UPPER AND LOWER BOUNDS FOR THE AREA OF A TRIANGLE FOR WHOSE SIDES TWO SYMMETRIC FUNCTIONS ARE KNOWN

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1. Introduction. Improving on inequalities given by Gerretsen **(2)**, Beatty **(1)** has proved that for the area Δ of any plane triangle with sides a, b, c the following inequalities hold:

$$\frac{(K-H)^2}{12} \geqslant \Delta^2 \geqslant \frac{(K-H)(3K-5H)}{12} \tag{1.1}$$

where

$$H = \tfrac{1}{2}(a^2+b^2+c^2), \quad K = bc+ca+ab; \tag{1.2}$$

the signs of equality in (1.1) only apply when the triangle is equilateral. Beatty has also remarked that the second inequality in (1.1) is of no value in case $5H \geqslant 3K$, since then the lower estimate which it gives for Δ^2 is not even positive.

As an improvement on the foregoing estimates, a proof of the following inequalities will be given here (§2):

$$\frac{s(s-q)^2(s+2q)}{27} \geqslant \Delta^2 \geqslant \frac{s(s+q)^2(s-2q)}{27}, \tag{1.3}$$

where

$$q = (a^2+b^2+c^2-bc-ca-ab)^{\frac{1}{2}}, \tag{1.4}$$

i.e.

$$q = (\tfrac{1}{2}Q)^{\frac{1}{2}}, \tag{1.5}$$

where

$$Q = (b-c)^2 + (c-a)^2 + (a-b)^2 \tag{1.6}$$

is the measure of "unequilaterality" already considered by Gerretsen **(2)**; and

$$s = \frac{a+b+c}{2} = \left(\frac{H+K}{2}\right)^{\frac{1}{2}} \tag{1.7}$$

is the semiperimeter of the triangle.[1]

The first (second) equality sign in (1.3) holds for isosceles triangles whose base is the smallest (largest) of the three sides; of course both equality signs apply when the triangle is equilateral, since then $q = 0$.

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[1]The use of s and q as the given symmetric functions of the sides throughout the present note (instead of Beatty's H and K) seems justified by the fact that it simplifies the formulation and proof of (1.3).

In our (1.3), as in Beatty's (1.1), the second inequality is useless when $5H \geqslant 3K$, i.e. when $q \geqslant \frac{1}{2}s$. It is easily seen that in this case triangles can be found with as small an area as one might wish; hence no other lower bound for Δ can then exist than the trivial one: $\Delta > 0$.

Apart from this exceptional case (and that of an equilateral triangle), (1.3) represents an improvement over (1.1), which might be rewritten in terms of s and q as follows:

$$\text{(1.8)} \qquad \frac{(s^2 - q^2)^2}{27} \geqslant \Delta^2 \geqslant \frac{(s^2 - q^2)(s^2 - 4q^2)}{27},$$

by using the easily verified formulae

$$\text{(1.9)} \qquad H = \tfrac{1}{3}(2s^2 + q^2), \quad K = \tfrac{1}{3}(4s^2 - q^2).$$

Indeed, Beatty's upper bound in (1.8) is higher than ours in (1.3), since the difference between them is

$$\text{(1.10)} \qquad \frac{(s^2 - q^2)^2}{27} - \frac{s(s-q)^2(s+2q)}{27} = \frac{q^2(s-q)^2}{27} \geqslant 0;$$

analogously from

$$\text{(1.11)} \quad \frac{s(s+q)^2(s-2q)}{27} - \frac{(s^2 - q^2)(s^2 - 4q^2)}{27} = \frac{2q^2(s+q)(s-2q)}{27} \geqslant 0$$

($q < \frac{1}{2}s$) it follows that our lower bound in (1.3) is higher than Beatty's in (1.8).

Incidentally it might be remarked that a new simple proof of (1.1) is thus obtained by deriving (1.8) from (1.3).

Moreover, it can be shown that in a certain sense our estimates (1.3) are best possible; indeed, the bounds given by (1.3) for Δ^2 are nothing else than the extremal values that can be attained by Δ^2 considered as a function of the three positive variables a, b, c subject to the constraining relations (1.4) and (1.7) and furthermore to the obvious inequalities

$$\text{(1.12)} \qquad b + c > a, \quad c + a > b, \quad a + b > c,$$

equivalent to

$$\text{(1.13)} \qquad a < s, \quad b < s, \quad c < s.$$

Of course, the given values of s and q (besides being positive) have to satisfy the inequality

$$\text{(1.14)} \qquad q < s$$

(equivalent to Beatty's $H < K$) as a consequence of the easily verified identity

$$\text{(1.15)} \qquad q^2 = s^2 - \frac{3}{s}\left(abc + \frac{\Delta^2}{s}\right).$$

To prove the foregoing statements it would only be necessary to find these extremal values of Δ^2 by standard methods of differential calculus, although some caution is necessary because of the inequalities (1.13).[2] It seems, however, that the proof of (1.3) given in §2 avoids this difficulty, and at the same time has the advantage of being purely algebraic. (The use of trigonometric functions is not essential, as will be pointed out at the end of §2.)

A final remark (§3) will be devoted to the problem of finding bounds for the product of the three sides of a triangle.

2. Proof of the inequalities (1.3). We start with the well known formulae (**3**, pp. 25–28) for the trigonometric solution of a reduced cubic equation

$$x^3 + 3Cx + D = 0 \tag{2.1}$$

with real roots x_1, x_2, x_3 satisfying the condition

$$x_1 + x_2 + x_3 = 0, \tag{2.2}$$

viz. (if the roots are numbered conveniently):

$$\begin{cases} x_1 = 2\sqrt{-C}\cos\Theta, \; x_2 = 2\sqrt{-C}\cos(\Theta + 120°), \\ \qquad\qquad\qquad\qquad x_3 = 2\sqrt{-C}\cos(\Theta + 240°), \end{cases} \tag{2.3}$$

where the auxiliary angle Θ is defined by

$$\cos 3\Theta = -\tfrac{1}{2}D(-C)^{-3/2}, \quad 0 \leqslant \Theta \leqslant 60°. \tag{2.4}$$

A possible choice fulfilling (2.2) is

$$x_1 = a - \tfrac{2}{3}s, \quad x_2 = b - \tfrac{2}{3}s, \quad x_3 = c - \tfrac{2}{3}s; \tag{2.5}$$

then

$$3C = x_2x_3 + x_3x_1 + x_1x_2 \tag{2.6}$$

will be related to q—defined in (1.4)—by

$$C = -q^2/9, \tag{2.7}$$

since

$$q^2 = \frac{(b-c)^2 + (c-a)^2 + (a-b)^2}{2} = \frac{(x_2-x_3)^2 + (x_3-x_1)^2 + (x_1-x_2)^2}{2}$$
$$= (x_1 + x_2 + x_3)^2 - 3(x_2x_3 + x_3x_1 + x_1x_2) = -9C.$$

Hence (2.3) goes over into

$$x_1 = \tfrac{2}{3}q\cos\Theta, \quad x_2 = \tfrac{2}{3}q\cos(\Theta + 120°), \quad x_3 = \tfrac{2}{3}q\cos(\Theta + 240°), \tag{2.8}$$

[2]Such a proof has been suggested by Beatty (in a letter to the editor): Since for a maximum or minimum of Δ^2 in the sense of differential calculus the jacobian

$$\frac{\partial(s, H, \Delta^2)}{\partial(a, b, c)}$$

must vanish, from (1.2), (1.7) and (1.15) it would follow rather immediately that the triangle is isosceles.

and from (2.4) and (2.7) it follows that

$$D = -\frac{2}{27} q^3 \cos 3\Theta. \tag{2.9}$$

Let us now express Δ^2 as a function of q, s, and Θ. We have:

$$\frac{\Delta^2}{s} = (s-a)(s-b)(s-c) = \left(\frac{s}{3} - x_1\right)\left(\frac{s}{3} - x_2\right)\left(\frac{s}{3} - x_3\right), \tag{2.10}$$

and from the identity

$$(x - x_1)(x - x_2)(x - x_3) = x^3 + 3Cx + D \tag{2.11}$$

it follows for $x = \frac{1}{3}s$ that

$$\frac{\Delta^2}{s} = \frac{s^3 + 27Cs + 27D}{27}; \tag{2.12}$$

finally, using the values of C and D given by (2.7) and (2.9), we are led to the formula[3]

$$\Delta^2 = \frac{s}{27}(s^3 - 3sq^2 - 2q^3 \cos 3\Theta). \tag{2.13}$$

From (2.13) it is now obvious that the maximum of Δ^2 is

$$\frac{s}{27}(s^3 - 3sq^2 + 2q^3) = \frac{s(s-q)^2(s+2q)}{27}, \tag{2.14}$$

and it is really attained by taking $\Theta = 60°$, i.e.—see (2.5) and (2.8)—for an isosceles triangle with sides

$$a = c = \frac{2s+q}{3}, \quad b = \frac{2(s-q)}{3}. \tag{2.15}$$

The minimum of Δ^2, corresponding to the second equality sign in (1.3), viz.

$$\frac{s(s^3 - 3sq^2 - 2q^3)}{27} = \frac{s(s+q)^2(s-2q)}{27}, \tag{2.16}$$

is obtained by taking $\Theta = 0$ in (2.13), i.e. for an isosceles triangle with sides

$$a = \frac{2(s+q)}{3}, \quad b = c = \frac{2s-q}{3}, \tag{2.17}$$

if that triangle exists, i.e., if the conditions (1.13) are fulfilled. From (2.17) it is immediately seen that this is only the case if $q < \frac{1}{2}s$; but it has already been mentioned above (§1) that there is no positive minimum of Δ^2 in the exceptional case $q \geqslant \frac{1}{2}s$ (i.e., $5H \geqslant 3K$ in Beatty's notation).

Finally it should be remarked that the use of trigonometric functions in the

[3]Professor Gordon Pall has pointed out that the following more general formula can be proved along the same lines:

$$(k-a)(k-b)(k-c) = \tfrac{8}{27}q^3(p^3 - \tfrac{3}{4}p - \tfrac{1}{4}\cos 3\Theta), \text{ where } p = (3k-2s)/2q.$$

foregoing proof might be avoided by proving instead of (2.13) the following algebraic identity:

$$(2.18)\quad \left\{\Delta^2 - \frac{s^2}{27}(s^2 - 3q^2)\right\}^2 = \left(\frac{s}{27}\right)^2\left\{4q^6 - 27(b-c)^2(c-a)^2(a-b)^2\right\}$$

and obtaining thus the following inequality equivalent to (1.3):

$$(2.19)\qquad \left|\Delta^2 - \frac{s^2}{27}(s^2 - 3q^2)\right| \leqslant \frac{2}{27}sq^3.$$

3. Bounds for the product *abc*. In virtue of (1.15), one might obtain from (1.3), or from (2.15) and (2.17), exact bounds also for the product *abc*:

$$(3.1)\qquad \frac{2}{27}(s-q)(2s+q)^2 \leqslant abc \leqslant \frac{2}{27}(s+q)(2s-q)^2,$$

giving us the solution of the following algebraic problem which might be of some interest in itself: to find maximum and minimum of the product of three positive variables a, b, c, if two of their symmetric functions, viz. s and q, are given. Since

$$(3.2)\qquad q^2 = 4s^2 - 3K,$$

it is easy to obtain from (3.1) the exact bounds for *abc* also in the case when the first two *elementary* symmetric functions, viz. $2s$ and K, are given:

$$(3.3)\quad \frac{2}{27}\left\{s(9K - 8s^2) - (4s^2 - 3K)^{3/2}\right\} \leqslant abc$$
$$\leqslant \frac{2}{27}\left\{s(9K - 8s^2) + (4s^2 - 3K)^{3/2}\right\}.$$

REFERENCES

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3. Lois W. Griffiths, *Introduction to the Theory of Equations* (2nd ed., New York, 1947).

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