the result for the case $n = 3$.

LEMMA 1. *Every 1-parameter subgroup of $SL(3, \mathbb{C})$ can be written as*

$$g\lambda(t)g^{-1} = g\begin{pmatrix} t^{n_0} & 0 & 0 \\ 0 & t^{n_1} & 0 \\ 0 & 0 & t^{n_2} \end{pmatrix} g^{-1},$$

for some $g \in SL(3, \mathbb{C})$, where $n_0 \geq n_1 \geq n_2$ and $n_0 + n_1 + n_2 = 0$. We will denote the above diagonal 1-PS, λ, by $\lambda_{(n_0,n_1)}$ and will assume that the integers are relative primes.

REMARK 1. If $n_0 \geq n_1 \geq n_2$ and $n_0 + n_1 + n_2 = 0$, then $\frac{1}{2} \leq -\frac{n_2}{n_0} \leq 2$.

In this paper we use the group $SL(3, \mathbb{C})$ instead of $PGL(3, \mathbb{C})$ because these are isogenous.

3. Foliations on $\mathbb{CP}^2$. This section provides the definitions and results that we need to know about the holomorphic foliations on $\mathbb{CP}^2$ for the development of the paper.

DEFINITION 4. A holomorphic foliation X on $\mathbb{CP}^2$ of degree d is a non-trivial morphism of vector bundles:

$$X : \mathcal{O}_{\mathbb{CP}^2}(1 - d) \to T\mathbb{CP}^2,$$

where $T\mathbb{CP}^2$ is the tangent bundle of $\mathbb{CP}^2$; modulo the multiplication by a non-zero scalar. Then the space of foliations of degree d is $\mathcal{F}_d := \mathbb{P}H^0(\mathbb{CP}^2, T\mathbb{CP}^2(d - 1))$, where $d \geq 0$.

PROPOSITION 1. (see *[8]*) *Every foliation $X \in \mathcal{F}_d$ can be written as*

$$X = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z} = \begin{pmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{pmatrix},$$

where $P, Q, R \in \mathbb{C}[x, y, z]$ are homogeneous of degree d, modulo multiplication by a non-zero scalar, and if we consider the radial foliation

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

then X *and* $X + F(x, y, z)E$ *represent the same foliation for all* $F \in \mathbb{C}[x, y, z]$ *homogeneous of degree* $d - 1$.

Definition 5. A point $p = (a : b : c) \in \mathbb{CP}^2$ is singular for the above foliation X if $(P(a, b, c), Q(a, b, c), R(a, b, c)) = (ka, kb, kc)$ for some $k \in \mathbb{C}$. The set of singular points of X will be denoted by $Sing(X)$.

Definition 6. Let $X \in \mathcal{F}_d$ and let p be an isolated singularity of X. Let

$$\begin{pmatrix} Q(y, z) = Q_m(y, z) + Q_{m+1}(y, z) + \cdots \\ R(y, z) = R_n(y, z) + R_{n+1}(y, z) + \cdots \end{pmatrix}$$

be a local generator of X in $p = (1 : 0 : 0)$, where Q_i, R_i are homogeneous of degree i, and Q_m, R_n are not identically zero. We define the Milnor number of p by $\mu_p(X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,p}}{<Q,R>}$ and the multiplicity of p by $m_p(X) := min\{m, n\}$.

In the sequel, if there is no confusion with point p, we will use the following notation $I(Q, R) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,p}}{<Q,R>}$.

Proposition 2. (see *[2]*) *Let* X *be a foliation on* $\mathbb{CP}^2$ *of degree* d *with isolated singularities, then*

$$d^2 + d + 1 = \sum_{p \in \mathbb{CP}^2} \mu_p(X).$$

Definition 7. A foliation X is non-degenerate if it has isolated singularities and every singular point has Milnor number 1.

Definition 8. An irreducible plane curve defined by a polynomial $F(x, y, z)$ is an algebraic solution for X or invariant by X if and only if there exists a polynomial $H(x, y, z)$ such that

$$P(x, y, z)\frac{\partial F(x, y, z)}{\partial x} + Q(x, y, z)\frac{\partial F(x, y, z)}{\partial y} + R(x, y, z)\frac{\partial F(x, y, z)}{\partial z}$$
$$= F(x, y, z)H(x, y, z).$$

Definition 9. A foliation X is a Riccati foliation if there exists a rational fibration (maybe with singular fibres) on a surface S, obtained from $\mathbb{CP}^2$ after a finite number of blow-ups, whose generic fibre is transverse to the lifted foliation of X in S.

Remark 2. A non-degenerate foliation X of degree d has a line solution L if and only if L has $d + 1$ singular points.

Proof. If the line $L = ax + by + cz$ has $d + 1$ singular points of $X = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z}$, then the polynomial of degree d, $aP(x, y, z) + bQ(x, y, z) + cR(x, y, z)$ and L have $d + 1$ common zeros. Hence by Bézout, L is a factor of this polynomial.

Suppose z is a solution for X, then we can write $X = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y}$, so $Sing X \cap \{(x : y : z) \in \mathbb{CP}^2 : z \neq 0\} = V(P(x, y, 1), Q(x, y, 1))$ has at most d^2 different points. Hence, $Sing X \cap V(z) = V(yP(x, y, z) - xQ(x, y, z), z)$ has $d + 1$ points. $\square$

4. Characterisation of unstable and stable Holomorphic Foliations of degree 2. The group $PGL(3,\mathbb{C})$ of automorphisms of $\mathbb{CP}^2$ is a reductive group (see page 48 of [**13**]) and it acts linearly on $\mathcal{F}_2$ by change of coordinates:

$$\begin{aligned} PGL(3,\mathbb{C}) \times \mathcal{F}_2 &\to \mathcal{F}_2 \\ (g, X) &\mapsto gX = DgX \circ (g^{-1}). \end{aligned}$$

Before establishing the results of this section it is useful to recall the following:

Generically a foliation does not have algebraic solutions, this is a theorem by Jouanolou and was completed by Neto and Soares (see [**10**] and [**12**]).

Theorem 3. *For $d \geq 2$, the subset $\{X \in \mathcal{F}_d : X$ has no algebraic solutions$\}$ is non-empty and dense in $\mathcal{F}_d$ and contains an open and dense subset.*

Consider the following projective subspaces in $\mathcal{F}_2$:

$$CN_1 = \mathbb{P}\left\langle xy\frac{\partial}{\partial x}, xz\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial x}, yz\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y} y^2\frac{\partial}{\partial z}\right\rangle$$

$$CN_2 = \mathbb{P}\left\langle xz\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial x}, yz\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, xz\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y}\right\rangle,$$

we know by Proposition 2 that a foliation with isolated singularities of degree 2 has seven singularities counting Milnor number. In this section we prove the following:

Theorem 4. *Let $X \in \mathcal{F}_2$ be a foliation with isolated singularities, then X is unstable if and only if it has one of the following properties:*
(i) *There exists a singular point p of multiplicity* 2, *or*
(ii) *has an invariant line with a unique singular point with multiplicity* 1 *and Milnor number* 5.

Moreover, a foliation X satisfies (i) *if and only if there exists $g \in SL(3,\mathbb{C})$ such that $gX \in CN_1$ and it satisfies* (ii) *if and only if there exists $g \in SL(3,\mathbb{C})$ such that $gX \in CN_2 - CN_1$.*

With this theorem we will see that every unstable foliation has an invariant line. We also have the following:

Theorem 5. *Let $X \in \mathcal{F}_2$ be a foliation with isolated singularities, then X is semistable but not stable if and only if the multiplicity of every singular point is one, and X has an invariant line L with one singular point with Milnor number* 3 *or* 4 *or L has two different singular points, one of them with Milnor number* 3, 4, 5 *or* 6.

Corollary 1. *A foliation $X \in \mathcal{F}_2$ with isolated singularities is stable if and only if every singular point of X has multiplicity one and if we have one of the following: X does*

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not have invariant lines, or if L is an invariant line for X, then

(1) *L has two different singularities, one with Milnor number 2 and the other with Milnor Number 1; or*
(2) *L has two different singularities, both with Milnor number 2; or*
(3) *L has three different singularities.*

To prove these results we need the following:

LEMMA 2. *Let X be a foliation of degree 2. Then, X is unstable if and only if there exists $g \in SL(3,\mathbb{C})$ such that gX is unstable with respect to $\lambda_1 := \lambda_{(2,-1)}$ or with respect to $\lambda_2 := \lambda_{(4,1)}$.*

Proof. Consider the monomial foliations: $X_{i_0j_0} = x^{2-j_0}y^{j_0-i_0}z^{i_0}\frac{\partial}{\partial x}$, $X_{i_1j_1} = x^{2-j_1}y^{j_1-i_1}z^{i_1}\frac{\partial}{\partial y}$ and $X_{i_2j_2} = x^{2-j_2}y^{j_2-i_2}z^{i_2}\frac{\partial}{\partial z}$, where $j_l \geq i_l$, $l = 0, 1, 2$ and $j_l, i_l = 0, 1, 2$. The weight of the foliation $X_{i_lj_l}$ with respect to $\lambda_{(n_0,n_1)}$ is $n_l - n_0(2-j_l) - n_1(j_l - i_l) - n_2 i_l$.

It is easy to check the following: $\mu(X_{i_0j_0}, \lambda_1) > 0$ if and only if $j_0 = 1, 2$; $\mu(X_{i_1j_1}, \lambda_1) > 0$ if and only if $j_1 = 2$ and $\mu(X_{i_2j_2}, \lambda_1) > 0$ if and only if $j_2 = 2$.

We conclude that a foliation X, which is a linear combination of $X_{i_lj_l}$, is unstable with respect to λ_1 if and only if its non-zero foliations $X_{i_lj_l}$ satisfy $j_0 = 1$ or $j_0 = 2$, $j_1 = 2$ and $j_2 = 2$. For λ_2 we have

$$\mu(X_{i_0j_0}, \lambda_2) > 0 \Leftrightarrow 3j_0 + 6i_0 > 4 \Leftrightarrow j_0 = i_0 = 1 \quad \text{or } j_0 = 2,$$
$$\mu(X_{i_1j_1}, \lambda_2) > 0 \Leftrightarrow 3j_1 + 6i_1 > 7 \Leftrightarrow j_1 = i_1 = 1 \quad \text{or } j_1 = i_1 = 2 \text{ or } j_1 = 2, i_1 = 1,$$
$$\mu(X_{i_2j_2}, \lambda_2) > 0 \Leftrightarrow 3j_2 + 6i_2 > 13 \Leftrightarrow j_2 = i_2 = 2.$$

On the other hand, in every case the conditions to have $n_l - n_0(2-j_l) - n_1(j_l - i_l) - n_2 i_l > 0$ are as follows:

$$\begin{array}{lll}
j_0 = i_0 = 0 \quad \text{is not possible}, & j_1 = i_1 = 0 \Leftrightarrow \dfrac{n_2}{n_0} < -3, & j_2 = i_2 = 0 \Leftrightarrow \dfrac{n_2}{n_0} > 2, \\
j_0 = 1, i_0 = 0 \Leftrightarrow \dfrac{n_2}{n_0} > -1, & j_1 = 1, i_1 = 0 \quad \text{is not possible}, & j_2 = 1, i_2 = 0 \Leftrightarrow \dfrac{n_2}{n_0} > 0, \\
j_0 = i_0 = 1 \Leftrightarrow \dfrac{n_2}{n_0} < 0, & j_1 = i_1 = 1 \Leftrightarrow \dfrac{n_2}{n_0} < -1, & j_2 = i_2 = 1 \quad \text{is not possible}, \\
j_0 = 2, i_0 = 0 \Leftrightarrow \dfrac{n_2}{n_0} > -\dfrac{3}{2}, & j_1 = 2, i_1 = 0 \Leftrightarrow \dfrac{n_2}{n_0} > -1, & j_2 = 2, i_2 = 0 \Leftrightarrow \dfrac{n_2}{n_0} > -\dfrac{2}{3}, \\
j_0 = 2, i_0 = 1 \Leftrightarrow \dfrac{n_2}{n_0} > -2, & j_1 = 2, i_1 = 1 \Leftrightarrow \dfrac{n_2}{n_0} < 0, & j_2 = 2, i_2 = 1 \Leftrightarrow \dfrac{n_2}{n_0} > -1, \\
j_0 = i_0 = 2 \Leftrightarrow \dfrac{n_2}{n_0} < \dfrac{1}{2}, & j_1 = i_1 = 2 \Leftrightarrow \dfrac{n_2}{n_0} < -\dfrac{1}{3}, & j_2 = i_2 = 2 \Leftrightarrow \dfrac{n_2}{n_0} < 0.
\end{array}$$

We can suppose that $n_0 \geq n_1 \geq n_2$, then $-\frac{1}{2} \geq \frac{n_2}{n_0} \geq -2$; with the above we conclude that if $-\frac{2}{3} < \frac{n_2}{n_0} < -\frac{1}{3}$ or $-\frac{3}{2} < \frac{n_2}{n_0} < -1$, then we obtain maximal sets of generators

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for unstable foliations, which are as follows:

$$\begin{aligned}
CN_1 &= \left\{X : \mu\big(X, \lambda_{(n_0,n_1)}\big) > 0 \quad \forall \frac{n_2}{n_0} \in \left(-\frac{2}{3}, -\frac{1}{3}\right)\right\} \\
&= \mathbb{P}\left\langle xy\frac{\partial}{\partial x}, xz\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial x}, yz\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y} y^2\frac{\partial}{\partial z}\right\rangle \\
CN_2 &= \left\{X : \mu\big(X, \lambda_{(n_0,n_1)}\big) > 0 \quad \forall \frac{n_2}{n_0} \in \left(-\frac{3}{2}, -1\right)\right\} \\
&= \mathbb{P}\left\langle xz\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial x}, yz\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, xz\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y}\right\rangle
\end{aligned}$$

To finish the proof we must note that $CN_1 = \{X : \mu(X, \lambda_1) > 0\}$ and $CN_2 = \{X : \mu(X, \lambda_2) > 0\}$. $\square$

Proof of Theorem 4. Let

$$X = \begin{pmatrix} a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2 \\ b_0x^2 + b_1xy + b_2xz + b_3y^2 + b_4yz + b_5z^2 \\ c_0x^2 + c_1xy + c_3y^2 \end{pmatrix}$$

a generic foliation of degree 2 with isolated singularities. In the affine chart U_0 we have the following foliation:

$$X_0 = \begin{pmatrix} b_0 + (b_1 - a_0)y + b_2z + (b_3 - a_1)y^2 + (b_4 - a_2)yz + b_5z^2 - a_3y^3 - a_4y^2z - a_5yz^2 \\ c_0 + c_1y - a_0z + c_3y^2 - a_1yz - a_2z^2 - a_3y^2z - a_4yz^2 - a_5z^3 \end{pmatrix},$$

if $p_0 = (1 : 0 : 0)$, then $m_{p_0}(X) > 1$ if and only if $a_0 = b_0 = b_1 = b_2 = c_0 = c_1 = 0$, i.e. if and only if $X \in CN_1$.

Now, let $X \in CN_2 - CN_1$,

$$X = \begin{pmatrix} a_2xz + a_4yz + a_3y^2 + a_5z^2 \\ xz + b_4yz + b_5z^2 \\ 0 \end{pmatrix}.$$

The point p_0 is singular for this foliation and in the affine chart U_0, the foliation X is

$$X_0 = \begin{pmatrix} z + (b_4 - a_2)yz + b_5z^2 - a_3y^3 - a_4y^2z - a_5yz^2 \\ -a_2z^2 - a_3y^2z - a_4yz^2 - a_5z^3 \end{pmatrix}.$$

Let

$$\begin{aligned}
Q(y, z) &= z + (b_4 - a_2)yz + b_5z^2 - a_3y^3 - a_4y^2z - a_5yz^2, \\
R(y, z) &= -a_2z^2 - a_3y^2z - a_4yz^2 - a_5z^3.
\end{aligned}$$

Since X has isolated singularities, $a_3 \neq 0$, and with this it is easy to see that $I(Q, R) = \mu_{p_0}(X) = 5$. Note that p_0 is the unique singular point in the invariant line $z = 0$.

Now we will suppose that the foliation X has an invariant line with a unique singular point with Milnor number 5. We can suppose that the line is z and that the point is p_0. Then X has the form:

$$X = \begin{pmatrix} a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2 \\ b_0x^2 + b_1xy + b_2xz + b_3y^2 + b_4yz + b_5z^2 \\ 0 \end{pmatrix}$$

Condition 1. If p_0 is singular, then $b_0 = 0$.
Condition 2. If $(0 : 1 : 0)$ is not singular, then $a_3 \neq 0$, let us suppose $a_3 = 1$.
If $a_0 \neq 0$, in the affine chart U_0 we have:

$$X_0 = \begin{pmatrix} (b_1 - a_0)y + b_2z + (b_3 - a_1)y^2 + (b_4 - a_2)yz + b_5z^2 - y(y^2 + a_4yz + a_5z^2) \\ -z(a_0 + a_1y + a_2z + y^2 + a_4yz + a_5z^2) \end{pmatrix},$$

therefore

$$\mu_{p_0}(X) = I(z, (b_1 - a_0)y + (b_3 - a_1)y^2 - y^3) \leq 3,$$

hence $a_0 = 0$ and

$$X_0 = \begin{pmatrix} b_1y + b_2z + (b_3 - a_1)y^2 + (b_4 - a_2)yz + b_5z^2 - y(y^2 + a_4yz + a_5z^2) \\ -z(a_1y + a_2z + a_3y^2 + a_4yz + a_5z^2) \end{pmatrix}.$$

We must note that $(y, 0)$ is singular for this local vector field if and only if $b_1y + (b_3 - a_1)y^2 - y^3 = 0$ and this polynomial has its three roots equal to zero if and only if $b_1 = b_3 - a_1 = 0$, therefore,

$$X_0 = \begin{pmatrix} b_2z + (b_4 - a_2)yz + b_5z^2 - y(y^2 + a_4yz + a_5z^2) \\ -z(a_1y + a_2z + a_3y^2 + a_4yz + a_5z^2) \end{pmatrix}.$$

Moreover, because $m_{p_0}(X) = 1$, then $b_2 \neq 0$ and $\mu_{p_0}(X) = I(z, y^3) + I(a_1y + a_2z + y^2 + a_4yz + a_5z^2, b_2z + (b_4 - a_2)yz + b_5z^2 - y(y^2 + a_4yz + a_5z^2) = 4$ if and only if $a_1 \neq 0$. Then

$$X = \begin{pmatrix} a_2xz + y^2 + a_4yz + a_5z^2 \\ b_2xz + b_4yz + b_5z^2 \\ 0 \end{pmatrix},$$

and $I(a_2z + y^2 + a_4yz + a_5z^2, b_2z + (b_4 - a_2)yz + b_5z^2 - y(y^2 + a_4yz + a_5z^2) = I(a_2 z + y^2 + a_4yz + a_5z^2, b_2z + b_4yz + b_5z^2) = I(y^2, z) = 2$. Therefore $\mu_{p_0}(X) = 5$ and $X \in CN_2 - CN_1$. □

REMARK 3. An unstable foliation $X \in \mathcal{F}_2$ with isolated singularities has an invariant line.

Proof. If α is such that $\alpha^3c_3 - \alpha^2b_3 - \alpha b_4 - b_5 = 0$, then $y - \alpha z$ is an invariant line for a foliation in CN_1; and z is an invariant line for a foliation in CN_2. □

LEMMA 3. *Let $X \in \mathcal{F}_2$ with the following properties: Every singularity of X has multiplicity one and X has an invariant line with a unique singularity p. Then $\mu_p(X) = 5, 4$ or 3.*

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Proof. We follow the notation of Theorem 4. If z is the invariant line, $p_1 = (0:1:0)$ is singular and $p_0 = (1:0:0)$ is not singular, then in the affine chart U_1 we have

$$X_1 = \begin{pmatrix} a_0x^2 + a_1x + a_2xz + a_4z + a_5z^2 - x(x^2 + b_1x + b_2xz + b_3 + b_4z + b_5z^2) \\ -z(x^2 + b_1x + b_2xz + b_3 + b_4z + b_5z^2) \end{pmatrix}$$
$$= \begin{pmatrix} (a_1 - b_3)x + a_4z + (a_0 - b_1)x^2 + (a_2 - b_4)xz + a_5z^2 - x^3 - a_2x^2z - b_5xz^2 \\ -z(b_3 + b_1x + b_4z + x^2 + a_2xz + b_5z^2) \end{pmatrix}.$$

Note that $(x, 0)$ is singular in this chart if and only if $(a_1 - b_3)x + (a_0 - b_1)x^2 - x^3 = 0$, and this polynomial has its three roots equal to zero if and only if $a_1 = b_3$ and $a_0 = b_1$. Therefore $\mu_{p_1}(X) \geq I(z, x^3 + zH(x, z)) = 3$, and we can conclude that $\mu_{p_1}(X) = 3$ if and only if $b_3 \neq 0$.

If $b_3 = 0$, and if $m_{p_1}(X) = 1$, then we must have $a_4 \neq 0$. Hence $\mu_{p_1}(X) = 4$ if and only if $b_1 \neq 0$; moreover, $\mu_{p_1}(X) = 3 + I_p(z, x^2) = 5$ if and only if $b_1 = 0$ (in this last case the foliation is unstable). □

Proof of Theorem 5. We know that a foliation X is semistable but not stable if and only if X is not unstable and there exists $g \in SL(3, \mathbb{C})$ and a diagonal 1-PS, λ, of $SL(3, \mathbb{C})$ such that $\mu(X, g\lambda g^{-1}) = 0$. The weights of the monomial foliations with respect to the one-parameter subgroup $\lambda_{(1,r)}$, where $-\frac{1}{2} \leq r \leq 1$ are

	x^2	xy	xz	y^2	yz	z^2
$\frac{\partial}{\partial x}$	-1	$-r$	$1+r$	$1-2r$	2	$3+2r$
$\frac{\partial}{\partial y}$	$r-2$	-1	$2r$	$-r$	$1+r$	$3r+2$
$\frac{\partial}{\partial z}$	$-r-3$	$-2-2r$	-1	$-1-3r$	$-r$	$1+r.$

The unique cases where we can have any weight equal to zero with the condition $r \in \mathbb{Q} \cap [-\frac{1}{2}, 1]$ are the following:

$r=0$	x^2	xy	xz	y^2	yz	z^2
$\frac{\partial}{\partial x}$	-1	0	1	1	2	3
$\frac{\partial}{\partial y}$	-2	-1	0	0	1	2
$\frac{\partial}{\partial z}$	-3	-2	-1	-1	0	1

$r=\frac{1}{2}$	x^2	xy	xz	y^2	yz	z^2
$\frac{\partial}{\partial x}$	-1	$-\frac{1}{2}$	$\frac{3}{2}$	0	2	4
$\frac{\partial}{\partial y}$	$-\frac{3}{2}$	-1	1	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$
$\frac{\partial}{\partial z}$	$-\frac{7}{2}$	-3	-1	$-\frac{5}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$

$r=-\frac{1}{3}$	x^2	xy	xz	y^2	yz	z^2
$\frac{\partial}{\partial x}$	-1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$
$\frac{\partial}{\partial y}$	$-\frac{7}{3}$	-1	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	3
$\frac{\partial}{\partial z}$	$-\frac{8}{3}$	$-\frac{4}{3}$	-1	0	$\frac{1}{3}$	$\frac{2}{3}.$

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The monomial foliations with non-negative weights for $r = \frac{1}{2}$ also have non-negative weights for $r = 0$, and the monomial foliations with non-negative weights for $r = -\frac{1}{3}$ are in CN_1. Then, the unique case to consider is

$$X = \begin{pmatrix} a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2 \\ b_2xz + b_3y^2 + b_4yz + b_5z^2 \\ 0 \end{pmatrix},$$

the point $p_0 = (1 : 0 : 0)$ is singular and the line $z = 0$ is invariant for the foliation. In the affine chart U_0, X is

$$X_0 = \begin{pmatrix} b_2z + (b_3 - a_1)y^2 + (b_4 - a_2)yz + b_5z^2 - a_3y^3 - a_4y^2z - a_5yz^2 \\ -z(a_1y + a_2z + a_3y^2 + a_4yz + a_5z^2) \end{pmatrix},$$

the multiplicity of p_0 is one because otherwise the foliation is unstable, then we can consider $b_2 = 1$. The Milnor number of p_0 is

$$\mu_{p_0}(X) = I(z, (b_3 - a_1)y^2 - a_3y^3) + I(a_1y + a_2z + a_3y^2 + a_4yz \\ + a_5z^2, z + (b_3 - a_1)y^2 + (b_4 - a_2)yz + b_5z^2 - a_3y^3 - a_4y^2z - a_5yz^2).$$

We will see now the remaining singular points in $z = 0$. We have that $(\alpha : 1 : 0)$ is singular if and only if $(a_1 - b_3)\alpha + a_3 = 0$, therefore the line has at most two singular points.

Case 1. $a_1 \neq b_3$, then $(\frac{a_3}{b_3 - a_1} : 1 : 0)$ is the other singular point in z different from p_0 and $\mu_{p_0}(X) \geq 2 + 1$, this value could be 3, 4, 5 or 6.

Case 2. $a_1 = b_3$, then $a_3 \neq 0$, if not, $z = 0$ will be a curve of singularities. Therefore $z = 0$ has only the singularity p_0 and by Lemma 3 its Milnor number is 3 or 4. □

5. An unstable Foliation is a Riccati Foliation. In this section we prove the following:

THEOREM 6. *The generic unstable foliation on $\mathbb{CP}^2$ of degree 2 is Riccati.*

Now we give the proof of this result.

LEMMA 4. *Let X be a foliation of $\mathbb{CP}^2$ of degree 2 with isolated singularities. Then $X \in CN_1$ if and only if X is transversal with respect to the rational fibration given by the flow of λ_1 and the Milnor number of $p_0 = (1 : 0 : 0)$ is greater than 1.*

Proof. Let X be a generic point in CN_1, i.e.

$$X = \begin{pmatrix} a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2 \\ b_3y^2 + b_4yz + b_5z^2 \\ c_3y^2 \end{pmatrix},$$

where $a_i, b_i, c_i \in \mathbb{C}$. Since $m_{p_0}(X) > 1$, then $\mu_{p_0}(X) \geq 4$.

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The associated foliation to the flow given by λ_1 is $X_{\lambda_1} = 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$. It has a line of singularities defined by $x = 0$, and its leaves are the lines passing through the singular point p_0. Then we must blow up once the point p_0 to separate the leaves and thus obtain a rational fibration, we will denote the corresponding foliation by $\widetilde{X}_{\lambda_1}$.

The foliation X in the affine chart U_0 is

$$X_0 = \begin{pmatrix} (b_3 - a_1)y^2 + (b_4 - a_2)yz + b_5z^2 - a_3y^3 - a_4y^2z - a_5yz^2 \\ c_3y^2 - a_1yz - a_2z^2 - a_3y^2z - a_4yz^2 - a_5z^3 \end{pmatrix}.$$

Lifting the foliation X with the same blow-up we obtain

$$\widetilde{X}_0 = \begin{pmatrix} w_1^2\big(a_3 + a_4w_2 + a_5w_2^2\big) - w_1\big((b_3 - a_1) + (b_4 - a_2)w_2 + b_5w_2^2\big) \\ -c_3 + b_3w_2 + b_4w_2^2 + b_5w_2^3 \end{pmatrix}.$$

Then the fibres of the rational fibration obtained by $\widetilde{X}_{\lambda_1}$ and the leaves of $\widetilde{X}$ are tangent only in the common leaves $w_2 - k = 0$, where k is a root of the polynomial $-c_3 + b_3w_2 + b_4w_2^2 + b_5w_2^3$.

In the line $x = 0$ we have the same property. We conclude that X is λ_1 − Riccati.

Now, let

$$X = \begin{pmatrix} a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2 \\ b_0x^2 + b_1xy + b_2xz + b_3y^2 + b_4yz + b_5z^2 \\ c_0x^2 + c_1xy + c_2xz + c_3y^2 + c_4yz + c_5z^2 \end{pmatrix}$$

be a foliation of degree 2 with isolated singularities, λ_1 − Riccati and such that $\mu_{p_0}(X) > 1$.

The first condition in the coefficients is $b_0 = c_0 = 0$ because p_0 is a singular point for the foliation.

In order to separate the leaves of X_{λ_1} we must blow up the point p_0, then we lift the foliation X with this blow-up and see the condition in the coefficients of X to have transversality with the rational fibration defined by X_{λ_1}. As before, we have

$$\widetilde{X}_0 = \begin{pmatrix} w_1^3\big(a_3 + a_4w_2 + a_5w_2^2\big) + w_1^2\big((a_1 - b_3) + (a_2 - b_4)w_2 - b_5w_2^2\big) \\ \quad + w_1((a_0 - b_1) - b_2w_2) \\ w_1\big(- c_3 + (b_3 - c_4)w_2 + (b_4 - c_5)w_2^2 + b_5w_2^3\big) + \big(- c_1 + (b_1 - c_2)w_2 + b_2w_2^2\big) \end{pmatrix}.$$

Then, to have transversality with the fibration defined by X_{λ_1}, either of the following conditions is satisfied:
$a_3 + a_4w_2 + a_5w_2^2 = 0$ and $-c_3 + (b_3 - c_4)w_2 + (b_4 - c_5)w_2^2 + b_5w_2^3 = 0$; or $-c_1 + (b_1 - c_2)w_2 + b_2w_2^2 = 0$. This means $a_3 = a_4 = a_5 = c_3 = b_5 = 0$, $b_3 = c_4$ and $b_4 = c_5$; or $c_1 = b_2 = 0$ and $b_1 = c_2$.
The first one implies

$$X = \begin{pmatrix} a_0x^2 + a_1xy + a_2xz \\ b_1xy + b_2xz \\ c_1xy + c_2xz \end{pmatrix},$$

but it has a curve of singularities defined by x. Thus, we must have the second condition, i.e.

$$X = \begin{pmatrix} a_0x^2 + a_1xy + a_2xz + a_3y^2 + a_4yz + a_5z^2 \\ b_3y^2 + b_4yz + b_5z^2 \\ c_3y^2 + c_4yz + c_5z^2 \end{pmatrix}.$$

The hypothesis $\mu_{p_0}(X) > 1$ for X implies $a_0 = 0$. Then $X \in CN_1$. $\square$

PROPOSITION 3. *Let Y be a generic foliation in CN_2. Then the Kodaira dimension of Y is* 1.

Proof. Let $Y \in CN_2$, then

$$Y = \begin{pmatrix} a_2xz + a_3y^2 + a_4yz + a_5z^2 \\ b_2xz + b_4yz + b_5z^2 \\ 0 \end{pmatrix},$$

we must reduce the singularities of Y in the sense of [**14**].

We note that the multiplicity of $p_0 = (1 : 0 : 0)$ is 5. Generically we need two blow-ups to obtain the reduced foliation, $X = Y_{\mathrm{red}}$, birational to Y. Let S be the surface where X has only reduced singularities, let $L \subset S$ be the strict transform of a line that does not pass through the point p_{01}, where we made the blow-up in $\widetilde{\mathbb{CP}^2}$, and let $L_1 \subset \widetilde{\mathbb{CP}^2}$ be the strict transform of line L_0 in $\mathbb{CP}^2$, which does not pass through p_0. We have the following:

Blowing-up	Exceptional divisor	Foliation	Canonical bundle
$\widetilde{\widetilde{\mathbb{CP}^2}} := S$	E	X	$K_X = \mathcal{O}_S(L - E)$
$\downarrow \pi$			
$p_{01} \in \widetilde{\mathbb{CP}^2}$	E_1	$\tilde{Y}$	$K_{\tilde{Y}} = \mathcal{O}_{\widetilde{\mathbb{CP}^2}}(L_1)$
$\downarrow \pi_1$			
$p_0 \in \mathbb{CP}^2$		Y	$K_Y = \mathcal{O}_{\mathbb{CP}^2}(1)$,

where $L \cdot E = 0$, $L \cdot L = 1$, $E^2 = -1$, $E \approx \mathbb{CP}^1$.

The linear part of the reduced foliation X, in the unique singular point in E is

$$DX(0,0) = \begin{pmatrix} 2b_2 & 0 \\ 0 & -b_2 \end{pmatrix},$$

then $\frac{2b_2}{-b_2} = -2 \notin \mathbb{Q}^+$.

In [**11**] the author gives the definition of the Kodaira dimension for reduced foliations and proves analogous results to the case of algebraic varieties. In this case the Kodaira dimension of the foliation X is calculated as follows:

$$kod(X) = \limsup_{n\to\infty} \frac{1}{\log n} \log h^0(S, \mathcal{O}_S(nL - nE)),$$

and to compute it we consider the Riemann–Roch formula for surfaces: Let $n \in \mathbb{Z}^{>0}$, we obtain

$$\begin{aligned} h^0(S, \mathcal{O}_S(n(L-E))) = {} & \frac{1}{2}(-nL \cdot K_S + nE \cdot K_S) + \chi(\mathcal{O}_S) + h^1(S, \mathcal{O}_S(n(L-E))) \\ & - h^2(S, \mathcal{O}_S(n(L-E))). \end{aligned}$$

If E_1 is the exceptional divisor of the first blowing-up, π_1, then

$$K_S = \pi^* K_{\widetilde{\mathbb{CP}^2}} + E = \pi^*(\pi_1^* K_{\mathbb{CP}^2} + E_1) + E = \pi^*(\pi_1^* \mathcal{O}_{\mathbb{CP}^2}(-3) + E_1) + E.$$

Also we have

$$\begin{aligned} L \cdot K_S &= L \cdot \pi^* K_{\widetilde{\mathbb{CP}^2}} = \pi^* L_1 \cdot \pi^* K_{\widetilde{\mathbb{CP}^2}} = L_1 \cdot K_{\widetilde{\mathbb{CP}^2}} = L_1 \cdot (\pi_1^*(\mathcal{O}_{\mathbb{CP}^2}(-3)) + E_1) \\ &= \pi_1^*(L_0) \cdot \pi_1^* \mathcal{O}_{\mathbb{CP}^2}(-3) = \mathcal{O}_{\mathbb{CP}^2}(1) \cdot \mathcal{O}_{\mathbb{CP}^2}(-3) = -3, \end{aligned}$$

and $E \cdot K_S = E \cdot (\pi^* K_{\widetilde{\mathbb{CP}^2}} + E) = -1$, $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{\mathbb{CP}^2}) = 1$.
Then,

$$h^0(S, \mathcal{O}_S(n(L-E))) = n + 1 + h^1(S, \mathcal{O}_S(n(L-E))) - h^2(S, \mathcal{O}_S(n(L-E))).$$

If we consider n sufficiently large, we can assume by Serre's Theorem (see [**15**]) that $h^1(S, \mathcal{O}_S(n(L-E))) = 0$ and $h^2(S, \mathcal{O}_S(n(L-E)) = 0$. Finally we obtain

$$kod(X) = \limsup_{n\to\infty} \frac{1}{\log n} \log h^0(S, \mathcal{O}_S(nL - nE)) = \limsup_{n\to\infty} \frac{\log(n+1)}{\log n} = 1.$$

□

LEMMA 5. *The canonical bundle, $K_X = \mathcal{O}_S(L - E)$, of the reduced foliation arising from $Y \in CN_2$ defines a linear system of dimension* 1 *without base points.*

Proof. Since $kod(X) = 1$, then $h^0(S, \mathcal{O}_S(L - E)) \leq 2$. We consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_S(L-E) \longrightarrow \mathcal{O}_S(L) \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

this induces the following sequence:

$$\begin{aligned} 0 \longrightarrow H^0(S, \mathcal{O}_S(L-E)) \longrightarrow H^0(S, \mathcal{O}_S(L)) \longrightarrow H^0(E, \mathcal{O}_E) \\ \longrightarrow H^1(S, \mathcal{O}_S(L-E)) \longrightarrow 0. \end{aligned}$$

Then, we have $h^0(S, \mathcal{O}_S(L-E)) = h^0(S, \mathcal{O}_S(L)) - h^0(E, \mathcal{O}_E) + h^1(S, \mathcal{O}_S(L-E)) = 2 + h^1(S, \mathcal{O}_S(L-E)) \geq 2$.

Therefore, $h^0(S, K_X) = 2$. If we assume that $s_0, s_1 \in H^0(S, K_X)$ and $(s_0)_0 \cap (s_1)_0 \neq \emptyset$, then $K_X^2 = (s_0)_0 \cdot (s_1)_0 > 0$, but $K_X^2 = (L-E)^2 = 0$. Therefore, $H^0(S, \mathcal{O}_S(L-E))$ does not have base points. □

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THEOREM 7. *The generic foliation Y in CN_2 is Riccati with respect to the rational fibration defined by the pencil of the canonical bundle of the reduced foliation birrational to Y.*

Proof. The above pencil defines a fibration $f : S \to \mathbb{CP}^1$. The generic fibre of this fibration is linearly equivalent to the divisor $L - E$, then the genus of this curve is

$$g = \frac{(L-E)^2 + (L-E)\cdot K_S + 2}{2} = 0.$$

We then see that f is a rational fibration that satisfies $K_X^* \cdot (L-E) = (E-L)\cdot(L-E) = 0$. Then the reduced foliation X is Riccati (see page 50 of [**2**]). □

6. Geometric Quotient of non-degenerate Foliations without Invariant Lines. By Theorem 1 we know that there exists a geometric quotient of stable foliations of degree 2 by the action of $PGL(3,\mathbb{C})$. In this section we prove that this quotient contains another geometric quotient, which is the coarse moduli space of polarised del Pezzo surfaces of degree 2 (see [**9**]).

For this we will need the following:

DEFINITION 10. A set of seven points in $\mathbb{CP}^2$ is said to be in general position if no three of these lie on one line and no six of them lie on one conic.

THEOREM 8. *The geometric quotient by $PGL(3,\mathbb{C})$ of non-degenerate holomorphic foliations of degree 2 without invariant lines is the coarse moduli space of polarised del Pezzo surfaces of degree 2.*

Proof. We begin by noting that the set $\mathcal{F}_2^0 = \{X \in \mathcal{F}_2 : X \text{ is non-degenerate without invariant lines}\}$ is an open $PGL(3,\mathbb{C})$-stable set of $\mathcal{F}_2$. To see that this set is open, it is sufficient to use part 2 of Theorem 5.2 of [**8**] and Theorem 2.5 of [**7**], because the first one says that the set of holomorphic foliations without invariant lines is open and the second one says the same for the set of non-degenerate holomorphic foliations. Also, due to Corollary 1 we have that a foliation in $\mathcal{F}_2^0$ is stable by $PGL(3,\mathbb{C})$. □

In [**7**] we have the following:

THEOREM 9. *Let $X \in \mathcal{F}_d$, $d > 1$ be a non-degenerate holomorphic foliation with singular set Z, and let $X_1 \in \mathcal{F}_d$ be another holomorphic foliation with singular set Z, then $X_1 = X$.*

Moreover, in the case of degree 2, the corollary 4.10 in [**3**] says that seven different points of $\mathbb{CP}^2$ are the singular set of a unique non-degenerate holomorphic foliation of degree 2 if and only if there are not present six points in a conic. On the other hand, in Remark 2 we prove that a non-degenerate foliation of degree d has an invariant line if and only if this line has $d+1$ singular points.

If U_7 is the open subspace of the punctual Hilbert scheme $Hilb^7(\mathbb{CP}^2)$ consisting of points that represent seven points in general position in $\mathbb{CP}^2$, then we can conclude that U_7 parametrises the non-degenerate holomorphic foliations of degree 2 without invariant lines. In U_7 we have the natural action by $PGL(3,\mathbb{C})$:

$$\begin{aligned} PGL(3,\mathbb{C}) \times U_7 &\to U_7 \\ (g, p_1 + \cdots + p_7) &\mapsto gp_1 + \cdots + gp_7, \end{aligned}$$

such that the diagram

$$\begin{array}{ccc} PGL(3,\mathbb{C})\times \mathcal{F}_2^0 & \to & \mathcal{F}_2^0 \\ \downarrow & & \downarrow \\ PGL(3,\mathbb{C})\times U_7 & \to & U_7 \end{array}$$

is commutative, therefore the geometric quotient of non-degenerate foliations of degree 2, without invariant lines by $PGL(3, \mathbb{C})$ is the geometric quotient of U_7 by $PGL(3, \mathbb{C})$.

Finally in [**9**] the author proves that the coarse moduli space of polarised del Pezzo surfaces of degree 2 is the geometric quotient by $PGL(3, \mathbb{C})$ of U_7.

COROLLARY 2. *The geometric quotient of stable foliations of degree* 2 *by* $PGL(3, \mathbb{C})$ *contains an open set isomorphic to the coarse moduli space of polarised del Pezzo surfaces of degree* 2.

Proof. The proof of this corollary is a consequence of the following: □

LEMMA 6. (see page 234 of [**5**]). *Let G be an algebraic group acting on an algebraic variety V. If (Y, ϕ) is a geometric quotient for this action and U is an open $G-$ stable subset of V, then $(\phi(U), \phi_|)$ is the geometric quotient for the action of G on U.*

LEMMA 7. (see Theorem 4.20 of [**5**]). *Let G be an affine algebraic group, V a regular $G-$ variety and assume that a geometric quotient (Y, ϕ) exists. Then, Y is unique up to isomorphism.*

If (Y, ϕ) is the geometric quotient of $\mathcal{F}_2$ by $PGL(3, \mathbb{C})$, then $(\phi(\mathcal{F}_2^0), \phi_|)$ is the geometric quotient of $\mathcal{F}_2^0$ by $PGL(3, \mathbb{C})$, which is the geometric quotient of U_7 by $PGL(3, \mathbb{C})$ and, by a property of geometric quotients, it is an open subset of Y.

REMARK 4. The foliation of degree 2, $(xz + y^2)\frac{\partial}{\partial x} - x^2\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}$, given in [**4**] has a unique singular point in $(0 : 0 : 1)$, which is not reduced with Milnor number 7 and does not have any invariant line, therefore X is a stable foliation.

This tells us that it is not possible to construct the good quotient of all the stable foliations through the singular set, like we do in the case of non-degenerate foliations without invariant lines. In the second section of [**9**] we can see the construction of a compactification of the coarse moduli space of del Pezzo surfaces using a subspace of $Hilb^7(\mathbb{CP}^2)$ allowing points with multiplicity at most 2.

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