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On Algebraically Maximal Valued Fields and Defectless Extensions

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Abstract. Let v be a Henselian Krull valuation of a field K. In this paper, the authors give some necessary and sufficient conditions for a finite simple extension of (K, v) to be defectless. Various characterizations of algebraically maximal valued fields are also given which lead to a new proof of a result proved by Yu. L. Ershov.

1 Introduction

Throughout this paper, by a valuation v of a field K we mean a Krull valuation, *i.e.,* v is a mapping from K onto $G \cup \{\infty\}$, where G is a totally ordered additively written abelian group such that for all x, y in K the following properties are satisfied:

(i) $v(x) = \infty$ if and only if $x = 0$;
(ii) $v(xy) = v(x) + v(y)$;
(iii) $v(x + y) \geqslant \min\{v(x), v(y)\}$.

The pair (K, v) is called a valued field and G the value group of v. The subring $\mathcal{O}_v = \{x \in K \mid v(x) \geqslant 0\}$ of K with unique maximal ideal $\mathcal{M}_v = \{x \in K \mid v(x) > 0\}$ is called the valuation ring of v and $\mathcal{O}_v/\mathcal{M}_v$ its residue field. A valuation v' is said to be an extension or prolongation of v to an overfield K' of K if v' coincides with v on K, in which case (K', v') is said to be an extension of (K, v). For a valued field extension $(K', v')/(K, v)$, if $G \subseteq G'$ and $\mathcal{O}_v/\mathcal{M}_v \subseteq \mathcal{O}_{v'}/\mathcal{M}_{v'}$ denote respectively the value groups and the residue fields of v, v', then the index $[G' : G]$ and the degree of the field extension $\mathcal{O}_{v'}/\mathcal{M}_{v'}$ over $\mathcal{O}_v/\mathcal{M}_v$ are called respectively the index of ramification and the residual degree of v'/v. An extension (K', v') of (K, v) is said to be immediate if the value groups and the residue fields of v' and v coincide, *i.e.,* the index of ramification and the residual degree of v'/v are both one. A valued field (K, v) is said to be Henselian if v has a unique prolongation to the algebraic closure of K. Henselian valued fields form an important class of valued fields and have several equivalent characterizations [3, 7] and [4, Theorem 4.1.3]. In this paper, we characterize some special types of Henselian valued fields.

In what follows, v is a Henselian valuation of a field K and $\tilde{v}$ is the unique prolongation of v to the algebraic closure $\widetilde{K}$ of K. In this paper, we prove that a valued field (K, v) is algebraically maximal, *i.e.*, it has no proper immediate algebraic extension if and only if the set $\{\tilde{v}(\theta - a) \mid a \in K\}$ has a maximum element for every θ in

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$\widetilde{K} \setminus K$. It is shown that the above characterization quickly yields a result proved by Yu. L. Ershov which states that (K, v) is algebraically maximal if and only if the set $\{v(f(a)) \mid a \in K\}$ has a maximum element for every polynomial $f(x)$ belonging to $K[x]$ [5, Proposition 1.5.4, p. 54, p. 259]. Furthermore, it is also shown that for any fixed θ in $\widetilde{K}$ which is algebraic over K of degree $n > 1$, each of the sets $M_j(\theta)$, $1 \leqslant j \leqslant n-1$, defined by

$$M_j(\theta) = \{\tilde{v}(\theta - \beta) \mid \beta \in \widetilde{K},\ [K(\beta) : K] \leqslant j\} \tag{1.1}$$

has a maximum if and only if $K(\theta)$ is a defectless extension of (K, v). Recall that a finite extension (K', v') of a Henselian valued field (K, v) (or briefly K'/K) is said to be defectless if $[K' : K] = ef$ where e, f are the index of ramification and the residual degree of v'/v.

Theorem 1.1 *Let v be a Henselian valuation of a field K and $\tilde{v}$ be the unique prolongation of v to the algebraic closure $\widetilde{K}$ of K. The following statements are equivalent.*

(i) *(K, v) is algebraically maximal.*
(ii) *For every θ in $\widetilde{K} \setminus K$, the set $\{\tilde{v}(\theta - a) \mid a \in K\}$ has a maximum.*
(iii) *For each monic irreducible polynomial $f(x) \in K[x]$, there exists an element a_f belonging to K such that $v(f(a_f)) \geq v(f(a))$ for every a in K.*
(iv) *For each polynomial $f(x)$ belonging to $K[x]$, there exists a_f belonging to K such that $v(f(a_f)) \geq v(f(a))$ for every a in K.*

The equivalence of (i) and (ii) above will be deduced from a slightly more general result to be proved as Theorem 1.2.

Theorem 1.2 *Let (K, v), $(\widetilde{K}, \tilde{v})$ be as in the above theorem and let θ be an element of $\widetilde{K} \setminus K$. Then the set $\{\tilde{v}(\theta - a) \mid a \in K\}$ has no maximum if and only if there exists γ belonging to $\widetilde{K} \setminus K$ with $[K(\gamma) : K] \leqslant [K(\theta) : K]$ such that $K(\gamma)/K$ is an immediate extension and $\tilde{v}(\gamma - a) = \tilde{v}(\theta - a)$ for every a in K.*

As regards defectless extensions, we prove the following.

Theorem 1.3 *Let (K, v), $(\widetilde{K}, \tilde{v})$ be as in Theorem 1.1 and let θ be an element of $\widetilde{K} \setminus K$ with the minimal polynomial $g(x)$ over K of degree n. The following statements are equivalent.*

(i) *$K(\theta)/K$ is a defectless extension.*
(ii) *The set $M_j(\theta) = \{\tilde{v}(\theta - \beta) \mid \beta \in \widetilde{K},\ [K(\beta) : K] \leq j\}$ has a maximum element for each number j not exceeding $n - 1$.*
(iii) *For $1 \leq j \leq n - 1$, the set $N_j(g) = \{\tilde{v}(g(\beta)) \mid \beta \in \widetilde{K},\ [K(\beta) : K] \leq j\}$ has a maximum element.*

Our proof in fact specifies the elements β_j with $[K(\beta_j) : K] \leqslant j$ such that $\max N_j(g) = \tilde{v}(g(\beta_j))$, $1 \leqslant j \leqslant n-1$ (see Remark 3.6).

It may be pointed out that some other characterizations of finite separable defectless extensions are given in [2, 8].

2 Proof of Theorem 1.1 and Theorem 1.2

In what follows, (K, v) and $(\widetilde{K}, \tilde{v})$ are as in Theorem 1.1. By the degree of an element α in $\widetilde{K}$, we shall mean the degree of the extension $K(\alpha)/K$ and shall denote it by $\deg \alpha$. For an element ξ in the valuation ring of $\tilde{v}$, $\bar{\xi}$ will stand for its $\tilde{v}$-residue, *i.e.*, the image of ξ under the canonical homomorphism from the valuation ring of $\tilde{v}$ onto its residue field. When there is no chance of confusion, we shall write $\tilde{v}(\alpha)$ as $v(\alpha)$ for α belonging to $\widetilde{K}$.

Proposition 2.1 *Suppose that the set* $M_1 = \{\tilde{v}(\alpha - a) \mid a \in K\}$ *does not have a maximum element for some* α *belonging to* $\widetilde{K} \setminus K$. *Then either* $K(\alpha)$ *is an immediate extension of* (K, v) *or there exists* β *belonging to* $\widetilde{K} \setminus K$ *with* $\deg \beta < \deg \alpha$ *such that* $\tilde{v}(\alpha - a) = \tilde{v}(\beta - a)$ *for each* a *in* K.

Proof Let M denote the set $\{\tilde{v}(\alpha - \beta) \mid \beta \in \widetilde{K}, \deg \beta < \deg \alpha\}$ containing M_1 and $\sup M$ its supremum. The proof is split into two cases.

Case 1: $\sup M_1 < \sup M$. Then there exists β belonging to $\widetilde{K}$ with $\deg \beta < \deg \alpha$ such that $\tilde{v}(\alpha - \beta) \geq \sup M_1$. Since M_1 does not have a maximum element, the above inequality shows that $\tilde{v}(\alpha - \beta) > \tilde{v}(\alpha - a)$ for every a in K. Therefore by the strong triangle law, for any element a of K, we have

$$\tilde{v}(\beta - a) = \min\{\tilde{v}(\beta - \alpha), \tilde{v}(\alpha - a)\} = \tilde{v}(\alpha - a).$$

Case 2: $\sup M_1 = \sup M$. Then M_1 is a cofinal subset of M. In this case we show that $K(\alpha)/K$ is an immediate extension. For this it is enough to prove that for any polynomial $h(x)$ belonging to $K[x]$ with $\deg h(x) < \deg \alpha$, there exists $c \in K$ such that

$$\tilde{v}\Big(\frac{h(\alpha)}{h(c)} - 1\Big) > 0. \tag{2.1}$$

Write $h(x) = a \prod_{j=1}^{t} (x - \gamma_j)$. Since $\deg \gamma_j \leqslant \deg h(x) < \deg\ \alpha$ and $\tilde{v}(\alpha - \gamma_j) \in M$, there exists an element $\tilde{v}(\alpha - a_s)$ of M_1 such that $\tilde{v}(\alpha - \gamma_j) < \tilde{v}(\alpha - a_s)$ for $1 \leq j \leq t$; consequently by the strong triangle law, we have

$$\tilde{v}(a_s - \gamma_j) = \min\{\tilde{v}(a_s - \alpha), \tilde{v}(\alpha - \gamma_j)\} = \tilde{v}(\alpha - \gamma_j) < \tilde{v}(\alpha - a_s).$$

On writing $\frac{h(\alpha)}{h(a_s)} = \prod_{j=1}^{t} \big(\frac{\alpha - \gamma_j}{a_s - \gamma_j}\big)$ as $\prod_{j=1}^{t} \big(1 + \frac{\alpha - a_s}{a_s - \gamma_j}\big)$ and using the above inequality, we see that $\tilde{v}\big(\frac{h(\alpha)}{h(a_s)} - 1\big) > 0$ which proves (2.1) with $c = a_s$. ■

Proof of Theorem 1.2 Suppose first that $\{\tilde{v}(\theta - a) \mid a \in K\}$ does not have a maximum element. Then by Proposition 2.1, either $K(\theta)/K$ is an immediate extension or there exists η belonging to $\widetilde{K} \setminus K$ with $\deg \eta < \deg \theta$ such that $v(\theta - a) = v(\eta - a)$ for every a in K. If $K(\theta)/K$ is an immediate extension, then we take $\gamma = \theta$, otherwise, applying Proposition 2.1 to η we see that there exists β belonging to $\widetilde{K} \setminus K$ with $\deg \beta < \deg \eta$ such that either $K(\beta)/K$ is an immediate extension or $v(\beta - a) = v(\eta - a) = v(\theta - a)$ for every a in K. The above process must terminate after a finite number of steps, giving us an element γ belonging to $\widetilde{K} \setminus K$ with $\deg\gamma \leqslant \deg\theta$ such that $K(\gamma)/K$ is an immediate extension and $v(\gamma - a) = v(\theta - a)$ for every a belonging to K.

Conversely, suppose that there exists γ belonging to $\widetilde{K} \setminus K$ such that $K(\gamma)/K$ is an immediate extension and $\tilde{v}(\gamma - a) = \tilde{v}(\theta - a)$ for every a in K. We now show that the set $S = \{\tilde{v}(\gamma - a) \mid a \in K\}$ has no maximum element. Suppose to the contrary that $\tilde{v}(\gamma - c)$, $c \in K$ is the maximum element of S. Since $K(\gamma)/K$ is an immediate extension, there exists b in K such that $\tilde{v}(\gamma - c) = v(b)$; also we can find $d \in K$ such that the $\tilde{v}$-residue of $\frac{\gamma - c}{b}$ equals the $\tilde{v}$-residue of d, *i.e.*, $\tilde{v}(\frac{\gamma-c}{b} - d) > 0$, which implies that $\tilde{v}(\gamma - c - bd) > v(b) = \tilde{v}(\gamma - c)$. This contradicts the choice of $\tilde{v}(\gamma - c)$. ■

Proof of Theorem 1.1 The equivalence of (i) and (ii) follows immediately from Theorem 1.2.

(ii) $\Rightarrow$ (iii). Let $f(x) = \prod_{i=1}^{n}(x - \alpha^{(i)})$ be any monic irreducible polynomial over K having a root α in $\widetilde{K}$. There exists c belonging to K such that $v(\alpha - c) = \max\{v(\alpha - a) \mid a \in K\}$. Since (K, v) is Henselian for any a in K, we have

$$v(f(a)) = nv(\alpha - a) \leq nv(\alpha - c) = v(f(c)).$$

(iii) $\Rightarrow$ (iv). Let $f(x)$ be any polynomial belonging to $K[x]$ with the factorization $bf_1(x)^{m_1} f_2(x)^{m_2} \cdots f_r(x)^{m_r}$ into powers of distinct monic irreducible polynomials over K. Let n_i denote the degree of $f_i(x)$ and θ_i be a root of $f_i(x)$. By (iii), there exist c_i belonging to K for $1 \leq i \leq r$ such that $v(f_i(c_i)) = \max\{v(f_i(a)) \mid a \in K\}$, *i.e.*, $v(\theta_i - c_i) = \max\{v(\theta_i - a) \mid a \in K\}$. It will be proved that for each d belonging to K, we have

$$v(f(d)) \leq \max_{1 \leq i \leq r}\{v(f(c_i))\}. \tag{2.2}$$

Fix any d in K. Choose an index $j \geq 1$ such that

$$v(c_j - d) = \max_{1 \leq i \leq r}\{v(c_i - d)\}. \tag{2.3}$$

We are going to prove that $v(f(d)) \leqslant v(f(c_j))$, which is the same as showing

$$\sum_{i=1}^{r} m_i n_i v(d - \theta_i) \leqslant \sum_{i=1}^{r} m_i n_i v(c_j - \theta_i).$$

This will follow as soon as it is shown that

$$v(d - \theta_i) \leqslant v(c_j - \theta_i), \quad 1 \leqslant i \leqslant r. \tag{2.4}$$

Note that $v(c_i - d) \geqslant \min\{v(c_i - \theta_i), v(\theta_i - d)\} = v(\theta_i - d)$. In view of (2.3) and the above inequality, we have

$$v(c_j - d) \geqslant v(c_i - d) \geqslant v(\theta_i - d), \quad 1 \leqslant i \leqslant r,$$

which gives $v(c_j-\theta_i) \geqslant \min\{v(c_j-d), v(d-\theta_i)\} = v(d-\theta_i)$. Thus (2.4) and hence (2.2) is proved.

(iv) $\Rightarrow$ (ii). Let θ be an element of $\widetilde{K} \setminus K$ and $f(x)$ be its minimal polynomial over K of degree n. By hypothesis, there exists an element a_f belonging to K such that $v(f(a_f)) \geqslant v(f(a))$ for each a in K. Since (K, v) is Henselian, the above inequality is equivalent to saying that $v(\theta - a_f) = \max\{v(\theta - a) \mid a \in K\}$. ■

3 Proof of Theorem 1.3

We retain the notations introduced in the opening lines of the second section. For a subfield L of $\widetilde{K}$, let v_L denote the valuation of L obtained by restricting $\tilde{v}$. As usual, def(L/K) will stand for the defect of a finite extension L of (K, v) defined by

$$\operatorname{def}(L/K) = [L:K]/ef,$$

where e, f are the index of ramification and the residual degree of v_L/v.

As in [6], a pair (θ, α) of elements of $\widetilde{K}$ is called a distinguished pair (more precisely, a (K, v)-distinguished pair) if the following three conditions are satisfied:

- $\deg\theta > \deg\alpha$;
- $\tilde{v}(\theta - \beta) \leqslant \tilde{v}(\theta - \alpha)$ for every β in $\widetilde{K}$ with $\deg\beta < \deg\theta$;
- whenever $\gamma \in \widetilde{K}$ with $\deg\gamma < \deg\alpha$, then $\tilde{v}(\theta - \gamma) < \tilde{v}(\theta - \alpha)$.

Remark 3.1 If (θ, α) is a distinguished pair and $\deg\theta = n$, then the set $M_{n-1}(\theta)$ defined by (1.1) has a maximum element, *viz.* $\tilde{v}(\theta - \alpha)$. Conversely if α is an element of smallest degree over K for which $\tilde{v}(\theta-\alpha)$ is the maximum of $M_{n-1}(\theta)$, then clearly (θ, α) is a distinguished pair.

The following already-known result will be used in the sequel; its proof is omitted [1, §3, p. 223], [2, Theorem 1.1(iii)].

Theorem A *Let (θ, α) be a (K, v)-distinguished pair. Then*

$$\operatorname{def}(K(\theta)/K) = \operatorname{def}(K(\alpha)/K).$$

We now prove the following.

Lemma 3.2 *Let (θ, α) be a (K, v)-distinguished pair with* $\deg\alpha = n_1$. *Then $M_j(\theta) = M_j(\alpha)$ for $1 \leqslant j \leqslant n_1 - 1$.*

Proof Let γ be any element of $\widetilde{K}$ with $\deg\gamma \leqslant j \leqslant n_1 - 1$. Then by the definition of a distinguished pair $\tilde{v}(\theta - \gamma) < \tilde{v}(\theta - \alpha)$; consequently, by the strong triangle law $\tilde{v}(\alpha - \gamma) = \min\{\tilde{v}(\alpha - \theta), \tilde{v}(\theta - \gamma)\} = \tilde{v}(\theta - \gamma)$, which proves the lemma. ■

The result stated below is proved implicitly in [1, §4] and explicitly in [2, Theorem 2.4]. Its proof is omitted.

Lemma 3.3 *Suppose that $K(\theta)/K$ is a defectless extension of degree $n > 1$. Then the set $M_{n-1}(\theta)$ has a maximum element.*

Lemma 3.4 *Let (θ,α) be a (K,v)-distinguished pair. Let $f(x)$, $g(x)$ be the minimal polynomials over K of α, θ respectively of degree n_1 and n. Then for any γ belonging to $\widetilde{K}$ with* $\deg\gamma \leqslant n_1 - 1$, *one has* $\tilde{v}(g(\gamma)) = \frac{n}{n_1}\tilde{v}(f(\gamma))$.

Proof Let $h(x)$ belonging to $K[x]$ be the minimal polynomial of γ of degree m. Write $g(x) = \prod_{j=1}^{n}(x-\theta^{(j)})$, $h(x) = \prod_{i=1}^{m}(x-\gamma^{(i)})$. Since $g(x)$, $h(x)$ are irreducible over the Henselian valued field (K,v), it follows that

$$(3.1)\qquad \tilde{v}(g(\gamma^{(i)})) = \tilde{v}(g(\gamma)),\quad \tilde{v}(h(\theta^{(j)})) = \tilde{v}(h(\theta)),\quad 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n.$$

Keeping in view (3.1) and the equality $\prod_{i=1}^{m} g(\gamma^{(i)}) = \pm\prod_{j=1}^{n} h(\theta^{(j)})$, it follows that $m\tilde{v}(g(\gamma)) = n\tilde{v}(h(\theta))$, *i.e.*,

$$(3.2)\qquad \tilde{v}(g(\gamma)) = \frac{n}{m}\tilde{v}(h(\theta)).$$

Writing $f(x) = \prod_{k=1}^{n_1}(x-\alpha^{(k)})$ and arguing as above, it can be seen that

$$(3.3)\qquad \tilde{v}(f(\gamma)) = \frac{n_1}{m}\tilde{v}(h(\alpha)).$$

Since $\deg\gamma \leqslant n_1 - 1$, it follows from the definition of a distinguished pair that $\tilde{v}(\theta-\gamma^{(i)}) < \tilde{v}(\theta-\alpha)$; consequently by the strong triangle law

$$\tilde{v}(\alpha-\gamma^{(i)}) = \min\{\tilde{v}(\alpha-\theta), \tilde{v}(\theta-\gamma^{(i)})\} = \tilde{v}(\theta-\gamma^{(i)}).$$

On summing over i, we see that $\tilde{v}(h(\alpha)) = \tilde{v}(h(\theta))$, which combined with (3.2) and (3.3) proves the lemma. ∎

The following lemma, needed for the proof of Theorem 1.3, is also of independent interest as pointed out in Remark 3.6.

Lemma 3.5 *Let (θ,α) be a (K,v)-distinguished pair and let $g(x)$ be the minimal polynomial of θ over K of degree n. For any β belonging to $\widetilde{K}$ with* $\deg\beta \leqslant n-1$, *one has* $\tilde{v}(g(\beta)) \leqslant \tilde{v}(g(\alpha))$.

Proof Let β be as above. Since $\tilde{v}(g(\beta)) = \tilde{v}(g(\beta'))$ for every K-conjugate β' of β, it may be assumed without loss of generality that

$$(3.4)\qquad \tilde{v}(\theta-\beta) = \max\{\tilde{v}(\theta-\beta') \mid \beta' \text{ runs over all } K\text{-conjugates of } \beta\}.$$

Write $g(x) = \prod_{i=1}^{n}(x-\theta^{(i)})$. It will be shown that for $1 \leqslant i \leqslant n$,

$$(3.5)\qquad \tilde{v}(\beta-\theta^{(i)}) \leqslant \tilde{v}(\alpha-\theta^{(i)}).$$

Since (θ,α) is a distinguished pair and $\deg\beta \leqslant n-1$, we have

$$(3.6)\qquad \tilde{v}(\alpha-\beta) \geqslant \min\{\tilde{v}(\alpha-\theta), \tilde{v}(\theta-\beta)\} = \tilde{v}(\theta-\beta).$$

Fix any i, $1 \leqslant i \leqslant n$. Since (K, v) is Henselian, $\tilde{v}(\theta^{(i)} - \beta) = \tilde{v}(\theta - \beta')$ for some K-conjugate β' of β. Therefore, using (3.6) and (3.4), we obtain

$$\tilde{v}(\alpha - \beta) \geqslant \tilde{v}(\theta - \beta) \geqslant \tilde{v}(\theta - \beta') = \tilde{v}(\theta^{(i)} - \beta). \tag{3.7}$$

It follows from (3.7) and the triangle law that

$$\tilde{v}(\alpha - \theta^{(i)}) \geqslant \min\{\tilde{v}(\alpha - \beta), \tilde{v}(\beta - \theta^{(i)})\} = \tilde{v}(\beta - \theta^{(i)}),$$

which proves (3.5) and hence the lemma. ■

Proof of Theorem 1.3 We prove the equivalence of (i) and (ii) and then of (ii) and (iii) by induction on n.

(i) $\Rightarrow$ (ii). If $K(\theta)/K$ is a defectless extension of degree 2, then the set $M_1(\theta) = \{v(\theta - a) \mid a \in K\}$ has a maximum element in view of Proposition 2.1. Assume that the result holds for all elements of degree not exceeding $n - 1$ and that $K(\theta)/K$ is a defectless extension of degree $n \geqslant 3$. Now by Lemma 3.3 and Remark 3.1, there exists an element θ_1 belonging to $\widetilde{K}$ such that (θ, θ_1) is a distinguished pair. Let n_1 denote the degree of θ_1. Applying Theorem A, we see that $K(\theta_1)/K$ is a defectless extension. By Lemma 3.2, $M_j(\theta) = M_j(\theta_1)$ for $1 \leqslant j \leqslant n_1 - 1$. Therefore by induction hypothesis, $M_j(\theta_1)$ and hence $M_j(\theta)$ has a maximum element for $1 \leqslant j \leqslant n_1 - 1$. Also it is clear from the definition of a distinguished pair that $v(\theta - \theta_1)$ is the maximum element of $M_j(\theta)$ when $n_1 \leqslant j \leqslant n - 1$, which completes the proof that (i) implies (ii).

(ii) $\Rightarrow$ (i). When $n = 2$, then using the hypothesis that the set $\{v(\theta - a) \mid a \in K\}$ has a maximum element and arguing as in the last lines of the proof of Theorem 1.2, we conclude that $K(\theta)/K$ is not an immediate extension and hence it is a defectless extension of degree 2. Suppose that θ has degree n and the result is true for all elements of degree $\leqslant n - 1$. Since $M_{n-1}(\theta)$ has a maximum element, there exists an element θ_1 of degree n_1 (say) such that (θ, θ_1) is a distinguished pair. By Lemma 3.2, $M_j(\theta) = M_j(\theta_1)$ for $1 \leqslant j \leqslant n_1 - 1$ and hence $M_j(\theta_1)$ has a maximum element. Therefore by induction hypothesis, $K(\theta_1)/K$ is a defectless extension, and hence so is $K(\theta)/K$, in view of Theorem A.

(ii) $\Rightarrow$ (iii). Let c be an element of K such that $v(\theta - c) = \max\{v(\theta - a) \mid a \in K\}$. Then in the case $n = 2$, the set $N_1(g) = \{v(g(a)) \mid a \in K\}$ has $2v(\theta - c)$ as maximum. Suppose that θ has degree n and the result is true for all elements of smaller degree. In view of the hypothesis, there exists an element θ_1 belonging to $\widetilde{K}$ such that (θ, θ_1) is a distinguished pair with $\deg \theta_1 = n_1$ (say). Then by Lemma 3.2, $M_j(\theta_1) = M_j(\theta)$ for $1 \leqslant j \leqslant n_1 - 1$. Therefore, by induction hypothesis, if $f(x)$ is the minimal polynomial of θ_1 over K, then the set $N_j(f) = \{v(f(\beta)) \mid \beta \in \widetilde{K}, \deg \beta \leqslant j\}$ will have a maximum element for $1 \leqslant j \leqslant n_1 - 1$. Recall that by virtue of Lemma 3.4, for β belonging to $\widetilde{K}$ with $\deg \beta \leqslant n_1 - 1$, $v(g(\beta)) = \frac{n}{n_1} v(f(\beta))$. So it follows that the sets $N_j(g)$ will also have a maximum element for $1 \leqslant j \leqslant n_1 - 1$. Furthermore, by Lemma 3.5, $v(g(\theta_1))$ is the maximum of $N_{n-1}(g)$, and hence it is also the maximum of $N_j(g)$ when $n_1 \leqslant j \leqslant n - 1$, which completes the proof of the desired assertion.

(iii) $\Rightarrow$ (ii). For $n = 2$, the set $N_1(g) = \{2v(\theta - a) \mid a \in K\}$ has a maximum by (iii), and hence $M_1(\theta)$ has a maximum element. Suppose that $\deg \theta = n$ and the result holds for elements of lower degree. Let α be an element of degree not exceeding $n-1$ such that $v(g(\alpha))$ is the maximum of the set $N_{n-1}(g)$. Replacing α by its K-conjugate, we can assume that

$$v(\theta-\alpha)=\max\{v(\theta-\alpha')\mid \alpha' \text{ runs over all } K\text{-conjugates of } \alpha\}. \tag{3.8}$$

We claim that $M_{n-1}(\theta)$ has $v(\theta-\alpha)$ as maximum element. Suppose to the contrary that there exists an element γ belonging to $\widetilde{K}$ of degree $\leqslant n-1$ such that

$$v(\theta-\alpha)<v(\theta-\gamma). \tag{3.9}$$

We shall obtain the desired contradiction by showing that

$$v(g(\gamma))>v(g(\alpha)). \tag{3.10}$$

To verify (3.10), note that in view of (3.9) and the strong triangle law, we have

$$v(\gamma-\alpha)=\min\{v(\gamma-\theta),v(\theta-\alpha)\}=v(\theta-\alpha). \tag{3.11}$$

Let $\theta^{(i)}$ be any K-conjugate of θ. Keeping in mind (3.11), (3.8), and the fact that $v(\alpha-\theta^{(i)})=v(\alpha'-\theta)\leqslant v(\alpha-\theta)$, we have

$$v(\gamma-\theta^{(i)})\geqslant\min\{v(\gamma-\alpha),v(\alpha-\theta^{(i)})\}=v(\alpha-\theta^{(i)}).$$

Summing over all K-conjugates $\theta^{(i)}$ of θ and using (3.9), we obtain (3.10). Hence the claim is proved.

So there exists θ_1 in $\widetilde{K}$ such that (θ,θ_1) is a distinguished pair. Let $f(x)$ be the minimal polynomial of θ_1 over K of degree n_1. Then by virtue of Lemma 3.4, for any β belonging to $\widetilde{K}$ with $\deg\beta\leqslant n_1-1$, we have

$$v(g(\beta))=\frac{n}{n_1}v(f(\beta)). \tag{3.12}$$

By hypothesis, the sets $N_j(g)$ have a maximum element for $1\leqslant j\leqslant n-1$. It now follows from (3.12) that $N_j(f)=\{v(f(\beta))\mid\beta\in\widetilde{K},\deg\beta\leqslant j\}$ has a maximum element for $1\leqslant j\leqslant n_1-1$. Therefore by induction hypothesis, $M_j(\theta_1)$ and hence $M_j(\theta)$ will have a maximum element for $1\leqslant j\leqslant n_1-1$. As $v(\theta-\theta_1)$ is the maximum element of $M_j(\theta)$ for $n_1\leqslant j\leqslant n-1$, we see that (iii) $\Rightarrow$ (ii). ■

Remark 3.6 Suppose that $K(\theta)/K$ is a defectless extension. In view of Lemma 3.3, there exists θ_1 such that (θ,θ_1) is a distinguished pair. By successive applications of Lemma 3.2, there exist distinguished pairs $(\theta,\theta_1),(\theta_1,\theta_2),\dots,(\theta_{r-1},\theta_r)$ with θ_r in K and $\deg\theta_i=n_i$ (say). Using induction on $n_0=\deg\theta$ and applying Lemmas 3.5 and 3.4, it can be quickly shown (as in the proof of (ii) $\Rightarrow$ (iii) above) that $\max N_j(g)=v(g(\theta_i))$ when $n_i\leqslant j\leqslant n_{i-1}-1, 1\leqslant i\leqslant r$.

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