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ERRATUM

Nair–Tenenbaum uniform with respect to the discriminant – ERRATUM

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In this note we wish to correct several mistakes which appeared in our paper [1]. Fortunately, all the results claimed in that reference may be recovered, with small modifications to the statements and the argument. We are grateful to Nathan Ng for drawing our attention to an issue in the application of Theorem 6 to divisor sums, and to Régis de la Bretèche for helpful discussions on the problem addressed here.

Congruence conditions. We use the notation as well as page, equation and theorem numbers from [1]. The most problematic part of our paper is the lower bound in Theorem 6, which is incorrect as stated, and as a consequence the upper bounds in Theorem 5 and Corollaries 1–2 are not sharp as claimed (although they are still valid). We explain here how to recover a lower bound on the sum considered in Theorem 6, together with a matching upper bound in Theorems 5 and 6, by modifying the function $\widehat{\rho}_{\mathbf{R}}(a_1, \ldots, a_r)$ which encodes certain polynomial congruences.

The source of the error in Theorem 6 is at the end of p. 423, where the last displayed equation is wrong: the conditions $P^+(a_1 \cdots a_r) < x^{\varepsilon_3}$, $a_i \| R_i(n)$ for all i and $p | Q(n)$, $p \nmid a_1 \ldots a_r \Rightarrow p > x^{\varepsilon_3}$ alone are insufficient to guarantee that the a_i are the unique integers such that $R_i(n) = a_i b_i$ with $P^+(a_i) < x^{\varepsilon_3} \leqslant P^-(b_i)$ for all i. Indeed, under these conditions, if $p | (R_i(n)/a_i)$ then $p \nmid a_i$, but we may well have $p | \prod_{j \neq i} a_j$, preventing us from concluding that $p > x^{\varepsilon_3}$. To address this we introduce the following property, dependent on integers $a_1, \ldots, a_r, n$:

$$(\mathcal{P}_{a_1,\ldots,a_r,n}) : \ \forall 1 \leqslant i \leqslant r, \quad a_i \| R_i(n) \quad \text{and} \quad p | \textstyle\prod_{j \neq i} a_j, \ p \nmid a_i \Rightarrow p \nmid R_i(n). \qquad (0{\cdot}1)$$

The workaround we find is to replace the function $\widehat{\rho}_{\mathbf{R}}$ defined in (2·8), in all instances where it appears, by the function $\breve{\rho}_{\mathbf{R}}$ defined by

$$\breve{\rho}_{\mathbf{R}}(a_1, \ldots, a_r) = \#\{n \bmod [a_1\kappa(a_1), \ldots, a_r\kappa(a_r)] : \ (\mathcal{P}_{a_1,\ldots,a_r,n}) \ \}; \qquad (0{\cdot}2)$$

we explain shortly why this is possible. First observe that, by the Chinese remainder theorem, $\breve{\rho}_{\mathbf{R}}$ is also multiplicative, and since $\breve{\rho}_{\mathbf{R}} \leqslant \widehat{\rho}_{\mathbf{R}}$ it also satisfies the upper bound (2·9). Therefore $\breve{\rho}_{\mathbf{R}}$ behaves as $\widehat{\rho}_{\mathbf{R}}$ for analytic purposes, and it remains to show that the substitution can be performed. In our paper, the function $\widehat{\rho}_{\mathbf{R}}$ arises exclusively through applications of Lemma 6, and this lemma may be modified to state that

$$\sum_{\substack{x<n\leqslant x+y:\\ (\mathcal{P}_{a_1,\ldots,a_r,n})\\ p\mid Q(n),\ p\nmid a_1\ldots a_r,\ p\notin\Xi\ \Rightarrow\ p>z}} 1 \asymp y\frac{\breve{\rho}_{\mathbf{R}}(a_1,\ldots,a_r)}{[a_1\kappa(a_1),\ldots,a_r\kappa(a_r)]}\prod_{\substack{g<p\leqslant z\\ p\nmid a_1\ldots a_r}}\left(1-\frac{\rho(p)}{p}\right).$$

In all instances where Lemma 6 (or a variant thereof) is invoked, that is, on pp. 418–419 and on p. 423, the a_i arise from decompositions $R_i(n) = a_i b_i$, with $P^+(a_i) < q \leqslant P^-(b_i)$ $(1 \leqslant i \leqslant r)$ for some $q \geqslant 1$, so that $(\mathcal{P}_{a_1,\ldots,a_r,n})$ is satisfied. We may therefore use the above modified version of Lemma 6 on pp. 418–419, and substitute $\widehat{\rho}_{\mathbf{R}}$ with $\breve{\rho}_{\mathbf{R}}$ throughout Sections 4–6. We may also rectify the proof of Theorem 6 (with the previous substitution operated in its statement), by replacing the last displayed equation on p. 423 by

$$S \gg \sum_{\substack{a_1\ldots a_r\leqslant x^{\varepsilon_3}\\ p\in\Xi\ \Rightarrow\ p\mid a_1\ldots a_r}} \widetilde{F}(a_1,\ldots,a_r) \sum_{\substack{x<n\leqslant x+y\\ (\mathcal{P}_{a_1,\ldots,a_r,n})\\ p\mid Q(n),\ p\nmid a_1\ldots a_r,\ p\notin\Xi\ \Rightarrow\ p>x^{\varepsilon_3}}} 1,$$

and by invoking the modified version of Lemma 6 again.

Sub-multiplicativity. The proofs of Lemmas 2 and 3 also contain a small mistake: the sub-multiplicativity estimate (4·2) is invoked for integers a_i and $b_i = d_i t_i$ satisfying only $(a_i, b_i) = 1$ for all $1 \leqslant i \leqslant r$, but not necessarily

$$(a_1\cdots a_r, b_1\cdots b_r) = 1.$$

To address this, in the proof of Lemma 3 one should use the Möbius expansion

$$(n_1\cdots n_r)^\beta = \sum_{d\mid n_1\cdots n_r}\psi(d)$$

instead of distinct expansions $n_i^\beta = \sum_{d_i\mid n_i}\psi(d_i)$ for each i. Given $n_1,\ldots,n_r$ such that $d\mid n_1\cdots n_r$, write then $n_i = d_i a_i$ with $d_i\mid d^\infty$ and $(d, a_i) = 1$ for $1 \leqslant i \leqslant r$, so that $d\mid d_1\cdots d_r$. For $p \leqslant z$, the proof of Lemma 3 furnishes the bounds

$$\psi(p^s) \ll \chi\frac{\log p}{\log z} \quad (1\leqslant s\leqslant 8g) \quad \text{and} \quad \psi(p^s) \ll p^{s\beta} \quad (s\geqslant 1),$$

while Lemma 1 shows that, for $s \geqslant 1$,

$$\sum_{\nu_1+\cdots+\nu_r\geqslant s} T(p^{\nu_1},\ldots,p^{\nu_r}) \ll \min(p^{-1}, p^{-s/4g}).$$

Substituting the above expansion of $(n_1\cdots n_r)^\beta$ in the sum over $n_1,\cdots,n_r$, and using the above bounds in the same fashion as in the original proof, Lemma 3 follows, and it requires an appeal to (4·2) only for the integers a_i, d_i satisfying $(a_1\cdots a_r, d_1\cdots d_r) = 1$. The same fix can be applied to Lemma 2, up to replacing the product $\theta_1(n_1)\cdots\theta_r(n_r)$ in the assumptions of this lemma by $\theta(n_1\cdots n_r)$, where θ is a multiplicative function of one variable such that $\theta(p^\nu) = 1 + O(p^{-1})$ uniformly in p and ν (this is all that is needed in applications of this lemma). Substituting then $\theta(n_1\cdots n_r) = \sum_{d\mid n_1\cdots n_r}\lambda(d)$ in the sum considered, and using the same decomposition of the n_i as above, the result follows by a simple alteration of the original proof.

Polynomials. One last oversight concerns the primes dividing the leading coefficient of the main polynomial Q^*. For polynomials $P_1, P_2 \in \mathbb{Z}[X]$, we use the identity $\mathrm{Disc}(P_1, P_2) =$

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$\mathrm{Disc}(P_1)\,\mathrm{Disc}(P_2)\,\mathrm{Res}(P_1, P_2)^2$ on p. 422 (with $P_1 = R_i$, $P_2 = R_j$), however that identity is only valid for monic polynomials. When P_1 and P_2 are polynomials of respective degrees f_1 and f_2 and respective leading coefficients u_1 and u_2, the correct identity turns out to be

$$u_1^{f_2} u_2^{f_1}\,\mathrm{Disc}(P_1, P_2) = \mathrm{Disc}(P_1)\,\mathrm{Disc}(P_2)\,\mathrm{Res}(P_1, P_2)^2.$$

Let a^* denote the leading coefficient of Q^*, then by using this correct identity on p. 422 we may at least deduce that $\mathrm{Res}(R_i, R_j)|(a^*D^*)^\infty$ for every distinct i, j. Note also that the bound (2·5) is correct only under the more restrictive condition $p \nmid a^*D^*$. But it is then is easy to verify that Theorem 6 and Corollaries 1–2 hold as stated when the quantity D^* is replaced by a^*D^*.

Conclusion. For the convenience of the reader, we restate the main results of our paper [1], incorporating the above corrections. We again refer to this paper for the meaning of the notation used.

THEOREM (New Theorem 5). *Let $k \geqslant 1$ and suppose that $Q_1, \ldots, Q_k \in \mathbb{Z}[X]$ are primitive polynomials, and write $Q = Q_1 \cdots Q_k$. Let R_i be the irreducible factors of Q in $\mathbb{Z}[X]$, and g be its degree. Define also $\breve{\rho}_{\mathbf{R}}$ by* (0·1) *and* (0·2)*, and let $\alpha, \delta \in (0, 1]$ and $A, B \geqslant 1$ be parameters. Suppose finally that $0 < \varepsilon < \alpha/50g(g + \delta^{-1})$ and $F \in \mathcal{M}_k(A, B, \varepsilon)$. Then we have, uniformly in $x \geqslant C_0\|Q\|^\delta$ and $x^\alpha < y \leqslant x$,*

$$\sum_{x<n\leqslant x+y} F\big(|Q_1(n)|, \ldots, |Q_k(n)|\big)$$
$$\ll y \prod_{g<p\leqslant x}\left(1 - \frac{\rho(p)}{p}\right) \sum_{n_1\ldots n_r\leqslant x} \widetilde{F}(n_1, \ldots, n_r)\frac{\breve{\rho}_{\mathbf{R}}(n_1, \ldots, n_r)}{[n_1\kappa(n_1), \ldots, n_r\kappa(n_r)]},$$

where C_0 and the implicit constant depend at most on g, α, δ, A, B.

THEOREM (New Theorem 6). *Let $\eta > 0$ be a parameter. Under the same assumptions as above, and assuming furthermore that F is multiplicative and satisfies $F(n_1, \ldots, n_r) \gg \eta^{\Omega(n_1\ldots n_r)}$ uniformly in $n_1, \ldots, n_r \geqslant 1$, we have*

$$\sum_{x<n\leqslant x+y} F\big(|Q_1(n)|, \ldots, |Q_k(n)|\big) \asymp y \prod_{g<p\leqslant x}\left(1 - \frac{\rho(p)}{p}\right)\cdot \Delta_*\Delta$$

where

$$\Delta_* = \prod_{p|a^*D^*} \sum_{\substack{0\leqslant \nu_i\leqslant \deg(R_i)\\(1\leqslant i\leqslant r)}} \widetilde{F}(p^{\nu_1}, \ldots, p^{\nu_r})\frac{\breve{\rho}_{\mathbf{R}}(p^{\nu_1}, \ldots, p^{\nu_r})}{p^{\max(\nu_i)+1}},$$

$$\Delta = \prod_{\substack{p\leqslant x\\ p\nmid a^*D^*}}\left(1 + \sum_{i=1}^{r} \widetilde{F}^{(i)}(p)\frac{\rho_{R_i}(p)}{p}\right)$$

and where implicit constants depend at most on $g, \alpha, \delta, A, B, \eta$.

REFERENCE

[**1**] K. HENRIOT. Nair–Tenenbaum bounds uniform with respect to the discriminant. *Math. Proc. Camb. Phil. Soc.* **152** (2012), no. 3, 405–424. doi:10.1017/S0305004111000752

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