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SOLVABLE SUBGROUPS AND THEIR LIE ALGEBRAS IN CHARACTERISTIC p

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1. Introduction. Throughout this paper, G is a connected linear algebraic group over an algebraically closed field whose characteristic is denoted p. For any closed subgroup H of G, $\underline{h}$ denotes the Lie algebra of H and H^0 denotes the connected component of the identity of H.

A *Borel subalgebra* of $\underline{g}$ is the Lie algebra $\underline{b}$ of some Borel subgroup B of G. A *maximal torus* of $\underline{g}$ is the Lie algebra $\underline{t}$ of some maximal torus T of G. In [**4**], it is shown that the maximal tori of $\underline{g}$ are the maximal commutative subalgebras $\underline{t}$ of $\underline{g}$ consisting of semisimple elements, and the question was raised in § 14.3 as to whether the set of Borel subalgebras of $\underline{g}$ is the set of maximal triangulable subalgebras of $\underline{g}$.

In this paper, we give an example showing that this is not true and show for $p > 3$ that the set of Borel subalgebras of $\underline{g}$ is, rather, the set of those maximal solvable subalgebras of $\underline{g}$ which contain a maximal torus of $\underline{g}$. The upshot of this is that Borel subalgebras (as well as maximal tori) of $\underline{g}$ can, for $p > 3$, be characterized within the language of restricted Lie algebras (see [**6**], [**8**]). (It would be very interesting to know what happens for $p = 2, 3$. The situation there appears to be quite complicated, especially in characteristic $p = 2$, and requires methods different than those of this paper.)

We also examine the normalizer $N(\underline{t})$ and centralizer $C(\underline{t})$ in G of a maximal torus $\underline{t}$ of $\underline{g}$. The latter group $C(\underline{t})$, unexpectedly, is connected. (It is pointed out in [**1**] that $C(\underline{s})$ need not be connected for every torus $\underline{s}$). The *Weyl group* $W(\underline{t}) = N(\underline{t})/C(\underline{t})$ of $\underline{g}$ is isomorphic to the Weyl group $W(T) = N(T)/C(T)$ of G (although both $N(\underline{t})$, $C(\underline{t})$ generally are larger than $N(T)$, $C(T)$). Moreover, $W(\underline{t})$ acts transitively on the set of Borel subalgebras of $\underline{g}$.

2. Borel subalgebras of $\underline{g}$. We assume in this section that $p > 3$ and begin by stating without proof a simple proposition on root groups which can easily be verified by examining the root systems of the rank 1 and 2 groups A_1, $A_1 \times A_1$, A_2, B_2, G_2 generated by the root groups U_a.

2.1. Proposition. *Let G be semisimple with maximal torus T, set of roots $\underline{\underline{R}}$, root groups U_a and root spaces $\underline{g}_a = \underline{u}_a$ ($a \in \underline{\underline{R}}$). Then*

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1. $[\underline{g}_a, \underline{g}_b] = \{0\}$ *if and only if* $[U_a, U_b] = \{1\}$; *and*
2. $[U_a, U_b] \neq \{1\}$ *if and only if* $a + b \in \underline{\underline{R}} \cup \{0\}$; *so that*
3. $[\underline{g}_a, \underline{g}_b] \neq \{0\}$ *if and only if* $a + b \in \underline{\underline{R}} \cup \{0\}$.

We now establish the restricted Lie algebra characterization of Borel subalgebras.

THEOREM. *The Borel subalgebras of* $\underline{g}$ *are those maximal solvable subalgebras* $\underline{b}$ *of* $\underline{g}$ *which contain a maximal torus* $\underline{t}$ *of* $\underline{g}$.

Proof. Let $\underline{b}$ be a maximal solvable subalgebra of $\underline{g}$ containing a maximal torus $\underline{t}$ of $\underline{g}$. Letting R be the radical of $\bar{G}$ and $\pi : G \to \bar{G} = G/R$ the canonical homorphism, the differential $d\pi : \underline{g} \to \bar{\underline{g}} = \underline{g}/\underline{r}$ is surjective (see [**4**; p. 82]) and preserves maximal tori (see [**8**; 2.16]). It follows that the image $\bar{\underline{b}} = d\pi\underline{b}$ is a maximal solvable subalgebra of $\bar{\underline{g}}$ containing the maximal torus $\bar{\underline{t}} = d\pi\underline{t}$ of $\bar{\underline{g}}$. This shows that it suffices to prove the theorem for G semisimple, for then $\bar{\underline{b}}$ would be the Lie algebra of a Borel subgroup B of $\bar{G}$, whence $\underline{b}$ would be the Lie algebra of the Borel subgroup $B = \pi^{-1}(\bar{B})$ of G. Consider the root space decomposition $\underline{g} = \underline{t} + \sum_{a \in \underline{\underline{R}}} \underline{g}_a$. For $a \in \underline{\underline{R}}$, $\underline{g}_a$ is of dimension 1. Since $\underline{b} \supset \underline{t}$, it follows that $\underline{b} = \underline{t} + \sum_{a \in \underline{\underline{S}}} \underline{g}_a$ for some subset $\underline{\underline{S}}$ of $\underline{\underline{R}}$. Since $\underline{g}_{-a} + [\underline{g}_{-a}, \underline{g}_a] + \underline{g}_a$ is semisimple for all a (e.g. see [**4**; p. 12]) and since B is solvable, $\underline{\underline{S}}$ and $-\underline{\underline{S}} = \{-a | a \in \underline{\underline{S}}\}$ are disjoint. Furthermore, $\underline{\underline{S}}$ is *closed* in the sense that for any $a, b \in \underline{\underline{S}}$ for which $a + b \in \underline{\underline{R}}$, $a + b$ is also in $\underline{\underline{S}}$ (since $a + b \neq 0$ and $[\underline{g}_a, \underline{g}_b] = \underline{g}_{a+b} \subset \underline{b}$ by the above proposition). Since $\underline{\underline{S}}$ satisfies these two conditions, $\underline{\underline{S}}$ is contained in the set of positive roots for some ordering ([**3**; p. 163]), so that $\underline{b}$ is contained in and therefore equal to the corresponding Borel subalgebra. (We use here the maximal solvability of $\underline{b}$ in $\underline{g}$).

We now construct a maximal triangulable subalgebra of $\underline{g}$ which is not a Borel subalgebra of $\underline{g}$. For this, it is convenient and informative to state without proof the following proposition on which the example is based.

PROPOSITION. *Let* G *be semisimple and express* G *as* $G = G_1 G_2 \ldots G_n$ *(almost direct) where the* G_i *are almost simple closed normal subgroups of* G. *Let* x *be a nilpotent element of* $\underline{g}$, $\underline{b}$ *a Borel subalgebra of* $\underline{g}$ *containing a maximal torus of* $\underline{g}$, $\underline{u}$ *the ideal of nilpotent elements of* $\underline{b}$. *Then*

1. x *can be expressed uniquely as* $x = \sum_1^n x_i$ *where* x_i *is nilpotent and* $x_i \in \underline{g}_i$ *for* $1 \leqq i \leqq n$;
2. b *contains* x *if and only if* $\underline{b}$ *contains* x_i *for* $1 \leqq i \leqq n$;
3. $\underline{u} = \sum_1^n \underline{u} \cap \underline{g}_i$ *(direct)*.

To construct the example, take G to be semisimple of type $A_{p-1} \times A_{p-1}$ and of isogeny class such that $G = G_1 G_2$ (almost direct) where G_i is a closed connected normal subgroup of G of type A_{p-1} with $\underline{g}_i$ isomorphic to the Lie algebra $\underline{h}$ of linear transformations of trace 0 in a vector space V over F of dimension p for $i = 1, 2$ and where $\underline{g}_1 \cap \underline{g}_2$ is the center of $\underline{g}_i$ for $i = 1, 2$. (See [**4**], 10.4.) The Lie algebra $\underline{h}$ is unusual in that the center of $\underline{h}$ is spanned by the identity

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transformation t and $t = [m, n]$ where m and n are transformations defined in terms of a basis $e_1, \ldots, e_p$ by $e_j m = j e_{j+1}$ $(1 \leqq j \leqq p - 1)$, $e_p m = 0$, $e_j n = e_{j-1}$ $(2 \leqq j \leqq p)$, $e_1 n = 0$. Thus, there exist elements m_i, n_i in $\underline{g}_i$ such that $[m_i, n_i] = t_i$ is nonzero and spans the center of $\underline{g}_i$ for $i = 1, 2$. Since $\underline{g}_1$ and $\underline{g}_2$ have the same one-dimensional center, we can choose the m_i, n_i, t_i such that $[m_1, n_1] = t_1 = t_2 = [m_2, n_2]$. Let $m = m_1 + m_2$, $n = n_1 - n_2$. Note that $[\underline{g}_1, \underline{g}_2] = \{0\}$ since $[G_1, G_2] = \{1\}$. Since the m_i, n_i are nilpotent, m and n are therefore nilpotent. Moreover, $[m, n] = [m_1, n_1] - [m_2, n_2] = t_1 - t_2 = 0$! Thus, $\underline{u} = mF + nF$ is an abelian subalgebra of $\underline{g}$ consisting of nilponent elements. It is certainly triangulable, but it is not contained in a maximal solvable subalgebra $\underline{b}$ of $\underline{g}$ containing a maximal torus of $\underline{g}$. For if it were, the nilpotent elements $m = m_1 + m_2$ and $n = n_1 - n_2$ of $\underline{b}$ would have components m_1, m_2, n_1, $-n_2$ in $\underline{b}$ by the above proposition. But that would be impossible since $[\underline{b}, \underline{b}]$ consists of nilpotent elements (since $\underline{b}$ is triangulable) and therefore cannot contain the nonzero semisimple element $[m_1, n_1] = t_1$. Thus, no maximal triangulable or maximal solvable subalgebra of $\underline{g}$ containing $\underline{u}$ contains a maximal torus of $\underline{g}$. Furthermore, the subspace $\underline{u} + Fm_1 + Ft_1$ is a solvable subalgebra which is not triangulable since its derived algebra contains the nonzero semisimple element $[m_1, m] = [m_1, m_2] = t_1$.

3. The Weyl group of $\underline{g}$. For this section, we drop the assumption $p > 3$. Let $\underline{t}$ be a maximal torus of $\underline{g}$ and T a maximal torus of G with Lie algebra $\underline{t}$. We cannot precisely compare the normalizers $N(\underline{t}) = \{x \in G | x^{-1}\underline{t}x \subset t\}$ and $N(T) = \{x \in G | x^{-1}Tx \subset T\}$ or the centralizers

$$C(\underline{t}) = \{x \in G | x^{-1}tx = t \text{ for all } t \in \underline{t}\}$$

and $C(T) = \{x \in G | x^{-1}tx = t \text{ for all } t \in T\}$, but we can closely relate the *Weyl group* $W(\underline{t}) = N(\underline{t})/C(\underline{t})$ of g with the Weyl group $W(T) = N(T)/C(T)$ of G and establish that $C(\underline{t})$ is connected. We begin with the solvable case, which is settled by the following proposition. (This proposition is also proved in [**2**, Prop. 4.7]. Note that $\underline{t}$ need not be maximal in the proposition.)

PROPOSITION. *Let G be solvable. Then $N(\underline{t})$ and $C(\underline{t})$ are connected and $N(\underline{t}) = C(\underline{t})$.*

Proof. Let $x \in N(\underline{t})$ and note that $x^{-1}C(\underline{t})x = C(\underline{t})$. It follows that $x^{-1}Tx \subset C(\underline{t})_0$, so that $x^{-1}Tx = y^{-1}Ty$ for some $y \in C(\underline{t})_0$ by the conjugacy of the maximal tori $x^{-1}Tx$ and T of $C(\underline{t})_0$. But then $xy^{-1} \in N(T) = C(T) \subset C(\underline{t})_0$, so that x must be in $C(\underline{t})_0$. This is true for all $x \in N(\underline{t})$. It follows that $N(\underline{t}) = C(\underline{t}) = C(\underline{t})_0$, which was to be proved.

THEOREM. *$C(\underline{t})$ is connected for any maximal torus $\underline{t}$ of $\underline{g}$.*

Proof. Let B be a Borel subgroup of G containing T. Let $x \in C(\underline{t})$. The automorphism $\operatorname{Ad} x : y \to x^{-1}yx$ of $\underline{g}$ keeps fixed each element of $\underline{t}$ and therefore stabilizes each root space $\underline{g}_a$ of $\underline{t}$ in $\underline{g}$.

Since $\underline{b}$ is a sum of such root spaces, $\underline{b}$ is stable so that x is in the normalizer $N(\underline{b})$ of $\underline{b}$ in G. But $B = N(\underline{b})$ (see [**4**; 14.5] and [**2**; 2.5 and 2.6]), so that $x \in B$. Thus, we have $C(\underline{t}) \subset B$, so that $C(\underline{t})$ is connected by the above proposition for the solvable case.

THEOREM. *The Weyl group* $W(\underline{t}) = N(\underline{t})/C(\underline{t})$ *acts simply transitively on the set of Borel subalgebras of* $\underline{g}$ *containing* $\underline{t}$ *and* $W(T)$ *is isomorphic to* $W(\underline{t})$ *under the canonical mapping* $xC(T) \mapsto xC(\underline{t})$.

Proof. The Borel subgroups B of G and Borel subalgebras $\underline{b}$ of $\underline{g}$ are in $1 - 1$ correspondence relative to $B \mapsto \underline{b}$ and $\underline{b} \mapsto N(b)$ (see [**4**, 14.5], [**2**, 2.5 and 2.6]).

We have seen in the proof of the above theorem that $x^{-1}\underline{b}x = \underline{b}$ for $x \in C(\underline{t})$ and $\underline{b}$ a Borel subalgebra of $\underline{g}$ containing $\underline{t}$. Furthermore, $N(\underline{t})$ acts transitively on the set of Borel subalgebras $\underline{b}$ containing $\underline{t}$ by the above correspondence, since $N(\underline{t}) \supset N(T)$ and $N(T)$ acts transitively on the Borel subgroups of G containing T and therefore also on the Borel subalgebras of $\underline{g}$ containing $\underline{t}$. Suppose that $x \in N(\underline{t})$ and $x^{-1}\underline{b}x = \underline{b}$. Then $x \in N(\underline{b}) = B$, a Borel subgroup of G with Lie algebra $\underline{b}$ containing $\underline{t}$. By the above proposition for the solvable case, we therefore have $x \in C(\underline{t})$. Thus, $W(\underline{t}) = N(\underline{t})/C(\underline{t})$ acts simply transitively on the set of maximal solvable subalgebras $\underline{b}$ containing $\underline{t}$. Since $W(T) = N(T)/C(T)$ also acts simply transitively on the same set, it is now a simple exercise to show that the mapping $xC(T) \mapsto C(\underline{t})$ is an isomorphism from $W(T)$ to $W(\underline{t})$.

REFERENCES

1. A. Borel and T. A. Springer, *Rationality properties of linear algebraic groups*, Proc. Symp. Pure Math., vol. IX (1966), 26–32.

2. ——— *Rationality properties of linear algebraic groups II*, Tohoku Math. J. *20* (1968), 443–497.

3. N. Bourbaki, *Groupes et algèbres de Lie*, Chap. 4, 5, 6, Éléments de Mathématique *34*, (Hermann, Paris, 1968).

4. J. Humphreys, *Algebraic groups and modular Lie algebras*, Memoirs AMS No. *71*, AMS, Providence (1967).

5. ——— *Linear algebraic groups*, Graduate Texts in Mathematics *21*, (Springer Verlag, New York, 1975).

6. N. Jacobson, *Lie algebras*, (Interscience, New York, 1962).

7. D. J. Winter, *Abstract Lie algebras*, (MIT Press, Cambridge, 1972).

8. ——— *On the toral structure of Lie p-algebras*, Acta Math. *193* (1969), 69–81.

9. ——— *Solvable and nilpotent subalgebras of Lie algebras*, Bull. AMS *74* (1968), 754–758.

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