THE NO-THREE-IN-LINE PROBLEM

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Let S_n be the set of n^2 points with integer coordinates (x, y), $1 \leqslant x, y < n$. Let f_n be the maximum cardinal of a subset T of S_n such that no three points of T are collinear. Clearly $f_n \leqslant 2n$. For $2 \leqslant n \leqslant 10$ it is known ([2], [3] for $n = 8$, [1] for $n = 10$, also [4], [6]) that $f_n = 2n$, and that this bound is attained in 1, 1, 4, 5, 11, 22, 57, 51 and 156 distinct configurations for these nine values of n. On the other hand, P. Erdős [7] has pointed out that if n is prime, $f_n \geqslant n$, since the n points (x, x^2) reduced modulo n have no three collinear. We give a probabilistic argument to support the conjecture that there is only a finite number of solutions to the no-three-in-line problem. More specifically, we conjecture that

(1) (?) $$f_n \sim (2\pi^2/3)^{1/3}\, n.$$

THEOREM. <u>The number</u>, t_n, <u>of sets of</u> 3 <u>collinear points that can be chosen from</u> S_n <u>is</u>

$$t_n = \frac{3}{\pi^2} n^4 \log n + O(n^4).$$

<u>Proof</u>. The number of sets of 3 collinear points parallel to a coordinate axis is

(2) $$2n \binom{n}{3} = \frac{1}{3} n^2 (n-1)(n-2).$$

The number of such sets parallel to $x = \pm y$ is

(3) $$2\binom{n}{3} + 4\left\{\binom{n-1}{3} + \binom{n-2}{3} + \ldots + \binom{3}{3}\right\} = \frac{1}{6} n (n-1)^2 (n-2).$$

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We next count the triples chosen from $\{(a + sp,\ b + sq) : s = 0, 1, 2, \ldots\}$, where

$$1 \leqslant q < p \leqslant \left[\tfrac{1}{2}(n - 1)\right], \tag{4}$$

square brackets denoting integer part, and $(p, q) = 1$. Figure I illustrates the case $n = 60$, $p = 7$, $q = 5$. Define $r = [(n - 1)/p]$, so that $r = 8$ in this case. Points in regions marked 1 in Figure I, are in lines originating in the rectangle $1 \leqslant a \leqslant n - rp$, $1 \leqslant b \leqslant n - rq$, each line containing $r + 1$ points. Those in Regions 2 have $n - rp + 1 \leqslant a \leqslant p$

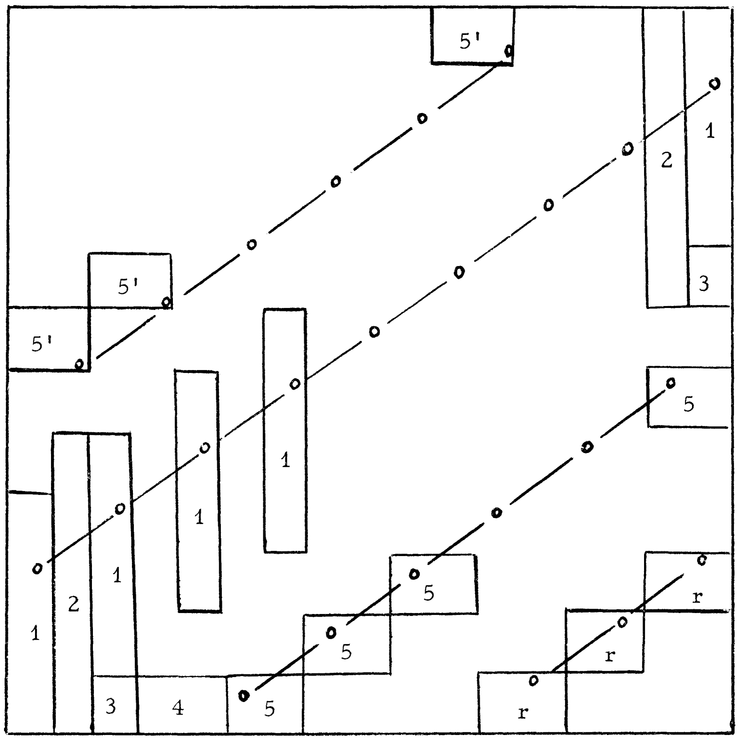

FIGURE I

and $1 \leqslant b \leqslant n - (r - 1)q$, and r points in each line. Triples arising from Regions 1 and 2 should be counted 4 times, to allow for the cases where one or both of p and q are negative. Triples arising from Regions j $(3 \leqslant j \leqslant r)$ are counted 8 times, for the same reason, together with the fact that they are each repeated (see the Regions 5' in Figure I). These regions have $n - (r + 3 - j)p + 1 \leqslant a \leqslant n - (r + 2 - j)p$ and $1 \leqslant b \leqslant q$, except for $j = 3$ where the range for a is $p + 1 \leqslant a \leqslant n - (r - 1)p$.

The lines in these cases contain $r + 3 - j$ points. The required number of triples is thus

$$4\{(n - rp)(n - rq)\binom{r+1}{3} + ((r+1)p - n)(n - (r-1)q)\binom{r}{3}\} + 8\{(n - rp)q\binom{r}{3} + pq\sum_{j=4}^{r}\binom{r+3-j}{3}\} = \frac{1}{3}r(r-1)\{6n^2 - 4n(p+q)(r+1) + pq(r+1)(3r+2)\},$$

summed for p and q in the range (4), and augmented by (2) and (3), so that

$$t_n = \frac{1}{6}n(n-1)(n-2)(3n-1)$$

$$+ \sum_{p=2}^{[\frac{1}{2}(n-1)]} \sum_{\substack{q=1 \\ (p,q)=1}}^{p-1} \frac{1}{3}r(r-1)\{6n^2 - 4n(p+q)(r+1) + pq(r+1)(3r+2)\}.$$

Using Euler's totient function, $\phi(p)$, and its properties [5]

$$\sum_{q=1}^{p-1} \phi(q) = \frac{1}{2}p\phi(p), \quad \sum_{p=1}^{m} \frac{\phi(p)}{p^2} = \frac{6}{\pi^2}\log m + O(1),$$

we obtain

$$t_n = \frac{1}{6}n(n-1)(n-2)(3n-1)$$

$$+ \sum_{p=2}^{[\frac{1}{2}(n-1)]} \frac{1}{6}r(r-1)\{12n^2 - 12np(r+1) + p^2(r+1)(3r+2)\}\phi(p)$$

$$= \frac{1}{2}n^4 \sum_{p=2}^{[\frac{1}{2}(n-1)]} \phi(p)/p^2 + O(n^4)$$

$$= \frac{3}{\pi^2}n^4 \log n + O(n^4),$$

and the theorem is proved.

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For large n, the probability that three points, chosen at random, should be in line is thus

$$\frac{3}{\pi^2} n^4 \log n \Big/ \binom{n^2}{3} \sim \frac{18 \log n}{\pi^2 n^2}$$

and the probability that three such points should not be in line is

$$1 - \frac{18 \log n}{\pi^2 n^2} + O\left(\frac{1}{n^2}\right) .$$

If we assume that the events are independent, the probability that 2n points contain no three in line is

$$\left\{1 - \frac{18 \log n}{\pi^2 n^2} + O\left(\frac{1}{n^2}\right)\right\}^{\binom{2n}{3}} = e^{-\frac{24}{\pi^2} n \log n + O(n)}$$

Hence, an estimate of the number of solutions to the no-three-in-line problem is given by

$$\binom{n^2}{2n} n^{-24n/\pi^2} e^{O(n)} .$$

which the use of Stirling's formula shows to be

(5) $$O(n^{-c_1 n} c_2^n) ,$$

where c_1 and c_2 are constants, with $c_1 = -2 + 24/\pi^2$. The expressior (5) supports the conjecture concerning the finiteness of the numbers of solutions.

If we repeat this argument with kn points in place of 2n, the corresponding value of c_1 in (5) is $-2 + 3k^3/\pi^2$, so that (5) tends to zero as $n \to \infty$, provided $k > (2\pi^2/3)^{1/3} \simeq 1.873856$, i.e. for large n, we expect to be able to select approximately $(2\pi^2/3)^{1/3} n$ points with no three in line, but no larger number.

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