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CHARACTERIZATIONS OF S-CLOSED HAUSDORFF SPACES

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Abstract

A topological space X is said to be S-closed if every cover of X by regular closed sets of X has a finite subcover. In this note some characterizations of S-closed Hausdorff spaces are obtained.

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1. Introduction

In 1976, Thompson [9] introduced the concept of S-closed spaces in terms of semiopen sets due to Levine [6]. The present author [7] defined subsets said to be S-closed relative to a topological space. For a topological space (X, τ), the family of open sets of (X, τ) whose complements are S-closed relative to (X, τ) was utilized as a base for a topology τ^* on X by Di Maio [2]. The purpose of the present note is to obtain some characterizations of S-closed Hausdorff spaces by utilizing τ^*, the family of semiopen sets and that of θ-semiopen sets due to Joseph and Kwack [5].

2. Preliminaries

Throughout the present note spaces always mean topological spaces. Let (X, τ) be a space and A be a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be *semiopen* [6] (respectively *regular closed*) if $A \subset Cl(Int(A))$ (respectively $A = Cl(Int(A)))$. The family of all semiopen (respectively regular closed) sets in (X, τ) is denoted by $SO(X, \tau)$ (respectively $RC(X, \tau))$. The complement of a semiopen set is said to be *semi-closed*. The intersection of all semi-closed sets containing A is called the *semi-closure* [1] of A and is denoted by $sCl(A)$.

DEFINITION 2.1. A subset A of a space (X, τ) is said to be *S-closed relative to* (X, τ) [7] if for every cover $\{U_\alpha | \alpha \in \nabla\}$ of A by semiopen sets of (X, τ), there exists a finite subset ∇_0 of ∇ such that $A \subset \bigcup\{Cl(U_\alpha) | \alpha \in \nabla_0\}$.

A space (X, τ) is said to be *S-closed* [9] if X is S-closed relative to (X, τ). It is shown in [4, Theorem 3.2] that a space (X, τ) is S-closed if and only if every regular closed cover of X has a finite subcover. We recall a space (X, τ^*) defined by Di Maio [2]. The family of all open sets of (X, τ) whose complements are S-closed relative to (X, τ) is a base for a topology τ^* on X. A space (X, τ) is said to be *extremally disconnected* (briefly E.D.) if $Cl(U) \in \tau$ for every $U \in \tau$.

DEFINITION 2.2. A space (X, τ) is said to be *weakly-Hausdorff* (briefly *weakly-T_2*) [8] if each point $x \in X$ is the intersection of regular closed sets of (X, τ).

Let A be a subset of a space (X, τ). A point $x \in X$ is said to be in the *θ-semiclosure* [5] of A, denoted by $\theta - sCl(A)$, if $A \cap Cl(U) \neq \varnothing$ for every $U \in SO(X, \tau)$ containing x. If $\theta - sCl(A) = A$, then A is said to be *θ-semiclosed*. The complement of a θ-semiclosed set is said to be *θ-semiopen*. By τ^+ we denote the family of all θ-semiopen sets in (X, τ). The following lemma is obvious from the definitions and will be often used in the sequel.

LEMMA 2.3. *The following are equivalent for a subset A of a space (X, τ):*

(a) $A \in \tau^+$;
(b) *for each $x \in A$, there exists $U \in SO(X, \tau)$ such that $x \in U \subset Cl(U) \subset A$;*
(c) *A is the union of regular closed sets of (X, τ).*

In general, τ^+ is not a topology on X. Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ [3, Example 0.4]. Then $\{a, c\}$ and $\{b, c\}$ are θ-semiopen in (X, τ) but $\{a, c\} \cap \{b, c\} \notin \tau^+$. However, we have the following lemma.

LEMMA 2.4. *The following are equivalent for a space (X, τ):*

(a) *(X, τ) is E.D.;*

(b) $SO(X, \tau)$ *is a topology on* X;
(c) τ^+ *is a topology on* X.

PROOF. (a) $\Rightarrow$ (b). Let $A, B \in SO(X, \tau)$. Since (X, τ) is E.D., $Cl(Int(A)) \in \tau$ and hence we have $A \cap B \subset Cl(Int(A)) \cap Cl(Int(B)) \subset Cl[Cl(Int(A)) \cap Int(B)] \subset Cl(Int(A \cap B))$. Therefore, we obtain $A \cap B \in SO(X, \tau)$. Now [6, Theorem 2] completes the proof.

(b) $\Rightarrow$ (c). Let $A, B \in \tau^+$ and $x \in A \cap B$. There exist $U, V \in SO(X, \tau)$ such that $x \in U \subset Cl(U) \subset A$ and $x \in V \subset Cl(V) \subset B$. Therefore, we have $x \in U \cap V \subset Cl(U \cap V) \subset Cl(U) \cap Cl(V) \subset A \cap B$ and $U \cap V \in SO(X, \tau)$. This shows that $A \cap B \in \tau^+$. Lemma 2.3 completes the proof.

(c) $\Rightarrow$ (a). Suppose that (X, τ) is not E.D. There exists $U \in \tau$ and $x \in X$ such that $x \in Cl(U) - Int(Cl(U))$. Let $A = Cl(U)$ and $B = X - Int(Cl(U))$, then $A, B \in RC(X, \tau)$ and hence $A, B \in \tau^+$. Since τ^+ is a topology, $x \in A \cap B \in \tau^+$. There exists $V \in SO(X, \tau)$ such that $x \in V \subset Cl(V) \subset A \cap B$. Since $V \subset B$, $Int(V) \subset Int(A) \cap B = \varnothing$. However, $x \in V \in SO(X, \tau)$ and hence $Int(V) \neq \varnothing$. This is a contradiction.

LEMMA 2.5. *If a space* (X, τ) *is E.D. and* $A \in SO(X, \tau)$, *then* $sCl(A) = \theta - sCl(A) = Cl(A)$.

PROOF. This is shown in [3, Lemma 0.3].

3. Characterizations

THEOREM 3.1. *The following are equivalent for a space* (X, τ):
(a) (X, τ^*) *is Hausdorff*;
(b) (X, τ^*) *is weakly-*T_2;
(c) (X, τ) *is S-closed Hausdorff*;
(d) $(X, SO(X, \tau))$ *is S-closed Hausdorff*;
(e) (X, τ^+) *is compact Hausdorff.*

PROOF. In the sequel, we denote the closure and the interior of a subset A of X with respect to the topology τ^* by $Cl_*(A)$ and $Int_*(A)$, respectively.

(a) $\Rightarrow$ (b). The proof is obvious.

(b) $\Rightarrow$ (c). Let (X, τ^*) be weakly-T_2. First, we shall show that (X, τ) is S-closed. Let x and y be distinct points of X. There exists $F \in RC(X, \tau^*)$ such that $x \in F$ and $y \notin F$. Since $Int_*(F) \neq \varnothing$, there exists $U, V \in \tau$ such that $\varnothing \neq U \subset X - F$, $\varnothing \neq V \subset Int_*(F)$ and $X - U$ and $X - V$ are S-closed relative to (X, τ). Since $X - F$ and $Int_*(F)$ are disjoint, $U \cap V = \varnothing$ and hence $X = (X - U) \cup (X - V)$ is S-closed relative to

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(X, τ) [7, Theorem 3.6]. Therefore, (X, τ) is S-closed. Next, we shall show that (X, τ) is weakly-T_2. For this purpose, we prove that $RC(X, \tau^*) \subset RC(X, \tau)$. Let $F \in RC(X, \tau^*)$. Since $\tau^* \subset \tau$, we have $Int_*(F) \subset Int(F)$ and hence $F = Cl_*(Int_*(F)) \subset Cl_*(Int(F)) \subset Cl_*(F) = F$. Therefore, we obtain $F = Cl_*(Int(F))$. Since $\tau^* \subset \tau$, $Cl(Int(F)) \subset Cl_*(Int(F))$ and hence $Cl(Int(F)) \subset F$. In order to show the opposite inclusion, we suppose that $x \notin Cl(Int(F))$. There exists $V \in \tau$ containing x such that $V \cap Int(F) = \varnothing$; hence $Int(Cl(V)) \cap Int(F) = \varnothing$. Since $X - Int(Cl(V)) \in RC(X, \tau)$ and (X, τ) is S-closed, it follows from [7, Theorems 3.3 and 3.4] that $X - Int(Cl(V)) = Cl(X - Cl(V))$ is S-closed relative to (X, τ). Therefore, we have $x \in Int(Cl(V)) \in \tau^*$ and hence $x \notin Cl_*(Int(F))$. Since $F = Cl_*(Int(F))$, we have $x \notin F$ and hence $F \subset Cl(Int(F))$. Consequently, we obtain $F \in RC(X, \tau)$. Therefore, it follows that (X, τ) is weakly-T_2. Moreover, an S-closed weakly-T_2 space is E.D. [4, Theorem 3.7]. Every regular closed set is clopen in an E.D. space. Therefore, (X, τ) is Hausdorff.

(c) $\Rightarrow$ (a). Since (X, τ) is S-closed Hausdorff, it follows from [9, Theorem 7] that (X, τ) is E.D. Let x and y be any distinct points of X. There exist $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$; hence $Cl(U) \cap Cl(V) = \varnothing$. Since $Cl(U)$ and $Cl(V)$ are clopen in (X, τ) and (X, τ) is S-closed, it follows from [7, Theorem 3.3] that $X - Cl(U)$ and $X - Cl(V)$ are S-closed relative to (X, τ). Therefore, we obtain $x \in Cl(U) \in \tau^*$, $y \in Cl(V) \in \tau^*$ and $Cl(U) \cap Cl(V) = \varnothing$. This shows that (X, τ^*) is Hausdorff.

(c) $\Rightarrow$ (d). Since (X, τ) is S-closed Hausdorff, (X, τ) is E.D. [9, Theorem 7] and by Lemma 2.4 $SO(X, \tau)$ is a topology on X. Let $\{V_\alpha | \alpha \in \nabla\}$ be any $SO(X, \tau)$-semiopen cover of X. For each $\alpha \in \nabla$, there exists $U_\alpha \in SO(X, \tau)$ such that $U_\alpha \subset V_\alpha \subset SO(X, \tau) - Cl(U_\alpha)$. By Lemma 2.5, $SO(X, \tau) - Cl(U_\alpha) = sCl(U_\alpha) = Cl(U_\alpha)$ and hence $V_\alpha \in SO(X, \tau)$ [6, Theorem 4]. Therefore, $\{V_\alpha | \alpha \in \nabla\}$ is a τ-semiopen cover of X. There exists a finite subset ∇_0 of ∇ such that $X = \bigcup\{Cl(V_\alpha) | \alpha \in \nabla_0\}$. It follows from Lemma 2.5 that $(X, SO(X, \tau))$ is S-closed. It is obvious that $(X, SO(X, \tau))$ is Hausdorff.

(d) $\Rightarrow$ (e). By Lemma 2.4, τ^+ is a topology on X. Let $\mathscr{V}$ be a cover of X by τ^+-open sets. Then each member of $\mathscr{V}$ is θ-semiopen in (X, τ). Every θ-semiopen set of (X, τ) is the union of regular closed sets of (X, τ). Every regular closed set of (X, τ) is semiopen and semiclosed in (X, τ) and hence clopen in $(X, SO(X, \tau))$. Therefore, $\mathscr{V}$ has a finite subcover. This shows that (X, τ^+) is compact. Next, we shall show that (X, τ^+) is Hausdorff. Let x and y be any distinct points of X. Since $(X, SO(X, \tau))$ is Hausdorff, there exists $U, V \in SO(X, \tau)$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$; hence $sCl(U) \cap V = \varnothing$. By Lemmas 2.4 and 2.5, $Cl(U) \cap V = \varnothing$ and hence

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$Cl(U) \cap Cl(V) = \varnothing$. Since $Cl(U)$ and $Cl(V)$ are regular closed in (X, τ) and $RC(X, \tau) \subset \tau^+$, (X, τ^+) is Hausdorff.

(e) $\Rightarrow$ (c). Since τ^+ is a topology on X, by Lemma 2.4 (X, τ) is E.D. First, we shall show that (X, τ) is S-closed. Let $\mathscr{V}$ be a cover of X by regular closed sets of (X, τ). Since $RC(X, \tau) \subset \tau^+$ and (X, τ^+) is compact, $\mathscr{V}$ has a finite subcover. This shows that (X, τ) is S-closed. Next, we shall show that (X, τ) is Hausdorff. Let x_1 and x_2 be any distinct points of X. Since (X, τ^+) is Hausdorff, there exists $V_1, V_2 \in \tau^+$ such that $x_1 \in V_1$, $x_2 \in V_2$ and $V_1 \cap V_2 = \varnothing$. Moreover, there exists $U_i \in SO(X, \tau)$ such that $x_i \in U_i \subset Cl(U_i) \subset V_i$ for $i = i, 2$. Therefore, we have $Cl(U_1) \cap Cl(U_2) = \varnothing$. Since $U_i \in SO(X, \tau)$, we have $Cl(U_i) = Cl(Int(U_i))$ for $i = 1, 2$. Thus, $Cl(U_1)$ and $Cl(U_2)$ are open in (X, τ) since (X, τ) is E.D. Therefore, (X, τ) is Hausdorff.

REMARK 3.2. Every S-closed weakly-T_2 space is E.D. [4, Theorem 3.7]. Each regular closed set is clopen in an E.D. space. Therefore, every S-closed weakly-T_2 space is Urysohn. The statement (c) in Theorem 3.1 is thus equivalent to each one of the following:

(c′) (X, τ) is S-closed weakly-T_2;

(c″) (X, τ) is S-closed Urysohn.

References

[1] S. Gene Crossley and S. K. Hildebrand, 'Semi-closure', *Texas J. Sci.* **22** (1971), 99–112.

[2] G. Di Maio, 'S-closed spaces, S-sets and S-continuous functions', *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **118** (1984), 125–134.

[3] M. Ganster, T. Noiri and I. L. Reilly, 'Weak and strong forms of θ-irresolute functions', *J. Inst. Math. Comp. Sci. Math. Ser.* **1** (1988), 19–29.

[4] R. A. Herrmann, 'RC-convergence', *Proc. Amer. Math. Soc.* **75** (1979), 311–317.

[5] J. E. Joseph and M. H. Kwack, 'On S-closed spaces', *Proc. Amer. Math. Soc.* **80** (1980), 341–348.

[6] N. Levine, 'Semi-open sets and semi-continuity in topological spaces', *Amer. Math. Monthly* **70** (1963), 36–41.

[7] T. Noiri, 'On S-closed subspaces', *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fi. Mat. Natur.* (8) **64** (1978), 157–162.

[8] T. Soundararajan, 'Weakly Hausdorff spaces and the cardinality of topological spaces', *General Topology and its Relations to Modern Analysis and Algebra III*, Proc. Conf. Kampur, 1968; pp. 301–306 (Academia, Prague, 1971).

[9] T. Thompson, 'S-closed spaces', *Proc. Amer. Math. Soc.* **60** (1976), 335–338.

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