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STATISTICS FOR POISSON MODELS OF OVERLAPPING SPHERES

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Abstract

In this paper we consider the stationary Poisson Boolean model with spherical grains and propose a family of nonparametric estimators for the radius distribution. These estimators are based on observed distances and radii, weighted in an appropriate way. They are ratio unbiased and asymptotically consistent for a growing observation window. We show that the asymptotic variance exists and is given by a fairly explicit integral expression. Asymptotic normality is established under a suitable integrability assumption on the weight function. We also provide a short discussion of related estimators as well as a simulation study.

Keywords: Stochastic geometry; spatial statistic; contact distribution function; Boolean model; spherical typical grain; point process; nonparametric estimation; radius distribution; asymptotic normality

2010 Mathematics Subject Classification: Primary 60D05; 60G57; 52A21
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1. Introduction

We consider a stationary random closed set Z in $\mathbb{R}^d$ ($d \geq 2$) which is given as a union of random balls of the form

$$Z := \bigcup_{n\geq 1} B(\xi_n, R_n), \tag{1.1}$$

where $B(x, r)$ is the closed Euclidean ball with radius $r \geq 0$ centered at $x \in \mathbb{R}^d$, $\Phi := \{\xi_n : n \geq 1\}$ is a stationary Poisson point process on $\mathbb{R}^d$, and the sequence $(R_n)_{n\geq 1}$ is independent of Φ and formed by independent nonnegative random variables with common distribution $\mathbb{G}$. Let R be a generic random variable with distribution $\mathbb{G}$. We will always assume that it has a finite $2d$th moment, that is,

$$\mathbb{E}R^{2d} < \infty.$$

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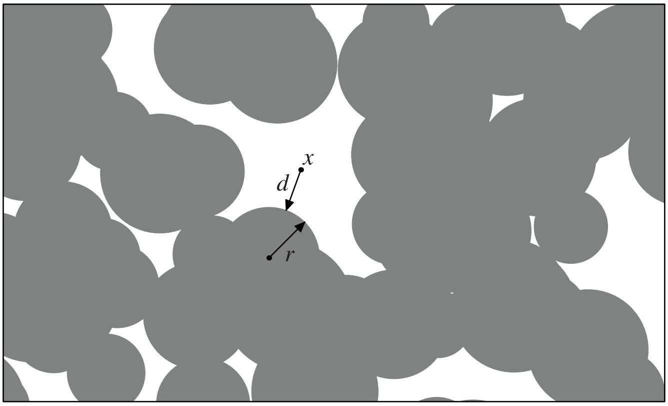

FIGURE 1: A simulated realization of a planar stationary Boolean model Z with spherical grains observed in a rectangular observation window. The symbol d denotes $d_{B^2}(x, Z)$ and r stands for $r_{B^2}(x, Z)$.

Definition (1.1) provides an important model in stochastic geometry with numerous applications in, e.g. physics and materials science. The set Z is called a *stationary Boolean model with spherical grains*. A simulated realization for $d = 2$ is shown in Figure 1.

It is a fundamental statistical problem to retrieve information on $\mathbb{G}$ based on an observation of Z in a bounded window W. Our aim in this paper is to propose and study a family of nonparametric estimators of $\mathbb{G}$. The nonparametric estimation of the radius distribution $\mathbb{G}$ has been studied before; see [4]–[6, Chapter 5.6], [19], [21], [25], or [26]. In [21] a kernel estimator is obtained by the method of tangent points. The asymptotic properties of this estimator are studied in [12]. For earlier work on statistics for the Boolean model, we refer the reader to [3, Chapter 3.4], [20], and the references therein.

In the following we assume that all random elements are defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a Borel set $A \subset \mathbb{R}^d$, we write $\Phi(A) := \text{card}\{n \geq 1 : \xi_n \in A\}$, and assume that Φ has a positive and finite intensity

$$\gamma := \mathbb{E}\Phi([0, 1]^d).$$

Throughout the paper, let B be a compact convex set which contains the origin o and a nondegenerate segment. We call B the *structuring element* or *gauge body*, but we point out that B need not be centrally symmetric or full dimensional. The B-distance from a point $x \in \mathbb{R}^d$ to a set $A \subset \mathbb{R}^d$ is

$$d_B(x, A) := \inf\{r \geq 0 : (x + rB) \cap A \neq \varnothing\} \in [0, \infty].$$

Clearly, if $\text{o} \in \text{int}\, B$, and A is nonempty and closed, then the infimum is a minimum. The most common structuring element is the Euclidean unit ball $B(\text{o}, 1)$, for which we also write B^d and which is based on the Euclidean norm denoted by $\|\cdot\|$. For given $x \notin Z$, almost surely, $d_B(x, Z) < \infty$ whenever R satisfies $\mathbb{P}(R > 0) > 0$. We always assume that this condition is fulfilled. Then, almost surely, there is a unique $n \in \mathbb{N}$ (that is, a ball $B(\xi_n, R_n)$) such that $(x + d_B(x, Z)B) \cap B(\xi_n, R_n) \neq \varnothing$ (see [14, Lemma 3.1] or [24, Lemma 9.5.1]). In this case, we define $r_B(x, Z)$ as R_n. In Figure 1 we illustrate the definitions of $d_B(x, Z)$ and $r_B(x, Z)$ for $x \notin Z$ and $B = B^2$.

For $s, r \geq 0$, we write $B_{s,r} := sB \oplus rB^d = \{sx + ry : x \in B,\ y \in B^d\}$ for the Minkowski sum of sB and rB^d. Let $|A|_d$ denote the d-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^d$, let

$\kappa_k := |B^k|_k = \pi^{k/2}/\Gamma(1+k/2)$ denote the volume of the k-dimensional unit ball, and write $V_j(B)$ for the jth intrinsic volume of B (see [24, Chapter 14.3]). Then, for $t \in \mathbb{R}^+ := [0, \infty)$, the *empty space function* F_B of Z is given by

$$\begin{aligned} F_B(t) &:= \mathbb{P}(d_B(\mathrm{o}, Z) \le t) \\ &= \mathbb{P}(Z \cap tB \neq \varnothing) \\ &= 1 - \exp\{-\gamma \mathbb{E}|B_{t,R}|_d\} \\ &= 1 - \exp\biggl\{-\gamma \sum_{j=0}^{d} \kappa_{d-j} V_j(B) t^j \mathbb{E} R^{d-j}\biggr\}. \end{aligned} \tag{1.2}$$

The empty space function is a useful summary statistic of random sets (see [3] and [8]). In the case of a strictly convex gauge body B a detailed study of F_B for (nonstationary) germ–grain models can be found in [13]. We denote the complementary empty space function by $\bar{F}_B(t) := 1 - F_B(t)$. As a consequence of [14, Theorem 3.2], for all measurable functions $\tilde{g} \colon [0, \infty] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\tilde{g}(0, r) = \tilde{g}(\infty, r) = 0$, $r \in \mathbb{R}^+$, and all $x \in \mathbb{R}^d$, we obtain

$$\mathbb{E}\tilde{g}(d_B(x, Z), r_B(x, Z)) = \gamma \int_0^\infty \int_0^\infty \tilde{g}(t, r) h_B(t, r) \bar{F}_B(t) \,\mathrm{d}t \,\mathbb{G}(\mathrm{d}r) \tag{1.3}$$

with

$$h_B(t, r) := \sum_{j=0}^{d-1} (j+1) \kappa_{d-1-j} V_{j+1}(B) r^{d-1-j} t^j$$

for $t, r \in [0, \infty)$; see also [24, Theorem 9.5.2]. Note that on the left-hand side of (1.3) the restriction to $\{0 < d_B(x, Z) < \infty\}$ is expressed by the condition $\tilde{g}(0, r) = \tilde{g}(\infty, r) = 0$.

For Borel sets $C \subset \mathbb{R}^+$ and $A \subset \mathbb{R}^d$, and a measurable function $f \colon [0, \infty] \to \mathbb{R}^+$ with $f(0) = f(\infty) = 0$, we define a random measure η_A by

$$\eta_A(C) := \int_A \mathbf{1}\{r_B(x, Z) \in C\} f(d_B(x, Z)) h_B(d_B(x, Z), r_B(x, Z))^{-1} \,\mathrm{d}x, \tag{1.4}$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function. Here we set $0/0 := 0$. Thus, in particular, the integration effectively extends over the complement $Z^{\mathrm{c}} := \{x \in \mathbb{R}^d : d_B(x, Z) > 0\}$ of Z. Throughout the paper, we will assume that

$$0 < \beta := \int_0^\infty f(t) \bar{F}_B(t) \,\mathrm{d}t < \infty. \tag{1.5}$$

In view of (1.2) this is a rather weak assumption on f. Moreover, we assume that the origin is an interior point of B if $\mathbb{P}(R = 0) > 0$. This assumption ensures that $h_B(t, r) > 0$ for $t \in (0, \infty)$ and $\mathbb{G}$-almost all $r \in \mathbb{R}^+$. By Fubini's theorem and (1.3), we obtain

$$\mathbb{E}\eta_A(C) = \gamma \, \beta \, |A|_d \, \mathbb{G}(C). \tag{1.6}$$

Consider a compact convex observation window $W \subset \mathbb{R}^d$ with $|W|_d > 0$. We propose a nonparametric estimator $\widehat{\mathbb{G}}$ for $\mathbb{G}$ based on the information contained in the data

$$\{(d_B(x, Z), r_B(x, Z)) \colon x \in W \setminus Z\}.$$

Note that this data may also require information from outside W. The estimator is given by

$$\widehat{\mathbb{G}}(C) := \frac{\eta_W(C)}{\eta_W(\mathbb{R}^+)}, \tag{1.7}$$

where $C \subset \mathbb{R}^+$ is a Borel set. If the denominator in (1.7) is 0 then the numerator is 0 as well, and we use the convention $0/0 := 0$. From (1.6) we see that $\mathbb{E}\eta_W(C) = \gamma\,\beta\,|W|_d\,\mathbb{G}(C)$ and $\mathbb{E}\eta_W(\mathbb{R}^+) = \gamma\,\beta\,|W|_d$. This means that $\widehat{\mathbb{G}}$ is a *ratio-unbiased* estimator of $\mathbb{G}$.

The paper is organized as follows. In Section 2 we study second-order properties of (1.4). Our Theorem 2.1 shows that the asymptotic variance exists, is positive, and is given by a fairly explicit integral expression. The proof is provided in Appendix A. Consequently, estimator (1.7) is asymptotically weakly consistent as the compact convex observation window W is expanding. Strong consistency follows from the spatial ergodic theorem. Section 3 contains the proof of asymptotic normality (see Theorem 3.2) under an integrability assumption on the function f. The proof of Theorem 3.2 is essentially based on the asymptotic normality of (1.4) established in Appendix B. In Section 4 we consider the estimator $\widehat{\mathbb{G}}$ in the plane and for the spherical case ($B = B(\mathrm{o}, 1)$) as well as for the linear case (B a segment). We also discuss some related estimators. A simulation study is performed to compare the behavior of different (discrete) versions of these estimators of the radius distribution $\mathbb{G}$.

2. Second-order properties

For a Borel set $A \subset \mathbb{R}^d$, we define the *restricted Boolean model* as

$$Z(A) := \bigcup_{\{n\,:\,\xi_n \in A\}} B(\xi_n, R_n).$$

Clearly, $Z(A)$ is not stationary unless $A = \mathbb{R}^d$. Furthermore, for $t \in \mathbb{R}^+$, the complementary empty space function of $Z(A)$ with respect to $x \in \mathbb{R}^d$ is defined by

$$\begin{aligned}
\bar{F}_B^A(x;t) &:= \mathbb{P}(d_B(x, Z(A)) > t) \\
&= \mathbb{P}((x + tB) \cap Z(A) = \varnothing) \\
&= \mathbb{E}\prod_{n\geq 1}(1 - \mathbf{1}\{(x + tB) \cap B(\xi_n, R_n) \neq \varnothing\}\mathbf{1}\{\xi_n \in A\}) \\
&= \exp\left\{-\gamma\mathbb{E}\int_{\mathbb{R}^d} \mathbf{1}\{(x + tB) \cap B(y, R) \neq \varnothing\}\mathbf{1}\{y \in A\}\,\mathrm{d}y\right\} \\
&= \exp\{-\gamma\mathbb{E}|(x + B_{t,R}) \cap A|_d\}.
\end{aligned}$$

In particular, we have $\bar{F}_B^{\mathbb{R}^d}(x;t) = \bar{F}_B(t)$.

For Borel sets $A_1, A_2 \subset \mathbb{R}^d$ and $t_1, t_2 \in \mathbb{R}^+$, it will be convenient to introduce the complementary *second-order empty space function* with respect to $x_1, x_2 \in \mathbb{R}^d$ as

$$\begin{aligned}
\bar{F}_B^{A_1,A_2}(x_1, x_2; t_1, t_2) &:= \mathbb{P}(d_B(x_1, Z(A_1)) > t_1,\ d_B(x_2, Z(A_2)) > t_2) \\
&= \mathbb{P}((x_1 + t_1B) \cap Z(A_1) = \varnothing,\ (x_2 + t_2B) \cap Z(A_2) = \varnothing) \\
&= \mathbb{E}\prod_{n\geq 1}(1 - \mathbf{1}\{(x_1 + t_1B) \cap B(\xi_n, R_n) \neq \varnothing\}\mathbf{1}\{\xi_n \in A_1\}) \\
&\qquad \times (1 - \mathbf{1}\{(x_2 + t_2B) \cap B(\xi_n, R_n) \neq \varnothing\}\mathbf{1}\{\xi_n \in A_2\}) \\
&= \exp\{-\gamma\mathbb{E}|((x_1 + B_{t_1,R}) \cap A_1) \cup ((x_2 + B_{t_2,R}) \cap A_2)|_d\}.
\end{aligned} \tag{2.1}$$

This function is related to the second-order contact distribution function studied in [1].

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In order to obtain a more concise statement in Lemma 2.1 below (and again in the proof of Theorem 3.1 below), for given Borel sets $A_1, A_2 \subset \mathbb{R}^d$, we introduce two functions, $I_1(A_1, A_2)$ and $I_2(A_1, A_2)$, depending on the arguments $(x_1, x_2, y, r) \in (\mathbb{R}^d)^3 \times \mathbb{R}^+$ and $(x_1, x_2, y_1, y_2, r_1, r_2) \in (\mathbb{R}^d)^4 \times (\mathbb{R}^+)^2$, respectively, defined by

$$I_1(A_1, A_2)(x_1, x_2, y, r) := \mathbf{1}\{y \in A_1 \cap A_2\}\bar{F}_B^{A_1,A_2}(x_1, x_2; d_B(x_1, B(y, r)), d_B(x_2, B(y, r)))$$

and

$$\begin{aligned}
&I_2(A_1, A_2)(x_1, x_2, y_1, y_2, r_1, r_2) \\
&\quad := \mathbf{1}\{y_1 \in A_1\}\mathbf{1}\{y_2 \in A_2\}[(1 - \mathbf{1}\{y_2 \in A_1\}\mathbf{1}\{d_B(x_1, B(y_2, r_2)) \leq d_B(x_1, B(y_1, r_1))\}) \\
&\qquad \times (1 - \mathbf{1}\{y_1 \in A_2\}\mathbf{1}\{d_B(x_2, B(y_1, r_1)) \leq d_B(x_2, B(y_2, r_2))\}) \\
&\qquad \times \bar{F}_B^{A_1,A_2}(x_1, x_2; d_B(x_1, B(y_1, r_1)), d_B(x_2, B(y_2, r_2))) \\
&\qquad - \bar{F}_B^{A_1}(x_1; d_B(x_1, B(y_1, r_1)))\bar{F}_B^{A_2}(x_2; d_B(x_2, B(y_2, r_2)))].
\end{aligned}$$

If the arguments of these two functions are clear from the context, they are sometimes omitted.

Lemma 2.1. *Let $A_1, A_2 \subset \mathbb{R}^d$ be Borel sets, and let $x_1, x_2 \in \mathbb{R}^d$. If $\tilde{g}\colon [0, \infty] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a measurable function with $\tilde{g}(0, r) = \tilde{g}(\infty, r) = 0$ for $r \in \mathbb{R}^+$, then*

$$\begin{aligned}
&\operatorname{cov}(\tilde{g}(d_B(x_1, Z(A_1)), r_B(x_1, Z(A_1))), \tilde{g}(d_B(x_2, Z(A_2)), r_B(x_2, Z(A_2)))) \\
&\quad = \gamma \int_0^\infty \int_{\mathbb{R}^d} \tilde{g}(d_B(x_1, B(y, r)), r)\tilde{g}(d_B(x_2, B(y, r)), r)I_1(A_1, A_2)(x_1, x_2, y, r)\, \mathrm{d}y\, \mathbb{G}(\mathrm{d}r) \\
&\qquad + \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{g}(d_B(x_1, B(y_1, r_1)), r_1)\tilde{g}(d_B(x_2, B(y_2, r_2)), r_2) \\
&\qquad\qquad \times I_2(A_1, A_2)(x_1, x_2, y_1, y_2, r_1, r_2)\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathbb{G}(\mathrm{d}r_1)\, \mathbb{G}(\mathrm{d}r_2).
\end{aligned}$$

Proof. For $n \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $i \in \{1, 2\}$, we define the event

$$D_n^{(i)}(x) := \bigg\{d_B\bigg(x, \bigcup_{\{k\neq n\colon \xi_k\in A_i\}} B(\xi_k, R_k)\bigg) > d_B(x, B(\xi_n, R_n))\bigg\}.$$

Then

$$\begin{aligned}
&\mathbb{E}\tilde{g}(d_B(x_1, Z(A_1)), r_B(x_1, Z(A_1)))\tilde{g}(d_B(x_2, Z(A_2)), r_B(x_2, Z(A_2))) \\
&\quad = \mathbb{E} \sum_{\{n\colon \xi_n\in A_1\}} \sum_{\{m\colon \xi_m\in A_2\}} \mathbf{1}_{D_n^{(1)}(x_1)\cap D_m^{(2)}(x_2)}\tilde{g}(d_B(x_1, B(\xi_n, R_n)), R_n) \\
&\qquad\qquad \times \tilde{g}(d_B(x_2, B(\xi_m, R_m)), R_m) \\
&\quad = \mathbb{E} \sum_{\{n\colon \xi_n\in A_1\cap A_2\}} \mathbf{1}_{D_n^{(1)}(x_1)\cap D_n^{(2)}(x_2)}\tilde{g}(d_B(x_1, B(\xi_n, R_n)), R_n)\tilde{g}(d_B(x_2, B(\xi_n, R_n)), R_n) \\
&\qquad + \mathbb{E} \sum\sum_{\{n\neq m\colon \xi_n\in A_1,\xi_m\in A_2\}} \mathbf{1}_{D_n^{(1)}(x_1)\cap D_m^{(2)}(x_2)}\tilde{g}(d_B(x_1, B(\xi_n, R_n)), R_n) \\
&\qquad\qquad \times \tilde{g}(d_B(x_2, B(\xi_m, R_m)), R_m) \\
&\quad =: J_1 + J_2.
\end{aligned}$$

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Applying Mecke's formula (see [24, Corollary 3.2.3]), we obtain

$$J_1 = \gamma \int_0^\infty \int_{A_1 \cap A_2} \tilde{g}(d_B(x_1, B(y, r)), r)\tilde{g}(d_B(x_2, B(y, r)), r) \\ \times \mathbb{P}(d_B(x_1, Z(A_1)) > d_B(x_1, B(y, r)), \\ d_B(x_2, Z(A_2)) > d_B(x_2, B(y, r)))\, \mathrm{d}y\, \mathbb{G}(\mathrm{d}r)$$

and

$$J_2 = \gamma^2 \int_0^\infty \int_0^\infty \int_{A_2} \int_{A_1} \tilde{g}(d_B(x_1, B(y_1, r_1)), r_1)\tilde{g}(d_B(x_2, B(y_2, r_2)), r_2) \\ \times \mathbb{E}[\mathbf{1}\{d_B(x_1, Z_{y_2}(A_1)) > d_B(x_1, B(y_1, r_1))\} \\ \times \mathbf{1}\{d_B(x_2, Z_{y_1}(A_2)) > d_B(x_2, B(y_2, r_2))\}] \\ \times\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathbb{G}(\mathrm{d}r_1)\, \mathbb{G}(\mathrm{d}r_2),$$

where $Z_{y_2}(A_1) = Z(A_1) \cup B(y_2, r_2)$ if $y_2 \in A_1$ and $Z_{y_2}(A_1) = Z(A_1)$ if $y_2 \notin A_1$. Analogously, $Z_{y_1}(A_2) = Z(A_2) \cup B(y_1, r_1)$ if $y_1 \in A_2$ and $Z_{y_1}(A_2) = Z(A_2)$ if $y_1 \notin A_2$. Hence,

$$J_2 = \gamma^2 \int_0^\infty \int_0^\infty \int_{A_2} \int_{A_1} \tilde{g}(d_B(x_1, B(y_1, r_1)), r_1)\tilde{g}(d_B(x_2, B(y_2, r_2)), r_2) \\ \times (1 - \mathbf{1}\{y_2 \in A_1\}\mathbf{1}\{d_B(x_1, B(y_2, r_2)) \le d_B(x_1, B(y_1, r_1))\}) \\ \times (1 - \mathbf{1}\{y_1 \in A_2\}\mathbf{1}\{d_B(x_2, B(y_1, r_1)) \le d_B(x_2, B(y_2, r_2))\}) \\ \times \mathbb{P}(d_B(x_1, Z(A_1)) > d_B(x_1, B(y_1, r_1)), \\ d_B(x_2, Z(A_2)) > d_B(x_2, B(y_2, r_2)))\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathbb{G}(\mathrm{d}r_1)\, \mathbb{G}(\mathrm{d}r_2).$$

Finally,

$$\mathbb{E}\tilde{g}(d_B(x_1, Z(A_1)), r_B(x_1, Z(A_1))) \\ = \mathbb{E} \sum_{\{n :\, \xi_n \in A_1\}} \mathbf{1}_{D_n^{(1)}(x_1)} \tilde{g}(d_B(x_1, B(\xi_n, R_n)), R_n) \\ = \gamma \int_0^\infty \int_{A_1} \tilde{g}(d_B(x_1, B(y_1, r_1)), r_1) \bar{F}_B^{A_1}(x_1; d_B(x_1, B(y_1, r_1)))\, \mathrm{d}y_1\, \mathbb{G}(\mathrm{d}r_1).$$

This completes the proof of Lemma 2.1.

Our aim is to analyze the second-order properties of the random measure η_A given by (1.4). For this reason, we work with the complementary second-order empty space function (2.1). For $A_1 = A_2 = \mathbb{R}^d$, $t_1, t_2 \in \mathbb{R}^+$, and $u = x_2 - x_1$, by the stationarity of Z, this function turns into

$$\bar{F}_B^{(2)}(u; t_1, t_2) := \mathbb{P}(d_B(\mathrm{o}, Z) > t_1,\ d_B(u, Z) > t_2) \\ = \exp\{-\gamma \mathbb{E}|B_{t_1,R} \cup (u + B_{t_2,R})|_d\} \\ = \bar{F}_B(t_1)\bar{F}_B(t_2) \exp\{\gamma \mathbb{E}\kappa_B(u; t_1, t_2, R)\}, \tag{2.2}$$

where

$$\kappa_B(u; t_1, t_2, r) := |B_{t_1,r} \cap (u + B_{t_2,r})|_d.$$

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Observe that, for any $u \in \mathbb{R}^d$ and $t_1, t_2 \in \mathbb{R}^+$, we have

$$\bar{F}_B^{(2)}(u; t_1, t_2) \geq \bar{F}_B(t_1)\bar{F}_B(t_2) \tag{2.3}$$

and

$$\bar{F}_B^{(2)}(u; t_1, t_2) \leq \exp\left\{-\frac{\gamma}{2}\mathbb{E}(|B_{t_1,R}|_d + |B_{t_2,R}|_d)\right\} = \sqrt{\bar{F}_B(t_1)\bar{F}_B(t_2)}. \tag{2.4}$$

These inequalities will be used subsequently. In addition, we will need the assumption that

$$\int_0^\infty f(t)\,\mathrm{e}^{-ct}\,\mathrm{d}t < \infty, \tag{2.5}$$

where $c := 4^{-1}\gamma\kappa_{d-1}V_1(B)\mathbb{E}R^{d-1} < \infty$ and $c > 0$ since $V_1(B) > 0$ (recall that B contains a nondegenerate line segment) and $\mathbb{P}(R > 0) > 0$.

Proposition 2.1. *Assume that (2.5) is satisfied. If $C \subset \mathbb{R}^+$ is a Borel set and $W_1, W_2 \subset \mathbb{R}^d$ are compact convex sets, then*

$$\operatorname{cov}(\eta_{W_1}(C), \eta_{W_2}(C)) = \int_{\mathbb{R}^d} |W_1 \cap (W_2 - u)|_d[\gamma\tau_1(C, u) + \gamma^2\tau_2(C, u)]\,\mathrm{d}u,$$

where

$$\tau_1(C, u) := \int_C \int_{\mathbb{R}^d} \frac{f(d_B(x, B(\mathrm{o}, r)))}{h_B(d_B(x, B(\mathrm{o}, r)), r)} \frac{f(d_B(u + x, B(\mathrm{o}, r)))}{h_B(d_B(u + x, B(\mathrm{o}, r)), r)} \times \bar{F}_B^{(2)}(u; d_B(x, B(\mathrm{o}, r)), d_B(u + x, B(\mathrm{o}, r)))\,\mathrm{d}x\,\mathbb{G}(\mathrm{d}r) \tag{2.6}$$

and

$$\tau_2(C, u) := \int_C \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(d_B(x_1, B(\mathrm{o}, r_1)))}{h_B(d_B(x_1, B(\mathrm{o}, r_1)), r_1)} \frac{f(d_B(x_2, B(\mathrm{o}, r_2)))}{h_B(d_B(x_2, B(\mathrm{o}, r_2)), r_2)} \times q(u; x_1, x_2, r_1, r_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathbb{G}(\mathrm{d}r_1)\,\mathbb{G}(\mathrm{d}r_2) \tag{2.7}$$

for $u \in \mathbb{R}^d$, and

$$\begin{aligned} q(u; x_1, x_2, r_1, r_2) := {} & \mathbf{1}\{d_B(x_2, B(u, r_2)) > d_B(x_1, B(\mathrm{o}, r_1))\} \\ & \times \mathbf{1}\{d_B(x_1, B(-u, r_1)) > d_B(x_2, B(\mathrm{o}, r_2))\} \\ & \times \bar{F}_B^{(2)}(u; d_B(x_1, B(\mathrm{o}, r_1)), d_B(x_2, B(\mathrm{o}, r_2))) \\ & - \bar{F}_B(d_B(x_1, B(\mathrm{o}, r_1)))\bar{F}_B(d_B(x_2, B(\mathrm{o}, r_2))) \end{aligned} \tag{2.8}$$

for $x_1, x_2 \in \mathbb{R}^d$ and $r_1, r_2 \in \mathbb{R}^+$.

Proof. To abbreviate the notation, we define the function

$$g(t, r) := \mathbf{1}\{r \in C\}f(t)h_B(t, r)^{-1} \tag{2.9}$$

for $t \in [0, \infty]$ and $r \in \mathbb{R}^+$, with the previous conventions in the cases where $t \in \{0, \infty\}$. Recall also that $h_B(t, r) > 0$ for $t \in (0, \infty)$ and $\mathbb{G}$-almost all $r \in \mathbb{R}^+$. Using Fubini's theorem and stationarity, we obtain

$$\begin{aligned} &\operatorname{cov}(\eta_{W_1}(C), \eta_{W_2}(C)) \\ &\quad = \int_{W_1} \int_{W_2} \operatorname{cov}(g(d_B(x_1, Z), r_B(x_1, Z)), g(d_B(x_2, Z), r_B(x_2, Z)))\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &\quad = \int_{\mathbb{R}^d} |W_1 \cap (W_2 - u)|_d \operatorname{cov}(g(d_B(\mathrm{o}, Z), r_B(\mathrm{o}, Z)), g(d_B(u, Z), r_B(u, Z)))\,\mathrm{d}u. \end{aligned}$$

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By Lemma 2.1 with $A_1 = A_2 = \mathbb{R}^d$, $x_1 = o$, and $x_2 = u$, we obtain

$$\operatorname{cov}(g(d_B(o, Z), r_B(o, Z)), g(d_B(u, Z), r_B(u, Z))) = J_1(u) + J_{21}(u) - J_{22},$$

where

$$\begin{aligned} J_1(u) := \gamma \int_0^\infty \int_{\mathbb{R}^d} & g(d_B(o, B(x, r)), r) g(d_B(u, B(x, r)), r) \\ & \times \bar{F}_B^{(2)}(u; d_B(o, B(x, r)), d_B(u, B(x, r))) \, \mathrm{d}x \, \mathbb{G}(\mathrm{d}r), \\ J_{21}(u) := \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & g(d_B(o, B(x_1, r_1)), r_1) g(d_B(o, B(x_2, r_2)), r_2) \\ & \times \mathbf{1}\{d_B(-u, B(x_2, r_2)) > d_B(o, B(x_1, r_1))\} \\ & \times \mathbf{1}\{d_B(u, B(x_1, r_1)) > d_B(o, B(x_2, r_2))\} \\ & \times \bar{F}_B^{(2)}(u; d_B(o, B(x_1, r_1)), d_B(o, B(x_2, r_2))) \\ & \times \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathbb{G}(\mathrm{d}r_1) \, \mathbb{G}(\mathrm{d}r_2), \end{aligned}$$

and

$$\begin{aligned} J_{22} := \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & g(d_B(o, B(x_1, r_1)), r_1) g(d_B(o, B(x_2, r_2)), r_2) \\ & \times \bar{F}_B(d_B(o, B(x_1, r_1))) \bar{F}_B(d_B(o, B(x_2, r_2))) \\ & \times \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathbb{G}(\mathrm{d}r_1) \, \mathbb{G}(\mathrm{d}r_2). \end{aligned}$$

Using the fact that $d_B(u, B(x, r)) = d_B(u - x, B(o, r))$ and the reflection invariance of the Lebesgue measure, we deduce that

$$\begin{aligned} J_1(u) = \gamma \int_0^\infty \int_{\mathbb{R}^d} & g(d_B(x, B(o, r)), r) g(d_B(u + x, B(o, r)), r) \\ & \times \bar{F}_B^{(2)}(u; d_B(x, B(o, r)), d_B(u + x, B(o, r))) \, \mathrm{d}x \, \mathbb{G}(\mathrm{d}r), \\ J_{21}(u) = \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & g(d_B(x_1, B(o, r_1)), r_1) g(d_B(x_2, B(o, r_2)), r_2) \\ & \times \mathbf{1}\{d_B(x_2, B(u, r_2)) > d_B(x_1, B(o, r_1))\} \\ & \times \mathbf{1}\{d_B(x_1, B(-u, r_1)) > d_B(x_2, B(o, r_2))\} \\ & \times \bar{F}_B^{(2)}(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2))) \\ & \times \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathbb{G}(\mathrm{d}r_1) \, \mathbb{G}(\mathrm{d}r_2), \end{aligned}$$

and

$$\begin{aligned} J_{22} = \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & g(d_B(x_1, B(o, r_1)), r_1) g(d_B(x_2, B(o, r_2)), r_2) \\ & \times \bar{F}_B(d_B(x_1, B(o, r_1))) \bar{F}_B(d_B(x_2, B(o, r_2))) \\ & \times \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathbb{G}(\mathrm{d}r_1) \, \mathbb{G}(\mathrm{d}r_2). \end{aligned}$$

The assertion now follows by recalling (2.9). The integrability of $\tau_1(C, \cdot)$ and $\tau_2(C, \cdot)$, which is explicitly stated in (A.1), will be shown in the proof of Theorem 2.1 given in Appendix A and is implied by assumption (2.5).

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Remark 2.1. Recall that $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. If $B = B^d$ is the unit ball then $d_{B^d}(u, B(x, r)) = (\|x - u\| - r)^+$,

$$h_{B^d}(t, r) = \sum_{j=0}^{d-1} d\kappa_d \binom{d-1}{j} r^{d-1-j} t^j = d\kappa_d (t + r)^{d-1},$$

and

$$\kappa_{B^d}(u; t_1, t_2, r) = |B(\mathrm{o}, t_1 + r) \cap B(u, t_2 + r)|_d.$$

Hence, $\tau_1(C, u)$ and $\tau_2(C, u)$ from Proposition 2.1 may be slightly simplified. In particular, we have

$$\begin{aligned}
&\tau_2(C, u)\\
&= \int_C \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f((\|x_1\| - r_1)^+)}{h_{B^d}((\|x_1\| - r_1)^+, r_1)} \frac{f((\|x_2\| - r_2)^+)}{h_{B^d}((\|x_2\| - r_2)^+, r_2)}\\
&\qquad \times [\mathbf{1}\{(\|x_2 - u\| - r_2)^+ > (\|x_1\| - r_1)^+\}\\
&\qquad\quad \times \mathbf{1}\{(\|x_1 + u\| - r_1)^+ > (\|x_2\| - r_2)^+\}\\
&\qquad\quad \times \bar{F}^{(2)}_{B^d}(u; (\|x_1\| - r_1)^+, (\|x_2\| - r_2)^+)\\
&\qquad\quad - \bar{F}_{B^d}((\|x_1\| - r_1)^+) \bar{F}_{B^d}((\|x_2\| - r_2)^+)] \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathbb{G}(\mathrm{d}r_1) \, \mathbb{G}(\mathrm{d}r_2)\\
&= \int_C \int_C \int_0^\infty \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{f(s_1)}{h_{B^d}(s_1, r_1)} (s_1 + r_1)^{d-1} \frac{f(s_2)}{h_{B^d}(s_2, r_2)} (s_2 + r_2)^{d-1}\\
&\qquad \times [\mathbf{1}\{(\|(s_2 + r_2)v_2 - u\| - r_2)^+ > s_1\}\\
&\qquad\quad \times \mathbf{1}\{(\|(s_1 + r_1)v_1 + u\| - r_1)^+ > s_2\}\\
&\qquad\quad \times \bar{F}^{(2)}_{B^d}(u; s_1, s_2) - \bar{F}_{B^d}(s_1) \bar{F}_{B^d}(s_2)]\\
&\qquad \times \mathcal{H}^{d-1}(\mathrm{d}v_1) \, \mathrm{d}s_1 \, \mathcal{H}^{d-1}(\mathrm{d}v_2) \, \mathrm{d}s_2 \, \mathbb{G}(\mathrm{d}r_1) \, \mathbb{G}(\mathrm{d}r_2)\\
&= \int_C \int_C \int_0^\infty \int_0^\infty f(s_1) f(s_2) \bigg[\frac{\mathcal{H}^{d-1}(\partial B(\mathrm{o}, s_2 + r_2) \cap B(u, s_1 + r_2)^{\mathrm{c}})}{\mathcal{H}^{d-1}(\partial B(\mathrm{o}, s_2 + r_2))}\\
&\qquad \times \frac{\mathcal{H}^{d-1}(\partial B(\mathrm{o}, s_1 + r_1) \cap B(-u, s_2 + r_1)^{\mathrm{c}})}{\mathcal{H}^{d-1}(\partial B(\mathrm{o}, s_1 + r_1))}\\
&\qquad \times \bar{F}^{(2)}_{B^d}(u; s_1, s_2) - \bar{F}_{B^d}(s_1) \bar{F}_{B^d}(s_2) \bigg]\\
&\qquad \times \mathrm{d}s_1 \, \mathrm{d}s_2 \, \mathbb{G}(\mathrm{d}r_1) \, \mathbb{G}(\mathrm{d}r_2),
\end{aligned}$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$, $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, and $\partial B(x, r)$ is the boundary of $B(x, r)$. We used the fact that $f((\|x\| - r)^+)$ is nonzero only if $\|x\| > r$. Then $x = (s + r)v$ for $s > 0$ and $v \in \mathbb{S}^{d-1}$.

Next we state a special case of [14, Theorem 2.1 and Remark 3.1] in the form needed in the present context. Let $\tilde{g} \colon \mathbb{R}^d \to [0, \infty]$ be measurable, and let $K, B \subset \mathbb{R}^d$ be convex bodies such that $\mathrm{o} \in B$ and K and B are in general relative position. Since in our application we will only need the case $K = rB^d$ for $r \in \mathbb{R}^+$, the assumption of general relative position will be satisfied for any choice of B. Then we have

$$\int_{\mathbb{R}^d} \mathbf{1}\{0 < d_B(z, rB^d) < \infty\} \tilde{g}(z)\,\mathrm{d}z$$
$$= \sum_{j=0}^{d-1} \binom{d-1}{j} \int_0^\infty \int t^{d-1-j} \tilde{g}(z+tb)\, \Theta_{j;d-j}(rB^d;\, B^*;\, \mathrm{d}(z,b))\,\mathrm{d}t,$$

where $B^* := -B$ and the mixed support measures $\Theta_{j;d-j}(rB^d; B^*; \cdot)$, $j \in \{0, \dots, d-1\}$, are finite Borel measures on $\mathbb{R}^{2d}$. Using [24, Equation (14.18)] (cf. [23, Equations (4.9) and (5.53)]) and [17, Middle of p. 327], for the total measures, we obtain $\Theta_{j;d-j}(rB^d; B^*; \mathbb{R}^{2d}) = r^j d\kappa_j V_{d-j}(B)/\binom{d}{j}$. In particular, for any measurable function $\tilde{f}\colon [0,\infty] \to [0,\infty]$ with $\tilde{f}(0) = \tilde{f}(\infty) = 0$, this yields

$$\int_{\mathbb{R}^d} \tilde{f}(d_B(z, rB^d))\,\mathrm{d}z = \int_0^\infty h_B(t,r)\tilde{f}(t)\,\mathrm{d}t. \tag{2.10}$$

We now turn to the asymptotic properties of the ratio-unbiased estimator (1.7). Our setting is similar to [20], where all limit theorems refer to a growing observation window in $\mathbb{R}^d$. More formally, we consider a sequence $(W_n)_{n\in\mathbb{N}}$ of compact, convex sets $W_n \subset \mathbb{R}^d$ such that $W_n \subset W_{n+1}$ for all $n \in \mathbb{N}$ and the inradius of W_n tends to ∞ as $n \to \infty$.

Theorem 2.1. *Assume that (2.5) is fulfilled. Then, for any Borel set $C \subset \mathbb{R}^+$,*

$$\frac{\operatorname{var} \eta_{W_n}(C)}{|W_n|_d} \to \sigma^2(C) \quad \text{as } n \to \infty. \tag{2.11}$$

The asymptotic variance is finite and given by

$$\sigma^2(C) = \gamma \int_{\mathbb{R}^d} \tau_1(C,u)\,\mathrm{d}u + \gamma^2 \int_{\mathbb{R}^d} \tau_2(C,u)\,\mathrm{d}u, \tag{2.12}$$

where $\tau_1(C,u)$ and $\tau_2(C,u)$ are defined in (2.6) and (2.7), respectively. Moreover, if $0 < \mathbb{G}(C) < 1$ then $\sigma^2(C) > 0$.

Proof. See Appendix A.

Remark 2.2. Assumption (2.5) is slightly stronger than (1.5).

Remark 2.3. Let $\widehat{\mathbb{G}}_n(C)$ be given by (1.7) with $W = W_n$. Theorem 2.1 implies that $\widehat{\mathbb{G}}_n(C)$ is asymptotically weakly consistent. Indeed, (1.6) and

$$\frac{\operatorname{var} \eta_{W_n}(C)}{|W_n|_d^2} \to 0 \quad \text{as } n \to \infty$$

ensure that $\eta_{W_n}(C)/|W_n|_d$ converges to $\gamma\,\beta\,\mathbb{G}(C)$ in probability as $n \to \infty$. Specifically,

$$\frac{\eta_{W_n}(\mathbb{R}^+)}{|W_n|_d} \to \gamma\beta \quad \text{in probability as } n \to \infty. \tag{2.13}$$

Hence, by the continuous mapping theorem, $\eta_{W_n}(C)/\eta_W(\mathbb{R}^+)$ converges to $\mathbb{G}(C)$ in probability as $n \to \infty$. This is in accordance with the following proposition which even shows that $\widehat{\mathbb{G}}_n(C)$ is asymptotically strongly consistent.

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Proposition 2.2. *For any Borel set $C \subset \mathbb{R}^+$, we have $\widehat{\mathbb{G}}_n(C) \to \mathbb{G}(C)$ $\mathbb{P}$-almost surely (abbreviated as $\mathbb{P}$-a.s.) as $n \to \infty$.*

Proof. The mapping $W \mapsto \eta_W(C)$ defined by (1.4) is a random measure on $\mathbb{R}^d$ depending on the Boolean model Z in a translation-invariant way. As the Boolean model is ergodic (see [24, Theorem 9.3.5]), we can apply the spatial ergodic theorem (see [16, Corollary 10.19]) to conclude that

$$\lim_{n\to\infty} |W_n|_d^{-1} \eta_{W_n}(C) = \mathbb{E}\eta_{[0,1]^d}(C) = \gamma\, \beta\, \mathbb{G}(C) \quad \mathbb{P}\text{-a.s.}$$

Applying this to the numerator as well as to the denominator in (1.7), we obtain the desired result.

The $\mathbb{P}$-almost sure uniform convergence for distribution functions can be obtained by a standard technique in analogy with the classical Glivenko–Cantelli theorem.

Proposition 2.3. *Let $\mathcal{C} = \{C \subset \mathbb{R}^+ : C = (a, b],\ 0 \le a < b\}$. Then*

$$\sup_{C\in\mathcal{C}} |\widehat{\mathbb{G}}_n(C) - \mathbb{G}(C)| \to 0 \quad \mathbb{P}\text{-a.s. as } n \to \infty.$$

Proof. Denote by $G(t)$ the distribution function corresponding to the radius distribution $\mathbb{G}$, i.e. $G(t) = \mathbb{G}([0, t])$, $t \ge 0$. Exploiting Proposition 2.2, and the right continuity and monotonicity of $G(t)$ and $\widehat{G}_n(t) = \widehat{\mathbb{G}}_n([0, t])$, $t \ge 0$, it can be shown that

$$\sup_{t\ge 0} |\widehat{G}_n(t) - G(t)| \to 0 \quad \mathbb{P}\text{-a.s. as } n \to \infty;$$

see, e.g. [9, Lemma 3] where a similar case is treated. The proof is completed by noting that $\widehat{\mathbb{G}}_n((a, b]) = \widehat{G}_n(b) - \widehat{G}_n(a)$ and $\mathbb{G}((a, b]) = G(b) - G(a)$.

3. Asymptotic normality

In this section we study the asymptotic normality of the ratio-unbiased estimator (1.7) for the radius distribution $\mathbb{G}$ of our stationary Boolean model Z with spherical grains. The proof will be based on approximation by m-dependent random fields. This idea comes from [11], where the same technique was used to prove the central limit theorem for random measures which are associated with the Boolean model in an additive way. In contrast to [11], the contribution of an individual grain to the random measure $A \mapsto \eta_A(C)$ is not determined by the grain alone, but does depend on a random number of other grains in a nontrivial manner. Therefore, the results of [11] do not apply in our setting.

We consider, for $n \in \mathbb{N}$ and a Borel set $C \subset \mathbb{R}^+$, the estimator

$$\widehat{\mathbb{G}}_n(C) = \frac{\eta_{W_n}(C)}{\eta_{W_n}(\mathbb{R}^d)},$$

where $W_n := [-n, n)^d$ and η_{W_n} is given in (1.4). First we concentrate on the asymptotic normality of the numerator $\eta_{W_n}(C)$. In addition to (1.5), we will need the integrability condition

$$\int_0^\infty (1 + t^d) f(t)\, \mathrm{d}t < \infty, \tag{3.1}$$

which is more restrictive than (2.5).

Theorem 3.1. *Assume that (1.5) and (3.1) are fulfilled. Then, for any Borel set $C \subset \mathbb{R}^+$,*

$$\sqrt{|W_n|_d}\left(\frac{\eta_{W_n}(C)}{|W_n|_d} - \gamma\,\beta\,\mathbb{G}(C)\right) \xrightarrow{\text{D}} N(0, \sigma^2(C)) \quad \text{as } n \to \infty,$$

where $\sigma^2(C)$ is given by (2.12).

Proof. See Appendix B.

Now we deal with the asymptotic normality of $\widehat{\mathbb{G}}_n(C)$.

Theorem 3.2. *Assume that (3.1) is satisfied. Let $W_n = [-n, n)^d$. If $C \subset \mathbb{R}^+$ is a Borel set then*

$$\sqrt{|W_n|_d}(\widehat{\mathbb{G}}_n(C) - \mathbb{G}(C)) \xrightarrow{\text{D}} N(0, \sigma^2_{\mathbb{G}}(C)) \quad \text{as } n \to \infty,$$

where

$$\sigma^2_{\mathbb{G}}(C) := \frac{1}{\gamma^2\beta^2}[(1 - \mathbb{G}(C))\sigma^2(C) + \mathbb{G}(C)\sigma^2(\mathbb{R}^+ \setminus C) - \mathbb{G}(C)(1 - \mathbb{G}(C))\sigma^2(\mathbb{R}^+)] \quad (3.2)$$

and $\sigma^2(\cdot)$ is given by (2.12). If $0 < \mathbb{G}(C) < 1$ then $\sigma^2_{\mathbb{G}}(C) > 0$.

Proof. Using (2.13) and Slutsky's theorem, the weak limit of

$$\sqrt{|W_n|_d}(\widehat{\mathbb{G}}_n(C) - \mathbb{G}(C))$$

coincides with the weak limit of

$$Y_n := \frac{1}{\gamma\,\beta\sqrt{|W_n|_d}}(\eta_{W_n}(C) - \eta_{W_n}(\mathbb{R}^+)\mathbb{G}(C)).$$

Observing that

$$\gamma\,\beta\,Y_n = \frac{1}{\sqrt{|W_n|_d}}\int_{W_n}(\mathbf{1}\{r_B(x, Z) \in C\} - \mathbb{G}(C))f(d_B(x, Z))h_B(d_B(x, Z), r_B(x, Z))^{-1}\,\mathrm{d}x,$$

we can proceed along the same lines as in the proof of Theorem 3.1 (see Appendix B) and obtain

$$\gamma\,\beta\,Y_n \xrightarrow{\text{D}} N(0, \gamma^2\beta^2\sigma^2_{\mathbb{G}}(C)) \quad \text{as } n \to \infty,$$

provided we can identify the asymptotic variance $\sigma^2_{\mathbb{G}}(C)$ of Y_n. Theorem 2.1 implies that

$$\gamma^2\beta^2\lim_{n\to\infty}\operatorname{var} Y_n = \sigma^2(C) + \mathbb{G}(C)^2\sigma^2(\mathbb{R}^+) - 2\mathbb{G}(C)\lim_{n\to\infty}\frac{1}{|W_n|_d}\operatorname{cov}(\eta_{W_n}(C), \eta_{W_n}(\mathbb{R}^+)). \quad (3.3)$$

Since $\eta_{W_n}(\cdot)$ is additive, from Theorem 2.1 we obtain

$$\begin{aligned}
&2\lim_{n\to\infty}\frac{1}{|W_n|_d}\operatorname{cov}(\eta_{W_n}(C), \eta_{W_n}(\mathbb{R}^+)) \\
&\quad = 2\sigma^2(C) + 2\lim_{n\to\infty}\frac{1}{|W_n|_d}\operatorname{cov}(\eta_{W_n}(C), \eta_{W_n}(\mathbb{R}^+ \setminus C)) \\
&\quad = 2\sigma^2(C) + \lim_{n\to\infty}\frac{1}{|W_n|_d}(\operatorname{var}\eta_{W_n}(\mathbb{R}^+) - \operatorname{var}\eta_{W_n}(C) - \operatorname{var}\eta_{W_n}(\mathbb{R}^+ \setminus C)) \\
&\quad = 2\sigma^2(C) + \sigma^2(\mathbb{R}^+) - \sigma^2(C) - \sigma^2(\mathbb{R}^+ \setminus C) \\
&\quad = \sigma^2(C) + \sigma^2(\mathbb{R}^+) - \sigma^2(\mathbb{R}^+ \setminus C).
\end{aligned}$$

Substituting this result into (3.3) yields (3.2) upon some simplification.

To prove the last assertion, we define $\tilde{g}(t,s) := (\mathbf{1}\{s \in C\} - \mathbb{G}(C))f(t)h_B(t,s)^{-1}$ and assume that $0 < \mathbb{G}(C) < 1$. For a convex body $W \subset \mathbb{R}^d$, we need to consider the variance of

$$H_W := \int_W \tilde{g}(d_B(x,Z), r_B(x,Z))\,\mathrm{d}x.$$

As in the proof of the positivity assertion in Theorem 2.1 (see Appendix A), we obtain

$$\operatorname{var} H_W \geq \gamma \int_0^\infty \int_{\mathbb{R}^d} \tilde{h}(y,r)^2\,\mathrm{d}y\,\mathbb{G}(\mathrm{d}r), \tag{3.4}$$

where

$$\begin{aligned}\tilde{h}(y,r) := {} & \mathbb{E}\int_W \mathbf{1}\{d_B(\mathrm{o}, B(y-x,r)) < d_B(\mathrm{o},Z)\}\tilde{g}(d_B(\mathrm{o},B(y-x,r)),r)\,\mathrm{d}x \\ & - \mathbb{E}\int_W \mathbf{1}\{d_B(\mathrm{o},B(y-x,r)) < d_B(\mathrm{o},Z)\}\tilde{g}(d_B(\mathrm{o},Z), r_B(\mathrm{o},Z))\,\mathrm{d}x.\end{aligned}$$

By (1.3) and the definition of $\tilde{g}$, the second expectation on the right-hand side of the above equation vanishes for all $y \in W$ and $r \geq 0$. Therefore,

$$\tilde{h}(y,r) = \int_W \bar{F}_B(d_B(\mathrm{o},B(y-x,r)))\tilde{g}(d_B(\mathrm{o},B(y-x,r)),r)\,\mathrm{d}x.$$

Again, as in the proof of Theorem 2.1, we let $\bar{C} := \mathbb{R}^+ \setminus C$ and obtain, from Jensen's inequality and (3.4),

$$\begin{aligned}\frac{\sqrt{\operatorname{var} H_W}}{\sqrt{|W|_d}} &\geq \frac{c}{|W|_d}\int_{\bar{C}}\int_W\int_W \bar{F}_B(d_B(\mathrm{o},B(y-x,r)))f(d_B(\mathrm{o},B(y-x,r))) \\ &\qquad\qquad \times h_B(d_B(\mathrm{o},B(y-x,r)),r)^{-1}\,\mathrm{d}x\,\mathrm{d}y\,\mathbb{G}(\mathrm{d}r) \\ &= \frac{c}{|W|_d}\int_{\bar{C}}\int_{\mathbb{R}^d} |W \cap (W-y)|_d \bar{F}_B(d_B(\mathrm{o},B(y,r)))f(d_B(\mathrm{o},B(y,r))) \\ &\qquad\qquad \times h_B(d_B(\mathrm{o},B(y,r)),r)^{-1}\,\mathrm{d}y\,\mathbb{G}(\mathrm{d}r),\end{aligned}$$

where $c > 0$ is a constant not depending on W. Hence, it is sufficient to show that

$$\int_{\bar{C}}\int_{\mathbb{R}^d} \bar{F}_B(d_B(\mathrm{o},B(y,r)))f(d_B(\mathrm{o},B(y,r)))h_B(d_B(\mathrm{o},B(y,r)),r)^{-1}\,\mathrm{d}y\,\mathbb{G}(\mathrm{d}r) > 0.$$

By (2.10), the above integral equals $\mathbb{G}(\bar{C})\int_{\mathbb{R}^d} \bar{F}_B(t)f(t)\,\mathrm{d}t$, which is positive by (1.5).

Remark 3.1. After some manipulation we obtain

$$\gamma^2\beta^2\sigma^2_{\mathbb{G}}(C) = \gamma\int_{\mathbb{R}^d}\tilde{\tau}_1(C,u)\,\mathrm{d}u + \gamma^2\int_{\mathbb{R}^d}\tilde{\tau}_2(C,u)\,\mathrm{d}u,$$

where

$$\begin{aligned}\tilde{\tau}_1(C,u) := \int_0^\infty\int_{\mathbb{R}^d} & \frac{f(d_B(\mathrm{o},B(x,r)))}{h_B(d_B(\mathrm{o},B(x,r)),r)}\,\frac{f(d_B(u,B(x,r)))}{h_B(d_B(u,B(x,r)),r)} \\ & \times \bar{F}^{(2)}_B(u; d_B(\mathrm{o},B(x,r)), d_B(u,B(x,r))) \\ & \times (\mathbf{1}\{r \in C\} - \mathbb{G}(C))^2\,\mathrm{d}x\,\mathbb{G}(\mathrm{d}r)\end{aligned}$$

and

$$\tilde{\tau}_2(C,u) := \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(d_B(x_1, B(o, r_1)))}{h_B(d_B(x_2, B(o, r_1)), r_1)} \frac{f(d_B(x_2, B(o, r_2)))}{h_B(d_B(x_2, B(o, r_2)), r_2)} \\ \times \mathbf{1}\{d_B(x_2, B(u, r_2)) \le d_B(x_1, B(o, r_1))\} \\ \times \mathbf{1}\{d_B(x_1, B(-u, r_1)) \le d_B(x_2, B(o, r_2))\} \\ \times \bar{F}_B^{(2)}(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2))) \\ \times (\mathbf{1}\{r_1 \in C\} - \mathbb{G}(C))(\mathbf{1}\{r_2 \in C\} - \mathbb{G}(C)) \\ \times \, dx_1 \, dx_2 \, \mathbb{G}(dr_1) \, \mathbb{G}(dr_2).$$

This relation can also be obtained directly by an analogue of the proof of Theorem 2.1.

Remark 3.2. Theorem 3.1 can be generalized to a multivariate setting. First we note that, for arbitrary Borel sets $C, C' \subset \mathbb{R}^+$,

$$\frac{\operatorname{cov}(\eta_{W_n}(C), \eta_{W_n}(C'))}{|W_n|_d} \to \sigma(C, C') \quad \text{as } n \to \infty, \tag{3.5}$$

where

$$\sigma(C, C') := \gamma \int_{\mathbb{R}^d} \tau_1(C \cap C', u) \, du + \gamma^2 \int_{\mathbb{R}^d} \tau_2(C, C', u) \, du.$$

Here $\tau_1(C, u)$ is given by (2.6), while $\tau_2(C, C', u)$ is obtained from (2.7) by replacing the second C on the right-hand side by C'. In the case $C \cap C' = \varnothing$, relation (3.5) follows from (2.11) and the identity

$$2 \operatorname{cov}(\eta_{W_n}(C), \eta_{W_n}(C')) = \operatorname{var} \eta_{W_n}(C \cup C') - \operatorname{var} \eta_{W_n}(C) - \operatorname{var} \eta_{W_n}(C').$$

In the general case, we can use the disjoint decompositions $C = (C \setminus (C \cap C')) \cup (C \cap C')$ and $C' = (C' \setminus (C \cap C')) \cup (C \cap C')$ to deduce, from the bilinearity of the covariance, the previous case, and a straightforward calculation, that

$$\sigma(C, C') = \gamma \int_{\mathbb{R}^d} \tau_1(C \cap C', u) \, du + \gamma^2 \int_{\mathbb{R}^d} \frac{1}{2}(\tau_2(C, C', u) + \tau_2(C', C, u)) \, du.$$

Checking the definition of the function q (see (2.8)), we obtain $q(u; x_1, x_2, r_1, r_2) = q(-u; x_2, x_1, r_2, r_1)$, and, hence, $\tau_2(C, C', u) = \tau_2(C', C, -u)$. Relation (3.5) then follows from the reflection invariance of the Lebesgue measure.

Let us now consider the random vectors $X_n := (\eta_{W_n}(C_1), \dots, \eta_{W_n}(C_k))$, where $C_1, \dots, C_k \subset \mathbb{R}^+$ are fixed Borel sets. Using the Cramér–Wold device and proceeding exactly as in the proof of Theorem 3.1 (see Appendix B), we can then show that

$$\sqrt{|W_n|_d}\left(\frac{X_n}{|W_n|_d} - \gamma\, \beta\, (\mathbb{G}(C_1), \dots, \mathbb{G}(C_k))\right) \xrightarrow{\text{D}} N(0, \Sigma) \quad \text{as } n \to \infty,$$

where $N(0, \Sigma)$ is a multivariate normal distribution with mean 0 and covariance matrix $\Sigma := (\sigma(C_i, C_j))$.

Theorem 3.2 can be extended in a similar way. We omit further details.

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4. The planar case

We mentioned at the beginning that the nonparametric estimator $\widehat{\mathbb{G}}$ which we discussed so far is based on the data $\{(d_B(x, Z), r_B(x, Z)) : x \in W \setminus Z\}$ and, therefore, may require information from outside the window W. To overcome this problem, a common procedure in spatial statistics is the so-called *minus sampling*, which can be used, e.g. if the radius distribution $\mathbb{G}$ is concentrated on an interval $[0, r_0]$, $0 < r_0 < \infty$. We can avoid such a condition by assuming that the function f is concentrated on an interval $[0, \varepsilon]$ with $\varepsilon > 0$. If we then assume that Z is observable in a window $W^{(\varepsilon)}$ which contains $W \oplus \varepsilon B$, then, for each $x \in W$, we have either $f(d_B(x, Z)) = 0$ or $d_B(x, Z) \le \varepsilon$, in which case the (a.s. unique) contact point $(x + d_B(x, Z)B) \cap Z$ lies in $W^{(\varepsilon)}$.

In this section we restrict attention to the planar case $d = 2$ and focus on spherical (that is, $B = B^2$) and on linear (that is, $B = [0, u]$ with a given unit vector $u \in \mathbb{R}^2$) structuring elements B. For simplicity, in the following considerations we concentrate on the window $W = [0, 1]^2$, and assume, as explained above, that f is concentrated on $[0, \varepsilon]$, $\varepsilon > 0$, and that Z is observed in $W^{(\varepsilon)} = W \oplus \varepsilon B$. Let $\tilde{C}_1, \dots, \tilde{C}_{\tilde{k}}$ be the (connected and relatively open) visible arcs in $\partial Z \cap W^{(\varepsilon)}$. We do not need to know whether some of these arcs belong to the same particle. In the following we consider the corresponding 'effective' arcs. These consist of the points of $\tilde{C}_i$ which are contained in $(x + d_B(x, Z)B) \cap Z$ for some $x \in W \setminus Z$ with $d_B(x, Z) \le \varepsilon$. The latter sets may be empty or not connected. Let $C_1, \dots, C_k$ be an enumeration of the nonempty and connected (relatively open) components of the effective arcs. For $i \in \{1, \dots, k\}$, let r_i be the radius and l_i the length of C_i, and let A_i be the set of points $x \in W \setminus Z$ with $d_B(x, Z) \le \varepsilon$ which project onto C_i in the sense that $(x + d_B(x, Z)B) \cap Z$ consists of a unique point and this point lies in C_i. Note that k, the arcs C_i, the sets A_i, and the subsequent notions depend on ε. See Figure 2 for an illustration. Our estimator $\widehat{\mathbb{G}}$ is now of the form

$$\widehat{\mathbb{G}} = \frac{1}{\sum_{i=1}^{k} w_i} \sum_{i=1}^{k} w_i \delta_{r_i},$$

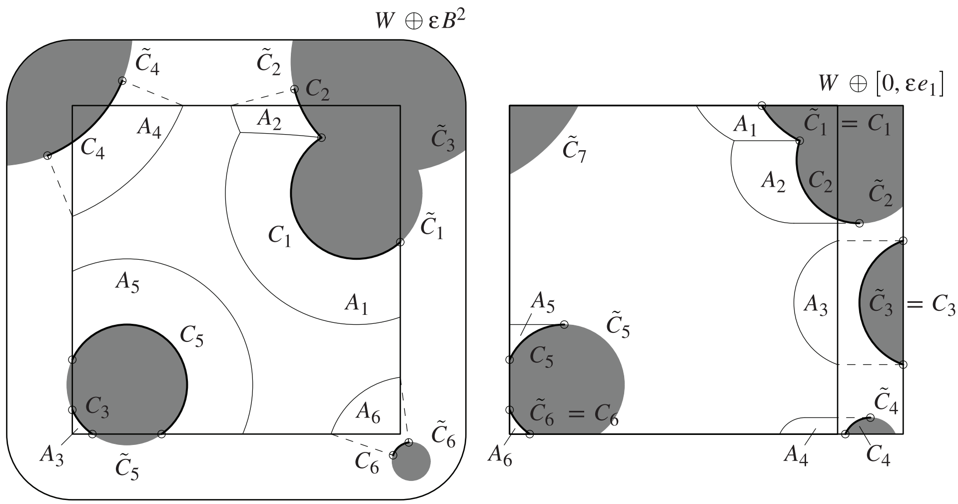

FIGURE 2: An illustration of the arcs $\tilde{C}_i$, the effective arcs C_i, and the sets A_i for spherical B (*left*) and linear B (*right*).

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where the weight w_i is given by

$$w_i = \int_{A_i} f(d_B(x, Z)) h_B(d_B(x, Z), r_B(x, Z))^{-1} \, \mathrm{d}x.$$

For $B = B^2$, we have $h_{B^2}(t, r) = 2\pi(t + r)$; see Remark 2.1. Therefore, upon choosing $f(t) = \varepsilon^{-1}\mathbf{1}\{t \le \varepsilon\}$, we obtain

$$w_i = \frac{1}{2\pi\varepsilon} \int_{A_i} \frac{1}{d_{B^2}(x, C_i) + r_i} \, \mathrm{d}x.$$

If we let $\varepsilon \to 0$ then the outer sampling window $W^{(\varepsilon)}$ shrinks to W and the effective arcs $C_1, \dots, C_k$ become subsets of W, so that in the limit only information in W is needed. The weights then converge to $w_i = l_i/(2\pi r_i)$ if $r_i > 0$ and to $w_i = 1$ if $r_i = l_i = 0$ (i.e. if C_i consists of one point only). Then the estimator becomes

$$\widehat{\mathbb{G}}_o(C) := \bigg(\sum_{i=1}^k \frac{l_i}{r_i}\bigg)^{-1} \sum_{i=1}^k \frac{l_i}{r_i} \delta_{r_i}$$

with l_i/r_i interpreted as 2π if $r_i = l_i = 0$. The estimator $\widehat{\mathbb{G}}_o(C)$ was discussed by Hall [6, Chapter 5.6] (more generally, he considered estimators of $\mathbb{E}A(R)$ for a given function A; $\widehat{\mathbb{G}}_o(C)$ corresponds to the case $A = \mathbf{1}_C$).

For $B = [0, u]$ (with $u \in \{\pm e_1, \pm e_2\}$), assuming (in the linear case) that $\mathbb{G}(\{0\}) = 0$ and, hence, $r_i > 0$, and again choosing $f(t) = \varepsilon^{-1}\mathbf{1}\{t \le \varepsilon\}$, we obtain $h_B(t, r) = 2r$ and

$$w_i = \frac{1}{2\varepsilon r_i} \int_{A_i} \mathrm{d}x = \frac{|A_i|_2}{2\varepsilon r_i},$$

which gives the estimator

$$\widehat{\mathbb{G}} = \bigg(\sum_{i=1}^k \frac{|A_i|_2}{r_i}\bigg)^{-1} \sum_{i=1}^k \frac{|A_i|_2}{r_i} \delta_{r_i}.$$

For example, if $u = e_1$, information in $[0, 1 + \varepsilon] \times [0, 1]$ would be required and the estimation is based on the areas of the regions $A_i \subset [0, 1]^2$, which depend on u and ε. The estimation can be improved by combining the estimators for $u = e_1, -e_1, e_2, -e_2$ which are available if Z is observed in $[-\varepsilon, 1 + \varepsilon]^2$.

In the limit $\varepsilon \to 0$, we obtain the estimator $\widehat{\mathbb{G}}_{l,u}$, $u \in \{\pm u_1, \pm u_2\}$, which is given by

$$\widehat{\mathbb{G}}_{l,u} := \bigg(\sum_{i=1}^k \frac{l_i(u)}{r_i}\bigg)^{-1} \sum_{i=1}^k \frac{l_i(u)}{r_i} \delta_{r_i}.$$

Here $l_i(u)$ is the length of the projection of the visible part of C_i in direction u (projected onto the line orthogonal to u). The combined estimator for $u = e_1, -e_1, e_2, -e_2$ is

$$\widehat{\mathbb{G}}_l := \tfrac{1}{4}(\widehat{\mathbb{G}}_{l,e_1} + \widehat{\mathbb{G}}_{l,-e_1} + \widehat{\mathbb{G}}_{l,e_2} + \widehat{\mathbb{G}}_{l,-e_2}).$$

If we do not have information from outside W then we may use a minus sampling approach and replace W by the eroded window $W_{\ominus\varepsilon} := \{x \in W : x + \varepsilon B \subset W\}$, i.e. we consider the estimator

$$\widehat{\mathbb{G}}_{\ominus\varepsilon}(C) := \frac{\eta_{W_{\ominus\varepsilon}}(C)}{\eta_{W_{\ominus\varepsilon}}(\mathbb{R}^+)}.$$

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Another possibility would be to use the naive approach which ignores edge effects. Then we have the *uncorrected estimator*

$$\widehat{\mathbb{G}}_{\mathrm{u}}(C) := \frac{\eta_{W,\mathrm{u}}(C)}{\eta_{W,\mathrm{u}}(\mathbb{R}^+)},$$

where

$$\eta_{W,\mathrm{u}}(C) := \int_W \mathbf{1}\{r_B(x, Z \cap W) \in C\} \frac{f(d_B(x, Z \cap W))}{h_B(d_B(x, Z \cap W), r_B(x, Z \cap W))} \, \mathrm{d}x.$$

If $B = [0, u]$ then it can happen that $d_B(x, Z \cap W) = \infty$. In that case we use our convention concerning fh_B^{-1}, i.e. the points x satisfying $d_B(x, Z \cap W) = \infty$ do not contribute to $\eta_{W,\mathrm{u}}(C)$. Besides minus sampling there exist more sophisticated methods of edge correction in the statistics of spatial point processes. We adopt the idea of local minus sampling that was originally applied in [7] to the estimation of the nearest-neighbor distance distribution function for stationary point processes (see also [8]). We use only points that are closer to Z than to the boundary of the window W. This gives the *Hanisch-type estimator*

$$\widehat{\mathbb{G}}_{\mathrm{H}}(C) := \frac{\eta_{W,\mathrm{H}}(C)}{\eta_{W,\mathrm{H}}(\mathbb{R}^+)},$$

where

$$\eta_{W,\mathrm{H}}(C) := \int_W \mathbf{1}\{r_B(x, Z) \in C\}\mathbf{1}\{d_B(x, Z) \le d_B(x, \partial W)\} \frac{f(d_B(x, Z))}{h_B(d_B(x, Z), r_B(x, Z))} \, \mathrm{d}x.$$

Note that, for $B = [0, u]$, the estimators $\widehat{\mathbb{G}}_{\mathrm{H}}$ and $\widehat{\mathbb{G}}_{\mathrm{u}}$ coincide.

In numerical implementations we can replace the integration with respect to the Lebesgue measure in (1.4) by an integration with respect to a discrete measure. This still gives a ratio-unbiased estimator of $\mathbb{G}$.

We compare the performance of the different estimators discussed above through computer simulations. We simulate a stationary planar Boolean model with spherical grains, given by (1.1). The observation window W is the unit square $[0, 1]^2$. The distribution $\mathbb{G}$ is assumed to be uniform on $(0.05, 0.1)$. We approximate the integrals over W by Riemannian sums over a rectangular grid of points $L_h \cap W$, where $L_h := \{((k - \frac{1}{2})h, (l - \frac{1}{2})h) : k, l \in \mathbb{N}\}$. For our purposes, we choose $h = \frac{1}{300}$.

We take $f(t) = \varepsilon^{-1}\mathbf{1}\{t \le \varepsilon\}$ for different choices of ε and compare the estimator $\widehat{\mathbb{G}}$, given by (1.7), with the estimators $\widehat{\mathbb{G}}_o$ (for spherical B) and $\widehat{\mathbb{G}}_l$ (for linear B) corresponding to the limiting case $\varepsilon \to 0$. The estimators $\widehat{\mathbb{G}}_{\ominus\varepsilon}$, $\widehat{\mathbb{G}}_{\mathrm{u}}$, and $\widehat{\mathbb{G}}_{\mathrm{H}}$ are also evaluated. For linear $B = [0, u]$, we always combine the corresponding estimators for $u = u_1, -u_1, u_2, -u_2$, which leads to a noticeable improvement.

The radius distribution $\mathbb{G}$ is uniquely determined by the distribution function $G(t) = \mathbb{G}([0, t])$, $t \ge 0$. We measure the quality of the estimators by the Kolmogorov–Smirnov distance

$$d_{\mathrm{KS}}(\widehat{G}, G) := \sup_{s \ge 0} |\widehat{G}(s) - G(s)|$$

and the Cramér–von Mises distance

$$d_{\mathrm{CvM}}(\widehat{G}, G) := \int_{0.05}^{0.1} (\widehat{G}(s) - G(s))^2 \, \frac{\mathrm{d}s}{0.05}.$$

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We generated 1000 independent realizations of the Boolean model Z with chosen intensity γ. For each realization, we determined several estimators under study. The sample means and their standard errors (given in parentheses in the tables) of corresponding Kolmogorov–Smirnov and Cramér–von Mises distances over 1000 simulations are presented in Table 1 for $\gamma = 25$ and Table 2 for $\gamma = 100$. By standard error we understand $s/\sqrt{1000}$, where $s =$

TABLE 1: Sample means and their standard errors (in parentheses) of distances between distribution functions computed from 1000 realizations of a Boolean model with intensity $\gamma = 25$ and uniform radius distribution on $(0.05, 0.1)$. For better readability, we multiply both the means and standard errors by 1000 in the case of the Cramér–von Mises distance.

	d_{KS}		$1000 d_{\mathrm{CvM}}$	
Estimator	Spherical B	Linear B	Spherical B	Linear B
$\widehat{\mathbb{G}},\ \varepsilon = 1$	0.18 (0.002)	0.15 (0.001)	8.36 (0.24)	5.30 (0.15)
$\widehat{\mathbb{G}},\ \varepsilon = 0.05$	0.18 (0.002)	0.17 (0.002)	7.89 (0.23)	7.41 (0.22)
$\widehat{\mathbb{G}},\ \varepsilon = 0.01$	0.17 (0.002)	0.17 (0.002)	7.58 (0.22)	7.53 (0.22)
$\widehat{\mathbb{G}}_o$ or $\widehat{\mathbb{G}}_l$	0.17 (0.002)	0.17 (0.002)	7.50 (0.22)	7.51 (0.22)
$\widehat{\mathbb{G}}_{\ominus\varepsilon},\ \varepsilon = 0.05$	0.20 (0.002)	0.18 (0.002)	9.69 (0.28)	8.04 (0.23)
$\widehat{\mathbb{G}}_{\ominus\varepsilon},\ \varepsilon = 0.01$	0.18 (0.002)	0.17 (0.002)	7.90 (0.23)	7.64 (0.22)
$\widehat{\mathbb{G}}_{\mathrm{u}},\ \varepsilon = 1$	0.19 (0.002)	0.18 (0.002)	9.01 (0.26)	8.39 (0.24)
$\widehat{\mathbb{G}}_{\mathrm{u}},\ \varepsilon = 0.05$	0.18 (0.002)	0.17 (0.002)	8.12 (0.24)	7.69 (0.22)
$\widehat{\mathbb{G}}_{\mathrm{u}},\ \varepsilon = 0.01$	0.17 (0.002)	0.17 (0.002)	7.62 (0.22)	7.58 (0.22)
$\widehat{\mathbb{G}}_{\mathrm{H}},\ \varepsilon = 1$	0.19 (0.002)	0.18 (0.002)	9.52 (0.28)	8.39 (0.24)
$\widehat{\mathbb{G}}_{\mathrm{H}},\ \varepsilon = 0.05$	0.18 (0.002)	0.17 (0.002)	8.54 (0.25)	7.69 (0.22)
$\widehat{\mathbb{G}}_{\mathrm{H}},\ \varepsilon = 0.01$	0.18 (0.002)	0.17 (0.002)	7.73 (0.22)	7.58 (0.22)

TABLE 2: Sample means and their standard errors (in parentheses) of distances between distribution functions computed from 1000 realizations of a Boolean model with intensity $\gamma = 100$ and uniform radius distribution on $(0.05, 0.1)$. For better readability, we multiply both the means and standard errors by 1000 in the case of the Cramér–von Mises distance.

	d_{KS}		$1000 d_{\mathrm{CvM}}$	
Estimator	Spherical B	Linear B	Spherical B	Linear B
$\widehat{\mathbb{G}},\ \varepsilon = 1$	0.14 (0.001)	0.13 (0.001)	4.95 (0.14)	4.14 (0.12)
$\widehat{\mathbb{G}},\ \varepsilon = 0.05$	0.14 (0.001)	0.13 (0.001)	4.82 (0.13)	3.90 (0.11)
$\widehat{\mathbb{G}},\ \varepsilon = 0.01$	0.13 (0.001)	0.12 (0.001)	3.97 (0.11)	3.71 (0.10)
$\widehat{\mathbb{G}}_o$ or $\widehat{\mathbb{G}}_l$	0.12 (0.001)	0.12 (0.001)	3.62 (0.10)	3.62 (0.10)
$\widehat{\mathbb{G}}_{\ominus\varepsilon},\ \varepsilon = 0.05$	0.15 (0.002)	0.13 (0.001)	5.81 (0.16)	4.17 (0.11)
$\widehat{\mathbb{G}}_{\ominus\varepsilon},\ \varepsilon = 0.01$	0.13 (0.001)	0.12 (0.001)	4.11 (0.11)	3.76 (0.10)
$\widehat{\mathbb{G}}_{\mathrm{u}},\ \varepsilon = 1$	0.14 (0.001)	0.14 (0.001)	5.06 (0.14)	4.59 (0.13)
$\widehat{\mathbb{G}}_{\mathrm{u}},\ \varepsilon = 0.05$	0.14 (0.001)	0.13 (0.001)	4.90 (0.14)	4.37 (0.13)
$\widehat{\mathbb{G}}_{\mathrm{u}},\ \varepsilon = 0.01$	0.13 (0.001)	0.13 (0.001)	3.98 (0.11)	3.89 (0.11)
$\widehat{\mathbb{G}}_{\mathrm{H}},\ \varepsilon = 1$	0.14 (0.001)	0.14 (0.001)	5.14 (0.14)	4.59 (0.13)
$\widehat{\mathbb{G}}_{\mathrm{H}},\ \varepsilon = 0.05$	0.14 (0.001)	0.13 (0.001)	5.02 (0.14)	4.37 (0.13)
$\widehat{\mathbb{G}}_{\mathrm{H}},\ \varepsilon = 0.01$	0.13 (0.001)	0.13 (0.001)	4.03 (0.11)	3.89 (0.11)

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$\sqrt{\sum_{i=1}^{1000}(d_i - \bar{d})^2/999}$ is the sample standard deviation. The results show that smaller values of ε are more preferable. The limiting estimators $\widehat{\mathbb{G}}_o$ and $\widehat{\mathbb{G}}_l$ produced the smallest error. They are outperformed only in the case of smaller intensity and linear B where our estimator, given by (1.7), with larger ε, gives better results. However, this estimator also uses information from outside W. Simulation studies for exponentially distributed radii (not presented) show very similar results. A change of resolution h has only a minor influence on the quality of the estimators. For intensity $\gamma \gg 100$, the deviation from the radius distribution increases, which is intuitively clear because many balls are covered so their radii are not available for the estimators.

Appendix A. Proof of Theorem 2.1

Suppose that $x \in W_n$ and $u \in \mathbb{R}^d$. If $x + u \notin W_n$ then $d_{B^d}(x, \partial W_n) \le \|u\|$. Hence, we obtain

$$|W_n|_d - |\{x \in W_n : d_{B^d}(x, \partial W_n) \le \|u\|\}|_d \le |W_n \cap (W_n - u)|_d \le |W_n|_d.$$

Thus, [16, Lemma 10.15(ii)] implies that

$$\frac{|W_n \cap (W_n - u)|_d}{|W_n|_d} \to 1 \quad \text{as } n \to \infty \quad \text{for any } u \in \mathbb{R}^d.$$

Therefore, Lebesgue's dominated convergence theorem and Proposition 2.1 yield (2.11) provided that

$$\int_{\mathbb{R}^d} \tau_1(C, u)\,\mathrm{d}u < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} |\tau_2(C, u)|\,\mathrm{d}u < \infty. \tag{A.1}$$

Using (2.4), we have

$$\begin{aligned}\int_{\mathbb{R}^d} \tau_1(C, u)\,\mathrm{d}u &\le \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(d_B(x, rB^d))}{h_B(d_B(x, rB^d), r)} \frac{f(d_B(y, rB^d))}{h_B(d_B(y, rB^d), r)} \\ &\qquad \times \sqrt{\bar{F}_B(d_B(x, rB^d))\bar{F}_B(d_B(y, rB^d))}\,\mathrm{d}x\,\mathrm{d}y\,\mathbb{G}(\mathrm{d}r) \\ &= \int_C \bigg(\int_{\mathbb{R}^d} \frac{f(d_B(x, rB^d))}{h_B(d_B(x, rB^d), r)} \sqrt{\bar{F}_B(d_B(x, rB^d))}\,\mathrm{d}x\bigg)^2 \mathbb{G}(\mathrm{d}r).\end{aligned}$$

An application of (2.10) shows that

$$\int_{\mathbb{R}^d} \frac{f(d_B(x, rB^d))}{h_B(d_B(x, rB^d), r)} \sqrt{\bar{F}_B(d_B(x, rB^d))}\,\mathrm{d}x = \int_0^\infty f(t)\sqrt{\bar{F}_B(t)}\,\mathrm{d}t,$$

and, thus, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \tau_1(C, u)\,\mathrm{d}u &\le \int_C \bigg(\int_0^\infty f(t)\sqrt{\bar{F}_B(t)}\,\mathrm{d}t\bigg)^2 \mathbb{G}(\mathrm{d}r) \\ &\le \mathbb{G}(C)\bigg(\int_0^\infty f(t)\,\mathrm{e}^{-2ct}\,\mathrm{d}t\bigg)^2 \\ &< \infty,\end{aligned}$$

where we have used the fact that $\bar{F}_B(t) \le \mathrm{e}^{-4ct}$ and assumption (2.5).

In order to show the second inequality in (A.1), we first rewrite $q(u; x_1, x_2, r_1, r_2)$ as the difference of two nonnegative terms, that is, $q = q_1 - q_2$ with

$$\begin{aligned} q_1(u; x_1, x_2, r_1, r_2) := \bar{F}_B^{(2)}(u; d_B(x_1, B(\mathrm{o}, r_1)), d_B(x_2, B(\mathrm{o}, r_2))) \\ - \bar{F}_B(d_B(x_1, B(\mathrm{o}, r_1)))\bar{F}_B(d_B(x_2, B(\mathrm{o}, r_2))), \end{aligned}$$

which is nonnegative by (2.3), and

$$\begin{aligned} q_2(u; x_1, x_2, r_1, r_2) := \bar{F}_B^{(2)}(u; d_B(x_1, B(\mathrm{o}, r_1)), d_B(x_2, B(\mathrm{o}, r_2))) \\ \times (1 - \mathbf{1}\{d_B(x_2, B(u, r_2)) > d_B(x_1, B(\mathrm{o}, r_1))\} \\ \times \mathbf{1}\{d_B(x_1, B(-u, r_1)) > d_B(x_2, B(\mathrm{o}, r_2))\}), \end{aligned}$$

for $u, x_1, x_2 \in \mathbb{R}^d$ and $r_1, r_2 \in \mathbb{R}^+$. Using (2.2), (2.4), and the inequality $1 - \mathrm{e}^{-a} \le a$ for $a \ge 0$, we obtain

$$\begin{aligned} q_1(u; x_1, x_2, r_1, r_2) &\le \bar{F}_B^{(2)}(u; d_B(x_1, B(\mathrm{o}, r_1)), d_B(x_2, B(\mathrm{o}, r_2))) \\ &\quad \times (1 - \exp\{-\gamma \mathbb{E}\kappa_B(u; d_B(x_1, B(\mathrm{o}, r_1)), d_B(x_2, B(\mathrm{o}, r_2)), R)\}) \\ &\le \sqrt{\bar{F}_B(d_B(x_1, B(\mathrm{o}, r_1)))\bar{F}_B(d_B(x_2, B(\mathrm{o}, r_2)))} \\ &\quad \times \gamma \mathbb{E}\kappa_B(u; d_B(x_1, B(\mathrm{o}, r_1)), d_B(x_2, B(\mathrm{o}, r_2)), R). \end{aligned}$$

Moreover, the inequality $1 - (1 - a)(1 - b) \le a + b$ for $a, b \ge 0$ and again (2.4) imply that

$$\begin{aligned} q_2(u; x_1, x_2, r_1, r_2) \le \sqrt{\bar{F}_B(d_B(x_1, B(\mathrm{o}, r_1)))\bar{F}_B(d_B(x_2, B(\mathrm{o}, r_2)))} \\ \times (\mathbf{1}\{d_B(x_2, B(u, r_2)) \le d_B(x_1, B(\mathrm{o}, r_1))\} \\ + \mathbf{1}\{d_B(x_1, B(-u, r_1)) \le d_B(x_2, B(\mathrm{o}, r_2))\}). \end{aligned}$$

Combining these bounds, we arrive at

$$\begin{aligned} \int_{\mathbb{R}^d} |\tau_2(C, u)| \,\mathrm{d}u \le \int_C \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & \frac{f(d_B(x_1, B(\mathrm{o}, r_1)))}{h_B(d_B(x_1, B(\mathrm{o}, r_1)), r_1)} \frac{f(d_B(x_2, B(\mathrm{o}, r_2)))}{h_B(d_B(x_2, B(\mathrm{o}, r_2)), r_2)} \\ & \times \sqrt{\bar{F}_B(d_B(x_1, B(\mathrm{o}, r_1)))\bar{F}_B(d_B(x_2, B(\mathrm{o}, r_2)))} \\ & \times [\gamma \mathbb{E}\kappa_B(u; d_B(x_1, B(\mathrm{o}, r_1)), d_B(x_2, B(\mathrm{o}, r_2)), R) \\ & \quad + \mathbf{1}\{d_B(x_2, B(u, r_2)) \le d_B(x_1, B(\mathrm{o}, r_1))\} \\ & \quad + \mathbf{1}\{d_B(x_1, B(-u, r_1)) \le d_B(x_2, B(\mathrm{o}, r_2))\}] \\ & \times \mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}u \,\mathbb{G}(\mathrm{d}r_1)\,\mathbb{G}(\mathrm{d}r_2). \end{aligned}$$

The above expression splits naturally into three summands which will be bounded from above separately. For the first bound, we observe that, by Fubini's theorem,

$$\mathbb{E}\int_{\mathbb{R}^d} \kappa_B(u; s_1, s_2, R)\,\mathrm{d}u = \mathbb{E}|B_{s_1,R}|_d |B_{s_2,R}|_d.$$

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Then we apply (2.10) to obtain

$$\begin{aligned}\int_C \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & \frac{f(d_B(x_1, B(o, r_1)))}{h_B(d_B(x_1, B(o, r_1)), r_1)} \frac{f(d_B(x_2, B(o, r_2)))}{h_B(d_B(x_2, B(o, r_2)), r_2)} \\ & \times \sqrt{\bar{F}_B(d_B(x_1, B(o, r_1)))\bar{F}_B(d_B(x_2, B(o, r_2)))} \\ & \times \gamma \mathbb{E}\kappa_B(u; d_B(x_1, B(o, r_1)), d_B(x_2, B(o, r_2)), R) \\ & \times \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}u \,\mathbb{G}(\mathrm{d}r_1) \,\mathbb{G}(\mathrm{d}r_2) \\ = \gamma \,\mathbb{G}(C)^2 \int_0^\infty \int_0^\infty & f(t_1)\sqrt{\bar{F}_B(t_1)} f(t_2)\sqrt{\bar{F}_B(t_2)}\, \mathbb{E}|B_{t_1,R}|_d |B_{t_2,R}|_d \,\mathrm{d}t_1 \,\mathrm{d}t_2.\end{aligned}$$

Choose $c_B > 0$ such that $B \subset c_B B^d$. Then $|B_{t,R}|_d \le \kappa_d(c_B t + R)^d$ and, hence, the Cauchy–Schwarz inequality, the convexity of $s \mapsto s^p$, $p \ge 1$, and $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, $a, b \ge 0$, yield

$$\begin{aligned}\mathbb{E}|B_{t_1,R}|_d |B_{t_2,R}|_d &\le c_1\sqrt{\mathbb{E}(c_B t_1 + R)^{2d}}\sqrt{\mathbb{E}(c_B t_2 + R)^{2d}} \\ &\le c_2(c_B^d t_1^d + \sqrt{\mathbb{E}R^{2d}})(c_B^d t_2^d + \sqrt{\mathbb{E}R^{2d}}) \\ &\le c_3(1 + t_1^d)(1 + t_2^d),\end{aligned}$$

where c_1, c_2, and c_3 denote finite constants independent of the expectation or t_1 and t_2. From this and (2.5), it follows again that the first summand is finite.

Since $d_B(x_2, B(u, r_2)) \le t_1$ if and only if $u \in x_2 + B_{t_1,r_2}$, applying Fubini's theorem and (2.10) (twice) we obtain, for the second summand,

$$\begin{aligned}\int_C \int_C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & \frac{f(d_B(x_1, B(o, r_1)))}{h_B(d_B(x_1, B(o, r_1)), r_1)} \frac{f(d_B(x_2, B(o, r_2)))}{h_B(d_B(x_2, B(o, r_2)), r_2)} \\ & \times \sqrt{\bar{F}_B(d_B(x_1, B(o, r_1)))\bar{F}_B(d_B(x_2, B(o, r_2)))} \\ & \times \mathbf{1}\{d_B(x_2, B(u, r_2)) \le d_B(x_1, B(o, r_1))\} \\ & \times \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}u \,\mathbb{G}(\mathrm{d}r_1) \,\mathbb{G}(\mathrm{d}r_2) \\ = \int_C \int_C \int_0^\infty \int_0^\infty & f(t_1)\sqrt{\bar{F}_B(t_1)} f(t_2)\sqrt{\bar{F}_B(t_2)} |B_{t_1,r_2}|_d \,\mathrm{d}t_1 \,\mathrm{d}t_2 \,\mathbb{G}(\mathrm{d}r_1) \,\mathbb{G}(\mathrm{d}r_2),\end{aligned}$$

which is finite by the same reasoning as above.

The third summand can be treated in exactly the same way.

To prove positivity of the asymptotic variance, we use the fact that the variance of any square-integrable function $H(\Psi)$ of the Poisson process $\Psi := \{(\xi_n, R_n) \colon n \ge 1\}$ satisfies the inequality

$$\operatorname{var} H(\Psi) \ge \gamma \int_0^\infty \int_{\mathbb{R}^d} (\mathbb{E}[H(\Psi \cup \{(y, r)\}) - H(\Psi)])^2 \,\mathrm{d}y \,\mathbb{G}(\mathrm{d}r);$$

see, e.g. [18, Theorem 4.2]. In our case this means that

$$\operatorname{var} \eta_W(C) \ge \gamma \int_0^\infty \int_{\mathbb{R}^d} \tilde{h}(y, r)^2 \,\mathrm{d}y \,\mathbb{G}(\mathrm{d}r), \tag{A.2}$$

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where

$$\begin{aligned}\tilde{h}(y, r) &:= \mathbb{E} \int_W [g(d_B(x, Z \cup B(y, r)), r_B(x, Z \cup B(y, r))) - g(d_B(x, Z), r_B(x, Z))] \, \mathrm{d}x \\ &= \mathbb{E} \int_W \mathbf{1}\{d_B(x, B(y, r)) < d_B(x, Z)\} \\ &\qquad \times [g(d_B(x, B(y, r)), r) - g(d_B(x, Z), r_B(x, Z))] \, \mathrm{d}x \\ &= \mathbb{E} \int_W \mathbf{1}\{d_B(\mathrm{o}, B(y - x, r)) < d_B(\mathrm{o}, Z)\} \\ &\qquad \times [g(d_B(\mathrm{o}, B(y - x, r)), r) - g(d_B(\mathrm{o}, Z), r_B(\mathrm{o}, Z))] \, \mathrm{d}x.\end{aligned}$$

Here the last identity follows from the stationarity of Z and g is as defined in (2.9). By (1.3),

$$\begin{aligned}\tilde{h}(y, r) = \gamma \int_W \int_0^\infty \int_0^\infty & \mathbf{1}\{d_B(\mathrm{o}, B(y - x, r)) < t\}[g(d_B(\mathrm{o}, B(y - x, r)), r) - g(t, s)] \\ & \times h_B(t, s) \bar{F}_B(t) \, \mathrm{d}t \, \mathbb{G}(\mathrm{d}s) \, \mathrm{d}x.\end{aligned}$$

Assume now that $0 < \mathbb{G}(C) < 1$, and let $\bar{C} := \mathbb{R}^+ \setminus C$. Recalling the definition of g given in (2.9), from (A.2) we obtain

$$\operatorname{var} \eta_W(C) \geq \gamma \int_0^\infty \int_{\mathbb{R}^d} h^*(y, r)^2 \mathbf{1}\{r \in \bar{C}, \ y \in W\} \, \mathrm{d}y \, \mathbb{G}(\mathrm{d}r),$$

where

$$\begin{aligned}h^*(y, r) &:= \gamma \int_W \int_0^\infty \int_0^\infty \mathbf{1}\{d_B(\mathrm{o}, B(y - x, r)) < t\} g(t, s) h_B(t, s) \bar{F}_B(t) \, \mathrm{d}t \, \mathbb{G}(\mathrm{d}s) \, \mathrm{d}x \\ &= \gamma \, \mathbb{G}(C) \int_W \int_0^\infty \mathbf{1}\{d_B(\mathrm{o}, B(y - x, r)) < t\} f(t) \bar{F}_B(t) \, \mathrm{d}t \, \mathrm{d}x.\end{aligned}$$

Applying Jensen's inequality with the normalization of $\mathbf{1}\{r \in \bar{C}, \ y \in W\} \, \mathrm{d}y \, \mathbb{G}(\mathrm{d}r)$, we obtain

$$\operatorname{var} \eta_W(C) \geq \frac{\gamma}{\mathbb{G}(\bar{C}) |W|_d} \left(\int_{\bar{C}} \int_W h^*(y, r) \, \mathrm{d}y \, \mathbb{G}(\mathrm{d}r) \right)^2.$$

Letting $a := \gamma^3 \mathbb{G}(C)^2 / \mathbb{G}(\bar{C}) > 0$ we obtain

$$\begin{aligned}&\frac{\operatorname{var} \eta_W(C)}{|W|_d} \\ &\quad \geq \frac{a}{|W|_d^2} \left(\int_{\bar{C}} \int_W \int_W \int_0^\infty \mathbf{1}\{d_B(\mathrm{o}, B(y - x, r)) < t\} f(t) \bar{F}_B(t) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}y \, \mathbb{G}(\mathrm{d}r) \right)^2 \\ &\quad = \frac{a}{|W|_d^2} \left(\int_{\bar{C}} \int_{\mathbb{R}^d} \int_0^\infty |W \cap (W - y)|_d \mathbf{1}\{d_B(\mathrm{o}, B(y, r)) < t\} f(t) \bar{F}_B(t) \, \mathrm{d}t \, \mathrm{d}y \, \mathbb{G}(\mathrm{d}r) \right)^2.\end{aligned}$$

Hence, it is sufficient to show that

$$\int_{\bar{C}} \int_0^\infty \left(\int_{\mathbb{R}^d} \mathbf{1}\{d_B(\mathrm{o}, B(y, r)) < t\} \, \mathrm{d}y \right) f(t) \bar{F}_B(t) \, \mathrm{d}t \, \mathbb{G}(\mathrm{d}r) > 0.$$

This is true since the inner integral is positive for all $r, t > 0$, and since both $\mathbb{G}(\bar{C})$ and $\int_0^\infty f(t) \bar{F}_B(t) \, \mathrm{d}t$ are positive.

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Appendix B. Proof of Theorem 3.1

We fix a Borel set $C \subset \mathbb{R}^+$ and skip the dependence on C in the notation. Let $E_z := [0, 1)^d + z$ for $z \in I_n := \{-n, \dots, n-1\}^d$. Then

$$\eta_{W_n} = \sum_{z\in I_n} \eta_{E_z} = \sum_{z\in I_n} \int_{E_z} g(d_B(x, Z), r_B(x, Z))\,\mathrm{d}x,$$

where g is given by (2.9). For some fixed integer m, we put $F_z := E_z \oplus [-m, m)^d$. We decompose η_{E_z} into the two random variables

$$\eta_z^{(m)} := \int_{E_z} g(d_B(x, Z(F_z)), r_B(x, Z(F_z)))\,\mathrm{d}x$$

and $\tilde\eta_z^{(m)} := \eta_{E_z} - \eta_z^{(m)}$. Let $\eta_{W_n}^{(m)} := \sum_{z\in I_n} \eta_z^{(m)}$ and $\tilde\eta_{W_n}^{(m)} := \sum_{z\in I_n} \tilde\eta_z^{(m)}$ so that $\eta_{W_n} = \eta_{W_n}^{(m)} + \tilde\eta_{W_n}^{(m)}$. It is easily seen that $\{\eta_u^{(m)} : u \in U\}$ and $\{\eta_v^{(m)} : v \in V\}$ are independent whenever $U, V \subset \mathbb{Z}^d$ are such that $\|u-v\|_\infty > 2m$ for each $u \in U$ and $v \in V$. Thus, the random variables $\eta_z^{(m)}$ for $z \in \mathbb{Z}^d$ constitute a stationary $2m$-dependent random field (cf. [10, Section 4.3.1]). The variance of $\eta_{W_n}^{(m)}$ is

$$\begin{aligned}\operatorname{var}\eta_{W_n}^{(m)} &= \operatorname{var}\sum_{z\in I_n}\eta_z^{(m)}\\ &= \sum_{z_1\in I_n}\sum_{z_2\in I_n}\operatorname{cov}(\eta_{z_1}^{(m)}, \eta_{z_2}^{(m)})\\ &= \sum_{z\in I_n-I_n} N_n(z)\operatorname{cov}(\eta_{\mathrm{o}}^{(m)}, \eta_z^{(m)}),\end{aligned}$$

where $N_n(z)$ is the cardinality of $\{(z_1, z_2) \in I_n \times I_n : z_2 - z_1 = z\}$, which may be bounded by $|W_n|_d = (2n)^d$ and $\lim_{n\to\infty} N_n(z)/|W_n|_d = 1$ for any $z \in \mathbb{Z}^d$. We define

$$(\sigma_n^{(m)})^2 := \frac{\operatorname{var}\eta_{W_n}^{(m)}}{|W_n|_d}.$$

Since $\operatorname{cov}(\eta_{\mathrm{o}}^{(m)}, \eta_z^{(m)}) = 0$ for $\|z\| > 2m$, the limit of $(\sigma_n^{(m)})^2$ as $n \to \infty$ exists and satisfies

$$(\sigma^{(m)})^2 := \lim_{n\to\infty}(\sigma_n^{(m)})^2 = \sum_{z\in\{-2m,\dots,2m\}^d}\operatorname{cov}(\eta_{\mathrm{o}}^{(m)}, \eta_z^{(m)}). \tag{B.1}$$

Next we show that $\mathbb{E}(\eta_{\mathrm{o}}^{(m)})^2 < \infty$. We put $A := [-m, m+1)^d$; hence,

$$\eta_{\mathrm{o}}^{(m)} = \int_{E_{\mathrm{o}}} g(d_B(x, Z(A)), r_B(x, Z(A)))\,\mathrm{d}x.$$

Proceeding as in the proof of Proposition 2.1 and bounding $\bar F_B^{A,A}(\cdot)$ as well as $(1 - \mathbf{1}\{\cdot\}\mathbf{1}\{\cdot\})$ by 1, we obtain

$$\begin{aligned}\mathbb{E}(\eta_{\mathrm{o}}^{(m)})^2 \le{}& \gamma\int_{E_{\mathrm{o}}}\int_{E_{\mathrm{o}}}\int_0^\infty\int_A g(d_B(y, B(x_1, r)), r)g(d_B(y, B(x_2, r)), r)\,\mathrm{d}y\,\mathbb{G}(\mathrm{d}r)\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &+\gamma^2\int_{E_{\mathrm{o}}}\int_{E_{\mathrm{o}}}\int_0^\infty\int_0^\infty\int_A\int_A g(d_B(y_1, B(x_1, r_1)), r_1)g(d_B(y_2, B(x_2, r_2)), r_2)\\ &\qquad\times\mathrm{d}y_1\,\mathrm{d}y_2\,\mathbb{G}(\mathrm{d}r_1)\,\mathbb{G}(\mathrm{d}r_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

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The right-hand side increases if A is replaced by $\mathbb{R}^d$. Arguing then as in the proof of Theorem 2.1, we obtain

$$\mathbb{E}(\eta_{\mathrm{o}}^{(m)})^2 \leq \gamma |E_{\mathrm{o}}|_d \bigg(\int_0^\infty f(t)\,\mathrm{d}t\bigg)^2 \mathbb{G}(C) + \bigg(\gamma |E_{\mathrm{o}}|_d \int_0^\infty f(t)\,\mathrm{d}t\, \mathbb{G}(C)\bigg)^2 < \infty.$$

Therefore, the central limit theorem for stationary m-dependent random fields (see, e.g. [22]) yields

$$\frac{1}{\sqrt{|W_n|_d}} \sum_{z\in I_n} (\eta_z^{(m)} - \mathbb{E}\eta_z^{(m)}) \xrightarrow{\mathrm{D}} N(0, (\sigma^{(m)})^2) \quad \text{as } n \to \infty.$$

In view of [2, Theorem 3.2], it remains to verify that

$$\lim_{m\to\infty} \sigma^{(m)} = \sigma(C) \tag{B.2}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\bigg(\frac{1}{\sqrt{|W_n|_d}} \bigg|\sum_{z\in I_n} (\tilde\eta_z^{(m)} - \mathbb{E}\tilde\eta_z^{(m)})\bigg| \geq \varepsilon\bigg) = 0 \quad \text{for any } \varepsilon > 0. \tag{B.3}$$

Define

$$\sigma_n^2 := \frac{\operatorname{var} \eta_{W_n}}{|W_n|_d}.$$

Then

$$|\sigma(C) - \sigma^{(m)}| \leq |\sigma(C) - \sigma_n| + |\sigma_n - \sigma_n^{(m)}| + |\sigma_n^{(m)} - \sigma^{(m)}|.$$

The first term goes to 0 as $n \to \infty$ by Theorem 2.1, and the last term goes to 0 as $n \to \infty$ as well for any $m \in \mathbb{N}$, by (B.1). By Minkowski's inequality, the middle term can be bounded as

$$|\sigma_n - \sigma_n^{(m)}| \leq \frac{1}{\sqrt{|W_n|_d}} \sqrt{\operatorname{var} \tilde\eta_{W_n}^{(m)}}.$$

Therefore, (B.2) follows if we can show that

$$\sup_{n\in\mathbb{N}} \frac{1}{|W_n|_d} \operatorname{var} \tilde\eta_{W_n}^{(m)} \to 0 \quad \text{as } m \to \infty. \tag{B.4}$$

By Chebyshev's inequality, (B.4) also implies (B.3). The variance in (B.4) satisfies

$$\frac{1}{|W_n|_d} \operatorname{var} \sum_{z\in I_n} \tilde\eta_z^{(m)} = \frac{1}{|W_n|_d} \sum_{z\in I_n - I_n} N_n(z) \operatorname{cov}(\tilde\eta_{\mathrm{o}}^{(m)}, \tilde\eta_z^{(m)}) \leq \sum_{z\in\mathbb{Z}^d} |\operatorname{cov}(\tilde\eta_{\mathrm{o}}^{(m)}, \tilde\eta_z^{(m)})|.$$

Therefore, the proof will be complete if we show that

$$\sum_{z\in\mathbb{Z}^d} |\operatorname{cov}(\tilde\eta_{\mathrm{o}}^{(m)}, \tilde\eta_z^{(m)})| \to 0 \quad \text{as } m \to \infty. \tag{B.5}$$

Consider a fixed $z \in \mathbb{Z}^d$. Then the covariance can be written as

$$\begin{aligned}
&\operatorname{cov}(\tilde\eta_{\mathrm{o}}^{(m)}, \tilde\eta_z^{(m)}) \\
&\quad = \operatorname{cov}(\eta_{E_{\mathrm{o}}}, \eta_{E_z}) - \operatorname{cov}(\eta_{\mathrm{o}}^{(m)}, \eta_{E_z}) - \operatorname{cov}(\eta_{E_{\mathrm{o}}}, \eta_z^{(m)}) + \operatorname{cov}(\eta_{\mathrm{o}}^{(m)}, \eta_z^{(m)}) \\
&\quad = \int_{E_{\mathrm{o}}} \int_{E_z} [c_{\mathbb{R}^d,\mathbb{R}^d}(x_1, x_2) - c_{F_{\mathrm{o}},\mathbb{R}^d}(x_1, x_2) - c_{\mathbb{R}^d,F_z}(x_1, x_2) + c_{F_{\mathrm{o}},F_z}(x_1, x_2)]\,\mathrm{d}x_2\,\mathrm{d}x_1,
\end{aligned}$$

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where, for Borel sets $A_1, A_2 \subset \mathbb{R}^d$ and $x_1, x_2 \in \mathbb{R}^d$,

$$c_{A_1,A_2}(x_1, x_2) := \operatorname{cov}(g(d_B(x_1, Z(A_1)), r_B(x_1, Z(A_1))), g(d_B(x_2, Z(A_2)), r_B(x_2, Z(A_2))))$$

is expressed in Lemma 2.1 as

$$\begin{aligned} c_{A_1,A_2}(x_1, x_2) &= \gamma \int_0^\infty \int_{\mathbb{R}^d} g(d_B(x_1, B(y, r)), r) g(d_B(x_2, B(y, r)), r) I_1(A_1, A_2)\, \mathrm{d}y\, \mathbb{G}(\mathrm{d}r) \\ &\quad + \gamma^2 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(d_B(x_1, B(y_1, r_1)), r_1) g(d_B(x_2, B(y_2, r_2)), r_2) \\ &\qquad \times I_2(A_1, A_2)\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathbb{G}(\mathrm{d}r_1)\, \mathbb{G}(\mathrm{d}r_2). \end{aligned}$$

Here we skip the arguments x_1, x_2, y, r and $x_1, x_2, y_1, y_2, r_1, r_2$ of the functions $I_1(A_1, A_2)$ and $I_2(A_1, A_2)$, respectively, which were defined before Lemma 2.1. It is natural to treat both parts of $c_{A_1,A_2}(x_1, x_2)$ separately. For this, we define

$$\begin{aligned} S_1 := \sum_{z \in \mathbb{Z}^d} \int_{E_\mathrm{o}} \int_{E_z} \int_0^\infty \int_{\mathbb{R}^d} & g(d_B(x_1, B(y, r)), r) g(d_B(x_2, B(y, r)), r) \\ & \times |I_1(\mathbb{R}^d, \mathbb{R}^d) - I_1(F_\mathrm{o}, \mathbb{R}^d) - I_1(\mathbb{R}^d, F_z) + I_1(F_\mathrm{o}, F_z)| \\ & \times \mathrm{d}y\, \mathbb{G}(\mathrm{d}r)\, \mathrm{d}x_2\, \mathrm{d}x_1 \end{aligned}$$

and

$$\begin{aligned} S_2 := \sum_{z \in \mathbb{Z}^d} \int_{E_\mathrm{o}} \int_{E_z} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & g(d_B(x_1, B(y_1, r_1)), r_1) g(d_B(x_2, B(y_2, r_2)), r_2) \\ & \times |I_2(\mathbb{R}^d, \mathbb{R}^d) - I_2(F_\mathrm{o}, \mathbb{R}^d) - I_2(\mathbb{R}^d, F_z) + I_2(F_\mathrm{o}, F_z)| \\ & \times \mathrm{d}y_1\, \mathrm{d}y_2\, \mathbb{G}(\mathrm{d}r_1)\, \mathbb{G}(\mathrm{d}r_2)\, \mathrm{d}x_2\, \mathrm{d}x_1. \end{aligned}$$

Observe that S_1 and S_2 depend on m via the dependence of F_o and F_z on m. It is possible to prove that both S_1 and S_2 tend to 0 as $m \to \infty$. For a detailed argument, which requires a careful distinction of several cases, we refer the reader to [15]. This shows that (B.5) holds, which completes the proof of Theorem 3.1.

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References

[1] Ballani, F. (2006). On second-order characteristics of germ-grain models with convex grains. *Mathematika* **53,** 255–285.
[2] Billingsley, P. (1999). *Convergence of Probability Measures*, 2nd edn. John Wiley, New York.
[3] Chiu, S. N., Stoyan, D., Kendall, W. S. and Mecke, J. (2013). *Stochastic Geometry and Its Applications*, 3rd edn. John Wiley, Chichester.

[4] Emery, X., Kracht, W., Egaña, Á. and Garrido, F. (2012). Using two-point set statistics to estimate the diameter distribution in Boolean models with circular grains. *Math. Geosci.* **44,** 805–822.
[5] Gille, W. (1995). Diameter distribution of spherical primary grains in the Boolean model from small-angle scattering. *Part. Part. Syst. Charact.* **12,** 123–131.
[6] Hall, P. (1988). *Introduction to the Theory of Coverage Processes*. John Wiley, New York.
[7] Hanisch, K.-H. (1984). Some remarks on estimators of the distribution function of nearest neighbour distance in stationary spatial point processes. *Math. Operationsforsch. Statist. Ser. Statist.* **15,** 409–412.
[8] Hansen, M. B., Baddeley, A. J. and Gill, R. D. (1999). First contact distributions for spatial patterns: regularity and estimation. *Adv. Appl. Prob.* **31,** 15–33.
[9] Heinrich, L. (1993). Asymptotic properties of minimum contrast estimators for parameters of Boolean models. *Metrika* **40,** 67–94.
[10] Heinrich, L. (2013). Asymptotic methods in statistics of random point processes. In *Stochastic Geometry, Spatial Statistics and Random Fields* (Lecture Notes Math. **2068**), Springer, Heidelberg, pp. 115–150.
[11] Heinrich, L. and Molchanov, I. S. (1999). Central limit theorem for a class of random measures associated with germ-grain models. *Adv. Appl. Prob.* **31,** 283–314.
[12] Heinrich, L. and Werner, M. (2000). Kernel estimation of the diameter distribution in Boolean models with spherical grains. *J. Nonparametr. Statist.* **12,** 147–176.
[13] Hug, D. and Last, G. (2000). On support measures in Minkowski spaces and contact distributions in stochastic geometry. *Ann. Prob.* **28,** 796–850.
[14] Hug, D., Last, G. and Weil, W. (2002). Generalized contact distributions of inhomogeneous Boolean models. *Adv. Appl. Prob.* **34,** 21–47.
[15] Hug, D., Last, G., Pawlas, Z. and Weil, W. (2013). Statistics for Poisson models of overlapping spheres. Preprint. Available at http://uk.arxiv.org/abs/1301.1499v1.
[16] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd edn. Springer, New York.
[17] Kiderlen, M. and Weil, W. (1999). Measure-valued valuations and mixed curvature measures of convex bodies. *Geom. Dedicata* **76,** 291–329.
[18] Last, G. and Penrose, M. D. (2011). Poisson process Fock space representation, chaos expansion and covariance inequalities. *Prob. Theory Relat. Fields* **150,** 663–690.
[19] Molchanov, I. S. (1990). Estimation of the size distribution of spherical grains in the Boolean model. *Biometrical J.* **32,** 877–886.
[20] Molchanov, I. S. (1997). *Statistics of the Boolean Model for Practitioners and Mathematicians*. John Wiley, Chichester.
[21] Molchanov, I. and Stoyan, D. (1994). Asymptotic properties of estimators for parameters of the Boolean model. *Adv. Appl. Prob.* **26,** 301–323.
[22] Rosén, B. (1969). A note on asymptotic normality of sums of higher-dimensionally indexed random variables. *Ark. Mat.* **8,** 33–43.
[23] Schneider, R. (2013). *Convex Bodies: The Brunn-Minkowski Theory*. 2nd edition. Cambridge University Press.
[24] Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin.
[25] Thovert, J.-F. and Adler, P. M. (2011). Grain reconstruction of porous media: application to a Bentheim sandstone. *Phys. Rev. E* **83,** 056116.
[26] Thovert, J.-F. *et al.* (2001). Grain reconstruction of porous media: application to a low-porosity Fontainebleau sandstone. *Phys. Rev. E* **63,** 061307.