PARALLEL CURVES

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In the Euclidean plane a curve C has a one-parameter family of parallel involutes and a unique evolute C^* which coincides with the locus of the centres of the osculating circles of C. If $\bar{C}$ is parallel to C, C^* is also the evolute of $\bar{C}$.

We will study parallel curves in n-dimensional Euclidean space and obtain generalizations of the properties given above.

DEFINITION. Curves C and $\bar{C}$ are *parallel* if there is a one-to-one correspondence between their points such that the tangents at corresponding points are parallel and such that the join of corresponding points is perpendicular to the tangents.

This definition was given by Da Cunha **(1)**.

It follows at once that parallelism is an equivalence relation.

We denote the position vector of a point on a curve C in n-space by r. We suppose that r is an $(n+1)$-times differentiable function of the arc length s of C. Let C have the moving n-hedron $\xi_1, \ldots, \xi_n$ and non-vanishing curvatures $k_1, \ldots, k_{n-1}$. We use corresponding notations for curves $\bar{C}$, $\tilde{C}$, etc.

If C and $\bar{C}$ are parallel, the distance between corresponding points is constant as we see by differentiating $(\bar{r} - r)^2$.

To find the curves $\bar{C}$ parallel to a given curve C we put

$$\bar{r} = r + \sum_1^n u_i \xi_i, \tag{1}$$

where $u_1, \ldots, u_n$ are scalar functions of s to be determined. Since $(\bar{r} - r)\xi_1 = 0$, $u_1 = 0$. Differentiating (1) and using the Frenet formulae

$$\xi_i' = -k_{i-1}\xi_{i-1} + k_i\xi_{i+1} \qquad (i = 1, \ldots, n), \tag{2}$$

in which $k_0 = k_n = 0$, $k_i > 0$, $(i = 1, \ldots, n-1)$, we obtain

$$\bar{\xi}_1\bar{s}' = (1 - k_1u_2)\xi_1 + \sum_2^n (u_i' - k_iu_{i+1} + k_{i-1}u_{i-1})\xi_i .$$

Since

$$\bar{\xi}_1 = \epsilon\xi_1 \qquad (\epsilon = \pm 1), \tag{3}$$

we have

$$\bar{s}' = \epsilon(1 - k_1u_2) \tag{4}$$

Received November 1, 1951; in revised form May 22,1952. In the first version of this paper only the case $n = 4$ was considered; most of the results being from a Ph.D. thesis written under the direction of Professor H. S. M. Coxeter. The referee suggested that the theorems be generalized and stated many of the results that could be proved. I take this opportunity to thank him for his assistance.

and

$$u_i' = -k_{i-1}u_{i-1} + k_i u_{i+1} \qquad (i = 2, \ldots, n). \tag{5}$$

These differential equations determine an $(n-1)$-parameter family F_{n-1} of parallel curves. We see that there is exactly one $\bar{C}$ through every point of the common normal $(n-1)$-space $H_{n-1}(s)$.

We assume that $\bar{s}$ is defined so that $\bar{s}' > 0$. Differentiating (3) we find

$$\bar{\xi}_i = \epsilon\xi_i, \quad \bar{k}_i\bar{s}' = k_i \qquad (i = 1, \ldots, n). \tag{6}$$

In connection with these equations for $\bar{k}_i$, it should be mentioned that if the curvatures never vanish, the sense of the vectors of the moving n-hedron of a curve will be chosen so that the curvatures are positive.

Let $C_\lambda : r_\lambda = r_\lambda(s)$ be a set of curves parallel to C. Since the distances $|r_\lambda - r_\mu|$ are constant, the figure consisting of the points r_λ will move rigidly as s varies. If a subfamily F_p of F_{n-1} intersects say $H_{n-1}(s_0)$ in a linear p-space then F_p intersects every $H_{n-1}(s)$ in a linear p-space $H_p(s)$. Thus the concept of linear dependence can be applied to parallel curves.

Lemma 1. *Let*

$$C_j\colon r_j = r + \sum_{i=1}^{n} u_{ij}\xi_i \qquad (j = 1, \ldots, p),$$

be p curves in F_{n-1}. $C, C_1, \ldots, C_p$ span an F_p if and only if the Wronskian of $u_{21}, \ldots, u_{2p}$ does not vanish.

Proof. The equations (5) imply that the u_{2i} are $(n-1)$-times continuously differentiable solutions of a linear homogeneous differential equation of order $n-1$ with continuous coefficients. Hence if the Wronskian vanishes, the u_{2i} are linearly dependent (**2**, p. 116). That is, there exist constants α_i, not all zero, such that

$$\sum_1^p u_{2i}\alpha_i = 0, \tag{7}$$

and since $(r_i - r)' = -k_1 u_{2i}\xi_1$, we have

$$\sum_1^p \alpha_i(r_i - r)' = 0$$

and

$$\sum_1^p \alpha_i(r_i - r) = r_0 = \text{a constant vector.} \tag{8}$$

Since $(r_i - r)\xi_1 = 0$, $r_0\xi_1 = 0$; so that if $r_0 \neq 0$, C is in an $(n-1)$-space which contradicts $k_{n-1} \neq 0$. Hence $r_0 = 0$ and the $r_i - r$ are dependent. On the other hand (7) can be obtained from (8) by differentiating.

Lemma 2. *The curves $C, C_1, \ldots, C_p$ of Lemma 1 span an F_p if and only if the determinant $|u_{ij}| \neq 0$ $(i = 2, \ldots, p+1; j = 1, \ldots, p)$.*

Proof. Using (5) we find that $k_2^{p-1}k_3^{p-2}\ldots k_p|u_{ij}|$ is the Wronskian of $u_{21}, \ldots, u_{2p}$.

LEMMA 3. *If C is on a hypersphere, every curve $\bar{C}$ parallel to C is on a concentric hypersphere.*

Proof. Let r_0 be the centre of the hypersphere on which C lies. Then $(r - r_0)\xi_1 = 0$ and $(\bar{r} - r_0)\bar{\xi}_1 = \epsilon(\bar{r} - r)\xi_1 + \epsilon(r - r_0)\xi_1 = 0$. Thus $\bar{C}$ is on a hypersphere with centre r_0.

DEFINITION.[1] C is a pth *involute* of C^*_p and C^*_p is a pth *evolute* of C if C is an orthogonal trajectory of the osculating p-spaces of $C^*_p (p = 1, \ldots , n - 1)$.

THEOREM 1. *The pth involutes of a curve form an F_p.*

Proof. Let $\tilde{C}: \tilde{r} = \tilde{r}(s)$ be a pth involute of C. We can write

$$\tilde{r} = r + \sum_{i=1}^{p} a_i\xi_i,$$

in which the a_i are to be determined so that $\xi_1\xi_i = 0$ $(i = 1, \ldots , p)$. These conditions are satisfied if and only if

$$(9) \qquad a_1' = k_1a_2 - 1, \quad a_i' = k_ia_{i+1} - k_{i-1}a_{i-1}, \quad a_p' = - k_{p-1}a_{p-1} \qquad (i = 2, \ldots , p - 1),$$

and when the a_i are chosen in this way, a_p does not vanish identically and $\xi_1 = \pm \xi_{p+1}$ whenever $a_p \neq 0$.

Let $\tilde{C}_{(1)}$ and $\tilde{C}_{(2)}$ be pth involutes of C. $\xi_{1(1)}$ is parallel to $\xi_{1(2)}$ since each is parallel to ξ_{p+1} and

$$(\tilde{r}_{(1)} - \tilde{r}_{(2)})\xi_{1(1)} = \pm (\tilde{r}_{(1)} - \tilde{r}_{(2)})\xi_{p+1} = 0.$$

Thus $\tilde{C}_{(1)}$ and $\tilde{C}_{(2)}$ are parallel.

Since (9) is a system of linear non-homogeneous differential equations for the a_i, we can determine $\tilde{r}_{(1)}, \ldots , \tilde{r}_{(p+1)}$ so that $\tilde{r}_{(1)} - \tilde{r}_{(i)}$ $(i = 2, \ldots , p + 1)$ are independent. Then if $\tilde{r}$ is any other pth involute $\tilde{r}_{(1)} - \tilde{r}$, $\tilde{r}_{(1)} - \tilde{r}_{(i)}$ are dependent. Thus the pth involutes form an F_p.

Next we find some necessary conditions in order that C^*_p shall be a pth evolute of C. Let $S^*_p(s)$ be the osculating p-space of C^*_p. Put

$$(10) \qquad r^*_p = r + \sum_{1}^{n} b_i\xi_i.$$

The b_i are to be determined so that $r^*_p - r$ is in S^*_p and so that ξ_1 is orthogonal to S^*_p. We see that $b_1 = 0$ and differentiating (10) and using $r^{*(i)}_p\xi_1 = 0$ $(i = 1, \ldots , p)$ we obtain

$$(11) \qquad b_i = c_i \qquad (i = 1, \ldots , p + 1),$$

where $c_1, \ldots , c_n$ are defined by

$$(12) \qquad c_1 = 0, \; k_1c_2 = 1, \; c_i' = - k_{i-1}c_{i-1} + k_ic_{i+1} \qquad (i = 2, \ldots , n - 1).$$

[1]At first only the cases $n = 4$, $p = 1$ and 3 were considered. The referee suggested that involutes and evolutes of other orders be introduced.

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Hence $b_1, \ldots, b_{p+1}$ are known. We will show later that the remaining b_i can be determined so that C^*_p is a pth evolute ($p \neq n-1$ if C is on a hypersphere).

THEOREM 2. *In general the curves of an F_p have exactly one common pth evolute C^*_p. There is an exception if and only if the members of F_p are on concentric hyperspheres whose common centre lies on all the $H_p(s)$. In this case there is no common pth evolute.*

Proof. Let $C, C_1, \ldots, C_p$ span F_p. If C^*_p is a common pth evolute of these curves, then C^*_p is a pth evolute of every member of F_p. For $r^*_p - r$, $r^*_p - r_i$, $(i = 1, \ldots, p)$ are in S^*_p and ξ_1 is orthogonal to S^*_p. If

$$\bar{r} = r + \sum_1^p \alpha_i(r_i - r)$$

is any other curve in F_p, $r^* - \bar{r}$ is in S^*_p and $\bar{\xi}_1$ is orthogonal to S^*_p. Thus as far as common pth evolutes are concerned we can replace F_p by $C, C_1, \ldots, C_p$.

Since $r^*_p - r$, $r^*_p - r_j$ $(j = 1, \ldots, p)$ are dependent and the $r_j - r$ are independent we can write

$$r^*_p = r + \sum_1^p \lambda_j(r_j - r). \tag{13}$$

Putting

$$r_j = r + \sum_{i=2}^n u_{ij}\xi_i$$

we have

$$r^*_p = r + \sum_{j=1}^p \sum_{i=2}^n u_{ij}\lambda_j\xi_i.$$

But C^*_p is a pth evolute of C. Hence by (11), $(r^*_p - r)\xi_i = c_i$ $(i = 1, \ldots, p+1)$. Thus

$$\sum_{j=1}^p \lambda_j u_{ij} = c_i \qquad (i = 2, \ldots, p+1). \tag{14}$$

By Lemma 2, the determinant $|u_{ij}|$ is not zero so these equations determine λ_j uniquely. Thus there is not more than one common pth evolute of the curves of F_p and if there is one it is the curve C^*_p given by (13) and (14).

Suppose now, first, that the vectors $r^{*(i)}_p$ $(i = 1, \ldots, p)$ are linearly independent. We then prove that C^*_p actually is a common pth evolute. Differentiating (13) and using $r^{*(i)}_p\xi_1 = 0$, we obtain

$$r^{*(i)}_p = \sum_{j=1}^p \lambda_j^{(i)}(r_j - r), \qquad (i = 1, \ldots, p).$$

Since the $r^{*(i)}_p$ are independent, we can solve these equations for the vectors $r_j - r$ in terms of $r^{*(i)}_p$. Now writing

$$r_j = r + (r_j - r) = r^*_p - \sum_{i=1}^p \lambda_i(r_i - r) + (r_j - r) = r^*_p + \text{a vector in } S^*_p,$$

we see that the point r_j is in S^*_p. Since we also have ξ_1 perpendicular to S^*_p, C^*_p is the common pth evolute.

Suppose next that the vectors $r^*_p{}^{(i)}$ are dependent so that C^*_p is less than p-dimensional. We can write

$$r^*_p{}^{(p)} = \sum_{i=1}^{p-1} d_i r^*_p{}^{(i)},$$

and differentiating this,

$$r^*_p{}^{(p+1)} = \sum_{i=1}^{p-1} d_i' r^*_p{}^{(i)} + d_i r^*_p{}^{(i+1)}.$$

Since $r^*_p{}^{(i)}\xi_1 = 0$ $(i = 1, \ldots, p)$, $r^*_p{}^{(p+1)}\xi_1 = 0$. Further differentiations yield $r^*_p{}^{(i)}\xi_1 = 0$ $(i = 1, \ldots, n)$. When we differentiate $r^*_p{}^{(i)}\xi_1 = 0$ we obtain $r^*_p{}^{(i)}\xi_2 = 0$ $(i = 1, \ldots, n-1)$. Continuing this, we have $r^*_p{}'\xi_j = 0$ $(j = 1, \ldots, n)$; hence $r^*_p{}' = 0$ and C^*_p reduces to a point. Thus there is no common pth evolute. By (13), r^*_p is on $H_p(s)$ and since $(r^*_p - r)\xi_1 = 0$, C is on a hypersphere with centre r^*_p.

Finally we show that if C is on a hypersphere with centre r_0 and if r_0 is in $H_p(s)$, there is no common pth evolute. We can write

$$r_0 = r + \sum_1^p \mu_i(r_i - r), \quad r_0^{(j)}\xi_1 = 0 \qquad (j = 1, \ldots, p).$$

But these are the conditions which determine r^*_p and λ_i. Thus $r^*_p = r_0$, and the result follows.

We observe that there is a $(1-1)$ correspondence between pth evolutes of C and p-spaces in $H_{n-1}(s)$ through r.

THEOREM 3. *The hypersphere with centre r^*_p and radius $|r^*_p - r|$ has at least $(p+1)$th order contact with C at r.*

Proof. The points of intersection of C and the hypersphere with centre $r^*_p(s_0)$ and radius $|r^*_p(s_0) - r(s_0)|$ are obtained by solving the equation

$$f(s) \equiv [r(s) - r^*_p(s_0)]^2 - [r(s_0) - r^*_p(s_0)]^2 = 0$$

for s. We find

$$f^{(i)}(s) = 2[r^*_p(s) - r^*_p(s_0)]\, r^{(i)}(s) \qquad (i = 1, \ldots, p+1).$$

Therefore

$$f^{(i)}(s_0) = 0 \qquad (i = 1, \ldots, p+1).$$

Consider the $(n-1)$th evolute $C^*_{n-1} = C^*$. Assuming that C is not on a hypersphere, C^* is the $(n-1)$th evolute of every member of F_{n-1} and by Theorem 3, C^* is the locus of the centres of the osculating hyperspheres of every curve in F_{n-1}. Thus the family of parallel curves has a common locus of centres of osculating hyperspheres. (This is true even in C if on a hypersphere).

Next we obtain the relationship between the moving n-hedrons of C and C^*. Since

$$r^* = r + \sum_2^n c_i\xi_i,$$

$r^{*\prime} = (c'_n + k_{n-1}c_{n-1})\xi_n$, and we see that C is on a hypersphere if and only if

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$c'_n + k_{n-1}c_{n-1} = 0$. We will assume that this expression never vanishes and that s^* is defined so that $s^{*\prime} < 0$. Then $s^{*\prime} = (c'_n + k_{n-1}c_{n-1})\epsilon^*$ and $\xi^*_1 = \epsilon^*\xi_n$, where $\epsilon^* = \pm 1$. Further differentiations yield

$$\xi^*_i = \epsilon^*\xi_{n+1-i}, \quad k^*_i s^{*\prime} = -k_{n-i} \qquad (i = 1, \ldots, n). \tag{15}$$

Definition. *The* pth *polar developable* D_p *of* F_{n-1} $(p = 1, \ldots, n-1)$ is the surface

$$z_p = r + \sum_1^{p+1} c_i\xi_i + \sum_{p+2}^{n} y_i\xi_i, \tag{16}$$

in which $s, y_{p+2}, \ldots, y_n$ are parameters.

A particular curve C of F_{n-1} has been used in this definition. In order to justify this we will prove that the same surface is obtained if we use any other curve

$$\bar{C}\colon \bar{r} = r + \sum_1^n u_i\xi_i$$

of F_{n-1}. We now have

$$\bar{z}_p = \bar{r} + \sum_1^{p+1} \bar{c}_i\bar{\xi}_i + \sum_{p+2}^{n} \bar{y}_i\bar{\xi}_i$$
$$= r + \sum_1^{p+1} (\epsilon\bar{c}_i + u_i)\xi_i + \sum_{p+2}^{n} (\epsilon\bar{y}_i + u_i)\xi_i.$$

Using (4), (5), (6) and (12) we obtain $\epsilon\bar{c}_i + u_i = c_i$ $(i = 1, \ldots, n)$ so that $\bar{z}_p = z_p$.

Theorem 4. (a) *The* pth *evolutes of a member of* F_{n-1} *are on* D_p.

(b) D_{p+1} *is the envelope of the* $(n - p - 1)$*-spaces which generate* D_p $(p = 1, \ldots, n-2)$.

(c) *If* C *is not on a hypersphere,* D_p *is generated by the* $(n - p - 1)$*-dimensional osculating spaces of* $C^* = D_{n-1}$.

(d) *If* F_p *has a* pth *evolute, this evolute is the locus of the point of intersection of* $H_p(s)$ *and the corresponding generator of* D_p.

(e) *The first evolutes of the curves in* F_{n-1} *are geodesics on* D_1.

(*f*) *If* $x = x(s)$ *is a geodesic on* D_1 *and is not a straight line,* x *is a first evolute of some member of* F_{n-1}.

Proof. (a) Compare (10), (11) and (16).

(b) The equations of a generator of D_p are $(R - r)\xi_i = c_i$ $(i = 1, \ldots, p+1)$. Differentiating these, we see that there is an envelope and that it is D_{p+1}.

(c) The $(n - p - 1)$-dimensional osculating spaces of C^* are

$$R = r^* + \sum_1^{n-p-1} y^*_i \xi^*_i$$

in which the y^*_i are parameters. This is

$$R = r + \sum_{1}^{n} c_i\xi_i + \sum_{p+2}^{n} \overset{*}{\epsilon}\overset{*}{y}_{n+1-i}\xi_i,$$

using (15). When we put

$$c_i + \overset{*}{\epsilon}\overset{*}{y}_{n+1-i} = y_i \qquad (i = p+2, \ldots, n),$$

R becomes z_p.

(d) Let $C, C_1, \ldots, C_p$ span F_p and let the point of intersection of $H_p(s)$ and the generator be R. Since R is in H_p,

$$R = r + \sum_{1}^{p} \mu_i(r_i - r).$$

The generator is $(R - r)\xi_j = c_j$ $(j = 1, \ldots, p+1)$. The unique solution of these equations is $\mu_i = \lambda_i$, $R = r^*_p$.

(e) The principal normal of C^*_1 is $\pm\,\xi_1$ which is normal to D_1. Hence C^*_1 is a geodesic.

(f) We will show that every first involute of $x = x(s)$ is parallel to C. Let $y = y(s)$ be a first involute of x. Since x is a geodesic, $\xi_{2(x)} = \pm\,\xi_1$ and since y is a first involute of x, $\xi_{1(y)} = \pm\,\xi_{2(x)}$. Thus $\xi_{1(y)}$ is parallel to ξ_1. Also $(y - r)\xi_1 = (y - x)\xi_1 + (x - r)\xi_1$. The first term of this is zero because $y - x$ is parallel to $\xi_{1(x)}$ and the second term is zero because x is on D_1. Hence $y = y(s)$ is parallel to C.

Next we want to develop D_1 on an $(n-1)$-space and determine the point of the $(n-1)$-space which corresponds to $(s, y_3, \ldots, y_n)$ of D_1. Since D_1 is the tangent hypersurface of C^*_{n-1} (provided C is not on a hypersphere) we will first see how to develop the tangent hypersurface of a given curve C on an $(n-1)$-space H. Vectors in H will be denoted by capital letters.

C will roll along a curve $R = R(s)$ in H and R will have arc length s. The point

$$z = r + \sum_{1}^{n-2} y_i\xi_i$$

of the tangent hypersurface is mapped on

$$Z = R + \sum_{1}^{n-2} y_i T_i$$

where $(T_1, \ldots, T_{n-1})$ is the moving $(n-1)$-hedron of R. Now using the fact that the line element is invariant under this transformation we find that k_i for R is equal to k_i for C $(i = 1, \ldots, n-2)$.

Turning now to the first polar developable of C, the point

$$z = \overset{*}{r} + \sum_{1}^{n-2} \overset{*}{y}_i\overset{*}{\xi}_i = r + c_2\xi_2 + \sum_{3}^{n} (c_i + \overset{*}{\epsilon}\overset{*}{y}_{n+1-i})\xi_i$$

of D_1 corresponds to

$$Z = \overset{*}{R} + \sum_{1}^{n-2} \overset{*}{y}_i\overset{*}{T}_i$$

where R^* and T^*_i are determined by

$$\frac{dR^*}{ds^*} = T^*_1, \quad \frac{dT^*_i}{ds^*} = -k^*_{i-1}T^*_{i-1} + k^*_iT^*_{i+1}, \quad \frac{dT^*_{n-1}}{ds^*} = -k^*_{n-2}T^*_{n-2}$$
$$(i = 1, \ldots, n-2).$$

Now

$$R^* = \int T^*_1\,ds^* = \int \epsilon^*(c_{n-1}k_{n-1} + c'_n)\,T^*_1\,ds$$

and after integrating by parts $n-1$ times we obtain

$$R^* = \epsilon^*\sum_2^n c_iT^*_{n+1-i}.$$

Let us put

$$\epsilon^*T^*_{n-i} = T_i, \quad c_j + \epsilon^*y^*_{n+1-j} = y_j \quad (i = 1, \ldots, n-1;\ j = 3, \ldots, n).$$

The point

$$r + c_2\xi_2 + \sum_3^n y_i\xi_i$$

of D_1 corresponds to

$$c_2T_1 + \sum_3^n y_iT_{i-1} \tag{17}$$

where $T_1, \ldots, T_{n-1}$ are solutions in H of

$$T_1' = k_2T_2, \quad T_i' = -k_iT_{i-1} + k_{i+1}T_{i+1} \qquad (i = 2, \ldots, n-1). \tag{18}$$

We have assumed above that C is not on a hypersphere. However, once we have (17) and (18) we can easily verify, by comparing line elements, that they are correct in this case also.

Let A be an arbitrary constant vector in H. The general solution of the differential equations (5) is

$$u_{i+1} = AT_i \qquad (i = 1, \ldots, n-1).$$

Thus we have a (1 − 1) correspondence between points of H and curves of F_{n-1}. The geometrical significance of the point A which corresponds to $\bar{C}$ of F_{n-1} is that if the whole n-space is moved rigidly when we are developing D_1, $\bar{C}$ cuts H in the fixed point A. This follows from the fact that the point of intersection of $\bar{C}$ and the tangent $(n-1)$-space of D_1 is

$$r + \sum_2^n u_i\xi_i;$$

and since $T_1 = \epsilon^*T^*_{n-1} = \epsilon^*\xi^*_{n-1} = \xi_2$, the transform of this point is

$$\sum_2^n u_iT_{i-1} = \sum_2^n (AT_{i-1})\,T_{i-1} = A.$$

THEOREM 5. *When D_1 is developed on H, the* pth *evolutes of curves of F_{n-1} become p-dimensional curves in H.*

Proof. We may assume that the pth evolute under consideration is the common pth evolute C^*_p of $C_1, \ldots, C_{p+1}$ where $C_1, \ldots, C_{p+1}$ are parallel curves which span an F_p. Let C_i correspond to the point A_i of H.

$$r^*_p = r_1 + \sum_2^{p+1} \lambda_i(r_i - r_1) = r + \sum_{j=2}^{n} (A_1 T_{j-1})\xi_j + \sum_{i=2}^{p+1}\sum_{j=2}^{n} \lambda_i\{(A_i - A_1)\,T_{j-1}\}\xi_j,$$

which transforms to

$$R^*_p = \sum_{j=2}^{n} (A_1 T_{j-1})\,T_{j-1} + \sum_{i=2}^{p+1}\sum_{j=2}^{n} \lambda_i\{(A_i - A_1)\,T_{j-1}\}\,T_{j-1} = A_1 + \sum_2^{p+1} \lambda_i(A_i - A_1).$$

Thus the transform is in the p-space determined by $A_1, \ldots, A_{p+1}$. (If the A_i were in a $(p-1)$-space, $C_1, \ldots, C_{p+1}$ would not span an F_p). If R^*_p were in a $(p-1)$-space the determinant $|\lambda_i^{(j)}|$ $(i = 2, \ldots, p+1; j = 1, \ldots, p)$ would be identically zero; the vectors $r^*_p{}^{(j)}$ would be dependent, and $C_1, \ldots, C_{p+1}$ would have no common pth evolute.

We now have a $(1-1)$ correspondence between pth evolutes of curves of F_{n-1} and p-spaces in H.

Consider the case $p = 1$. The transform of the common first evolute of C_1 and C_2 is $R^*_1 = A_1 + \lambda_2(A_2 - A_1)$, the straight line through A_1 and A_2. C^*_1 actually corresponds to a segment of this line. The segment includes A_1 if and only if $\lambda_2 = 0$ for some s. The family of first evolutes of C_1 corresponds to the family of straight lines through A_1. Thus we have the following theorem.

THEOREM 6. *The family of geodesics (which are not straight lines) through a point of D_1 is the family of first evolutes of some curve of F_{n-1}.*

THEOREM 7. *The angle between the tangents at corresponding points of two first evolutes of a given curve is constant. This angle is equal to the angle between the lines in H which correspond to the two first evolutes.*

Proof. Let the given curve be C and let $C^*_{1(i)}$ $(i = 1, 2)$ be the common first evolute of C and C_i. Since the tangent of $C^*_{1(i)}$ is parallel to $r_i - r$, the cosine of the angle between the tangents is

$$\frac{(r_1 - r)(r_2 - r)}{|r_1 - r||r_2 - r|} = \frac{\left[\sum_1^{n-1} (A_1 T_i)\xi_{i+1}\right]\left[\sum_1^{n-1} (A_2 T_j)\xi_{j+1}\right]}{\sqrt{\left[\sum_1^{n-1} (A_1 T_i)\xi_{i+1}\right]^2\left[\sum_1^{n-1} (A_2 T_j)\xi_{j+1}\right]^2}}$$
$$= \sum_1^{n-1} (A_1 T_i)(A_2 T_i) \Big/ \sqrt{\sum_1^{n-1} (A_1 T_i)^2 \sum_1^{n-1} (A_2 T_i)^2}$$
$$= A_1 A_2 / \sqrt{A_1^2 A_2^2}.$$

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