On an Asymptotic Integral

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1. This note gives an asymptotic evaluation of an integral of the form

$$I_n = \int_a^b \{f_n(x)\}^n \, g(x) \, dx, \tag{1}$$

as n tends to infinity, where $\{f_n(x)\}$ is a sequence of real-valued functions. The theorem to be established is a natural extension of B. Levi's generalised Laplace-Darboux theorem (1, 341-51); it gives a rule for evaluating a wider class of asymptotic integrals.

In what follows $(x, n) \to (\xi, \infty)$ denotes that x, n tend to ξ, ∞ respectively and independently. For instance, $\phi(x, n) \to 0$ as $(x, n) \to (\xi, \infty)$ means that for any given positive ϵ there exist positive numbers δ and N such that $|\phi(x, n)| < \epsilon$ whenever $|x - \xi| < \delta$ and $n > N$.

Theorem. Let $f_n(x)$ $[n = 1, 2, 3, \ldots]$ *and* $g(x)$ *be real functions such that*

(A) $g(x)$ *and each* $f_n(x)$ *are integrable over* (a, b);

(B) *there is a number* A, *independent of* x *and* n, *such that* $|f_n(x)| < A$, $|g(x)| < A$;

(C) $f_n(x)$ *attains a positive absolute maximum at* $x = \xi_n$;

(D) *for each positive number* d, *there is a positive number* δ (*depending on* d *but not on* n) *such that* $|f_n(x)| \leqq f_n(\xi_n) - \delta$ *whenever* $|x - \xi_n| \geqq d$ *and* $a \leqq x \leqq b$;

(E) *as* n *tends to infinity,* ξ_n *tends to* ξ, *where* $a < \xi < b$;

(F) $g(x)$ *is continuous at* $x = \xi$ *with* $g(\xi) \neq 0$;

(G) *there are positive constants* h *and* k *such that*

$$\lim_{(x,n) \to (\xi, \infty)} |f_n(x) - f_n(\xi_n)| \, / \, |x - \xi_n|^h = k. \tag{2}$$

Then for n *large*

$$I_n \sim (2/h) \, \Gamma(1/h) \, \{f_n(\xi_n)\}^n \, g(\xi) \, \{f_n(\xi_n)/nk\}^{1/h}. \tag{3}$$

2. The essential step in the proof is to determine a suitable small interval containing the variable point $x = \xi_n$ and then let n tend to infinity in order to get the dominant asymptotic value. Write

$$f_n(x) - f_n(\xi_n) = -k \mid x - \xi_n \mid^h \{1 + R(x, \xi_n)\} \tag{4}$$

so that by condition (G), $R(x, \xi_n) \to 0$ as $(x, n) \to (\xi, \infty)$. We now have

$$\log f_n(x) = \log f_n(\xi_n) + \log\{1 - k \mid x - \xi_n \mid^h \{f_n(\xi_n)\}^{-1}(1 + R)\}$$
$$= \log f_n(\xi_n) - k \mid x - \xi_n \mid^h \{f_n(\xi_n)\}^{-1}(1 + R + R'), \tag{5}$$

say,[1] where $R' = O(\mid x - \xi_n \mid^h)$ so that $R' \to 0$ as $(x, n) \to (\xi, \infty)$. Denote $R + R'$ by $\theta(x, \xi_n)$. Then, given any small positive nnmber ϵ, there are positive numbers Δ and M such that $\mid \theta \mid < \epsilon$ whenever $\mid x - \xi \mid < 2\Delta$ and $n > M$. We may take M so large that $\mid \xi - \xi_n \mid < \Delta$ whenever $n > M$. Thus $\mid x - \xi_n \mid < \Delta$ implies $\mid x - \xi_n \mid < 2\Delta$. We may also assume $a \leqq \xi_n - \Delta$, $\xi_n + \Delta \leqq b$ for $n > M$.

If we now use (5), if we make the change of variable

$$nk\{f_n(\xi_n)\}^{-1} \mid x - \xi_n \mid^h = t, \tag{6}$$

and if we denote $\theta(x, \xi_n)$ by $\theta_1(t)$ for $\xi_n \leqq x \leqq \xi_n + \Delta$ and by $\theta_2(t)$, for $\xi_n - \Delta \leqq x \leqq \xi_n$, we obtain

$$J_n \equiv \int_{\xi_n - \Delta}^{\xi_n + \Delta} \left\{\frac{f_n(x)}{f_n(\xi_n)}\right\}^n \left\{\frac{nk}{f_n(\xi_n)}\right\}^{1/h} dx$$
$$= \left\{\frac{nk}{f_n(\xi_n)}\right\}^{1/h} \int_{\xi_n - \Delta}^{\xi_n + \Delta} \exp\left[-nk(1 + \theta)\{f^n(\xi_n)\}^{-1} \mid x - \xi_n \mid^h\right] dx$$
$$= \frac{1}{h}\int_0^T \left[e^{-t[1 + \theta_1(t)]} + e^{-t[1 + \theta_2(t)]}\right] t^{1/h - 1} dt, \tag{7}$$

where $T = nk\{f_n(\xi_n)\}^{-1}\Delta^h$.

Now suppose that ϵ and Δ are fixed. Then T, which is not less than $nk\Delta^h/R$, tends to infinity with n. Since $\mid \theta_1 \mid < \epsilon$ and $\mid \theta_2 \mid < \epsilon$, we see from (7) that

$$\frac{2}{h}\left(\frac{1}{1+\epsilon}\right)^{1/h} \int_0^{T(1+\epsilon)} e^{-u} u^{1/h - 1} du \leqq J_n \leqq \frac{2}{h}\left(\frac{1}{1-\epsilon}\right)^{1/h} \int_0^{T(1-\epsilon)} e^{-u} u^{1/h - 1} du.$$

If we now let n tend to infinity, we find that

$$\frac{2}{h}\left(\frac{1}{1+\epsilon}\right)^{1/h} \Gamma\left(\frac{1}{h}\right) \leqq \underline{\lim}\, J_n \leqq \overline{\lim}\, J_n \leqq \frac{2}{h}\left(\frac{1}{1-\epsilon}\right)^{1/h} \Gamma\left(\frac{1}{h}\right). \tag{8}$$

[1] Here we use the fact that $f_n(\xi_n)$ cannot tend to zero as n tends to infinity. This follows from condition (D).

Let us now consider the integral

$$J_n^* \equiv \int_{\xi_n - \Delta}^{\xi_n + \Delta} \left\{\frac{f_n(x)}{f_n(\xi_n)}\right\}^n \left\{\frac{nk}{f_n(\xi_n)}\right\}^{1/h} g(x)\, dx. \tag{9}$$

Since $g(x)$ is continuous at $x = \xi$, we may assume that Δ is chosen so that[1]

$$g(\xi)(1-\epsilon) \leqq g(x) \leqq g(\xi)(1+\epsilon)$$

whenever $| x - \xi_n | < \Delta$. Thus from (8) and (9) we may infer that

$$\frac{2(1-\epsilon)}{h(1+\epsilon)^{1/h}} \Gamma\left(\frac{1}{h}\right) g(\xi) \leqq \underline{\lim}\, J_n^* \leqq \overline{\lim}\, J_n^* \leqq \frac{2(1+\epsilon)}{h(1-\epsilon)^{1/h}} \Gamma\left(\frac{1}{h}\right) g(\xi). \tag{10}$$

On the other hand, by hypothesis (D), there is a positive number δ (independent of n) such that $| f_n(x) | \leqq f_n(\xi_n) - \delta$ whenever $| x - \xi_n | \geqq \Delta$ and $a \leqq x \leqq b$. Hence for these values of x

$$\left|\frac{f_n(x)}{f_n(\xi_n)}\right| \leqq \left|\frac{f_n(\xi_n) - \delta}{f_n(\xi_n)}\right| \leqq \left| 1 - \frac{\delta}{A}\right| = \rho$$

say, where $0 < \rho < 1$. It now follows that, with Δ fixed,

$$J_n^{**} = \left(\int_a^{\xi_n - \Delta} + \int_{\xi_n + \Delta}^b\right) \left\{\frac{f_n(x)}{f_n(\xi_n)}\right\}^n \left\{\frac{nk}{f_n(\xi_n)}\right\}^{1/h} g(x)dx = O(\rho^n n^{1/h}) \to 0$$

as n tends to infinity. We may therefore replace J_n^* by $(J_n^* + J_n^{**})$ in equation (10). Since ϵ is arbitrary, and since $(J_n^* + J_n^{**})$ does not depend on ϵ, it now follows that

$$\lim_{n \to \infty} \left(J_n^* + J_n^{**}\right) = \frac{2}{h} \Gamma\left(\frac{1}{h}\right) g(\xi).$$

This is equivalent to (3), and our theorem is established.

3. Concrete examples are easily found for illustrating the use of the formula (3). A simple example is, for $n \to \infty$,

$$n^{1/s} \int_{-1/2}^{1/2} \left(1 + \frac{1}{n} - \left|x - \frac{1}{\sqrt{n}}\right|^s\right)^n \sin^{-1}(1 - |x|)\, dx \sim (1/s)\Gamma(1/s)\pi e,$$

where $s > 0$ and $0 \leqq \sin^{-1} y \leqq \frac{\pi}{2}$ $(0 \leqq y \leqq 1)$. As consequences of our theorem we now mention two important cases as follows:

[1] There are slight changes here if $g(\xi)$ is negative; $g(\xi)$ is not zero by hypothesis (F).

I. *Levi's case.* If $f_n(x) = f(x)$, then $\xi_n = \xi$ and the equation (2) becomes

$$\lim_{x \to \xi} | f(x) - f(\xi) | / | x - \xi |^h = k. \qquad (2)'$$

In this case we have

$$\int_a^b \left(f(x)\right)^n g(x)dx \sim \frac{2}{h} \Gamma\left(\frac{1}{h}\right)\left(f(\xi)\right)^n g(\xi)\left(\frac{f(\xi)}{nk}\right)^{1/h}. \qquad (3)'$$

II. *Laplace-Darboux case.* In the case of Levi, if $f(x)$ is continuous together with its derivatives $f'(x)$, $f''(x)$ so that $f'(\xi) = 0$, $f''(\xi) < 0$, and (2)′ is true for $h = 2$, $k = -\frac{1}{2}f''(\xi)$, then (3)′ reduces to the classical asymptotic formula of Laplace and Darboux ([2], [3], [4]) :

$$\int_a^b \left(f(x)\right)^n g(x)\, dx \sim \left(f(\xi)\right)^{n+1/2} g(\xi) \left(\frac{-2\pi}{nf''(\xi)}\right)^{1/2}. \qquad (3)''$$

4. Two remarks are worthy of mention. (i) In general the constants h and k may always be determined by means of the following equation

$$\lim_{n \to \infty} \lim_{x \to \xi_n} | f_n(x) - f_n(\xi_n) | / | x - \xi_n |^h = k. \qquad (2)^*$$

For it is easily seen that (2)* is implied by (2), in view of (4). In the case when ξ_n is a constant, (2)* and (2)′ are equivalent. (ii) If our hypothesis (F) is replaced by

$(F)^*$ *$g(x) \epsilon L(a, b)$ and $g(x)$ possesses limits on both sides of $x = \xi_n$ $(n = 1, 2, 3, \ldots)$,*

then by almost the same treatment as used before we easily obtain

$$I_n \sim (1/h)\, (g(\xi_n -) + g(\xi_n +))\, (f_n(\xi_n))^n (f_n(\xi_n)/nk)^{1/h}\, \Gamma(1/h). \qquad (3)^*$$

REFERENCES.

[1] Beppo Levi, *Publ. Inst., Math. Univ. Nac. Litoral*, 6 (1946).

[2] P. S. Laplace, *Oeuvres*, t. 7, p. 89. Paris (1886).

[3] G. Darboux, *Jour. de Math.* (3), 4 (1878).

[4] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Bd. 1, s. 78 (1925).

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