A CONVERSE TO LEBESGUE'S DOMINATED CONVERGENCE THEOREM

DWIGHT B. GOODNER

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Let (X, B, m) be a measure space and let $f(x)$ be a real-valued or complex-valued measurable function on X. A non-negative measurable function $s(x)$ will be said to dominate $f(x)$ provided $|f(x)| \leqq s(x)$ for almost all x in X. The function $s(x)$ will be said to dominate the sequence $\{f_n(x)\}_{n\in N}$, $N = \{1, 2, \cdots\}$, provided it dominates each $f_n(x)$ in the sequence. Unless otherwise specified, each integral will be over X with respect to m.

Lebesgue's theorem on dominated convergence [1], a cornerstone of modern analysis, says (cf. [3, p. 29]) that if the sequence $\{f_n(x)\}_{n\in N}$ of real-valued measurable functions on X is dominated by an integrable function, then

$$\int \liminf_{n\to\infty} f_n(x) \leqq \liminf_{n\to\infty} \int f_n(x)$$

and

$$\int \limsup_{n\to\infty} f_n(x) \geqq \limsup_{n\to\infty} \int f_n(x).$$

Furthermore, if $\lim_{n\to\infty} f_n(x)$ exists for almost all x in X, then

$$\int \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int f_n(x).$$

The purpose of this note is to prove a converse to Lebesgue's theorem. Our proof is based on the following theorem of B. C. Rennie [2]: *If the sequence $\{f_n(x)\}_{n\in N}$ of real-valued or complex-valued integrable functions tends almost everywhere to a function $f(x)$ on X, and if $\lim_{n\to\infty} \int g(x)f_n(x) = \int g(x)f(x)$ for each bounded measurable function $g(x)$ on X, then each infinite subsequence of $\{f_n(x)\}_{n\in N}$ contains an infinite sub-subsequence which is dominated by an integrable function.*

We note that if the $f_n(x)$'s are real-valued only, then in Rennie's proof it suffices to consider only those real-valued measurable functions $g(x)$ such that $|g(x)| = 1$ for each x in X.

THEOREM. *Let $\{f_n(x)\}_{n\in N}$ be a sequence of extended-real-valued integrable functions on X such that*

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$$r(x) = \limsup_{n\to\infty} f_n(x) \text{ and } s(x) = \liminf_{n\to\infty} f_n(x)$$

are integrable, and let

$$R_n(x) = (r \cup f_n)(x) = \max\{r(x), f_n(x)\}$$

and

$$S_n(x) = (s \cap f_n)(x) = \min\{s(x), f_n(x)\}$$

for each x and n. If

$$\lim_{n\to\infty} \int g(x)R_n(x) = \int g(x)r(x) \text{ and } \lim_{n\to\infty} \int g(x)S_n(x) = \int g(x)s(x)$$

for each measurable function $g(x)$ with range $\{1, -1\}$, then each infinite subsequence of $\{f_n(x)\}_{n\in N}$ contains an infinite sub-subsequence which is dominated by an integrable function.

Proof. Let A be the set of all x in X such that $r(x)$ or $s(x)$ or at least one $f_n(x)$ is infinite. Since A is of measure zero, we may redefine the functions on A without changing the values of their integrals on X and, hence, without loss of generality in our proof. For each x in A and each n in N let $f_n(x) = 0$. Then $r(x)$ and $s(x)$ and all $f_n(x)$, $R_n(x)$ and $S_n(x)$ are finite everywhere.

In the sequel $N(1)$, $N(2)$ and $N(3)$ will denote infinite subsets of N such that $N(1) \supset N(2) \supset N(3)$. Let $N(1)$ be any infinite subset of N. For each $n \in N(1)$ the function $R_n(x)$ is the least upper bound of two integrable functions and is, therefore, integrable. Since $r(x) = \limsup_{n\to\infty} f_n(x) = \limsup_{n\to\infty} R_n(x)$ and since $R_n(x) \geqq r(x)$ for each x in X, we see that $\lim_{n\to\infty} R_n(x) = r(x)$. By hypothesis $\lim_{n\to\infty} \int g(x)R_n(x) = \int g(x)r(x)$ for each measurable function $g(x)$ with range $\{1, -1\}$. Thus the sequence $\{R_n(x)\}_{n\in N}$ satisfies the hypotheses of Rennie's theorem for real-valued functions, and it follows that the subsequence $\{R_n(x)\}_{n\in N(1)}$ contains an infinite sub-subsequence $\{R_n(x)\}_{n\in N(2)}$ dominated by an integrable function $R(x)$.

By an argument similar to the one above we can show that $\{S_n(x)\}_{n\in N(2)}$ contains a subsequence $\{S_n(x)\}_{n\in N(3)}$ dominated by an integrable function $S(x)$. It follows that $-S(x) \leqq f_n(x) \leqq R(x)$ for all x and all $n \in N(3)$, and, hence, that the sub-subsequence $\{f_n(x)\}_{n\in N(3)}$ is dominated by an integrable function. Since $\{f_n(x)\}_{n\in N(1)}$ was a generic subsequence of $\{f_n(x)\}_{n\in N}$, the proof is complete.

We conclude by giving an example to show that the conditions of the theorem do not require the sequence $\{f_n(x)\}_{n\in N}$ to be convergent or to be dominated by an integrable function. For each n in N let the function $f_n(x)$ be defined on the half-open interval $(0, 1]$ by

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$$f_n(x) = \begin{cases} 0 & \text{if } 0 < x < 1/(n+1), \\ n & \text{if } 1/(n+1) \leqq x < 1/n, \\ (-1)^n & \text{if } 1/n \leqq x \leqq 1. \end{cases}$$

Each $f_n(x)$ is integrable with respect to Lebesgue measure on $(0, 1]$. It follows that

$$r(x) = \limsup_{n\to\infty} f_n(x) = 1,$$

$$s(x) = \liminf_{n\to\infty} f_n(x) = -1,$$

$$R_n(x) = (r \cup f_n)(x) = \begin{cases} n & \text{if } 1/(n+1) \leqq x < 1/n, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$S_n(x) = (s \cap f_n)(x) = -1.$$

We note that $r(x)$ and $s(x)$ are integrable on $(0, 1]$, and that

$$\lim_{n\to\infty} \int_0^1 R_n(x) = \int_0^1 r(x).$$

Let $g(x)$ be a measurable function on $(0, 1]$ such that $|g(x)| \equiv 1$. It follows that

$$\lim_{n\to\infty} \int_0^1 g(x) R_n(x) = \int_0^1 g(x) r(x),$$

and it is trivial that

$$\lim_{n\to\infty} \int_0^1 g(x) S_n(x) = \int_0^1 g(x) s(x).$$

Let D_k be a function on $(0, 1]$ which dominates $f_1(x), f_2(x), \cdots, f_k(x)$. Since

$$\int_0^1 D_k(x) \geqq \textstyle\sum_{n=1}^k 1/(n+1),$$

it follows that if a function $D(x)$ dominates the sequence $\{f_n(x)\}_{n\in N}$ on $(0, 1]$, then $D(x)$ is not integrable.

References

[1] H. Lebesgue, 'Sur l'intégration des fonctions discontinues', *Ann. Ecole Norm.* 27 (1910), 361–450.

[2] B. C. Rennie, 'On dominated convergence', *Jour. Australian Math. Soc.* 2 (1961–1962), 133–136.

[3] S. Saks, *Theory of the integral*, 2nd ed., Monografje Matematyczne 7, Warsaw, 1937.

Florida State University