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A GENERALIZED COMPARISON TEST

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Let $\sum c_j$ and $\sum d_j$ be, respectively, convergent and divergent series of positive terms and let $\sum a_j$ be a third series of positive terms. It is well known, [1, pg. 275] that $\sum a_j$ converges if $\limsup(a_j/c_j) < +\infty$, but diverges if $\liminf(a_j/d_j) > 0$. In this note we prove a generalized version of this comparison test that relies not on term-by-term comparison of the series, but on the relative densities of the terms of the series.

DEFINITION 1. Let $\{a_j\}_{j=1}^{\infty}$ be a null sequence of positive terms, let $x > 0$ and define

$$D(a, x) = \#\{j : a_j \geq x\}.$$

If $\{b_j\}_{j=1}^{\infty}$ is also a null sequence of positive numbers, we define

$$\bar{D}(b; a) = \overline{\lim_{x \to 0^+}} \frac{D(b, x)}{D(a, x)}$$

to be the upper density of $\{b_j\}$ relative to $\{a_j\}$. We take

$$\underline{D}(b; a) = \underline{\lim_{x \to 0^+}} \frac{D(b, x)}{D(a, x)}$$

to be the lower density of $\{b_j\}$ relative to $\{a_j\}$.

THEOREM 2. *Let $\sum c_j$ and $\sum d_j$ be, respectively, convergent and divergent series of positive terms (with $d_k \to 0$) and let $\sum a_j$ be a series of positive terms. Then*
(i) *if $\bar{D}(a; c) < +\infty$, then $\sum a_j$ converges.*
(ii) *if $\underline{D}(a; d) > 0$, then $\sum a_j$ diverges.*

Proof. We prove only (i) since the proof of (ii) is similar. We may assume, without loss of generality, that $\{a_j\}$ and $\{c_j\}$ are nonincreasing sequences. Let $\bar{D}(a; c) = d$. Then there is a positive number ϵ such that $0 < x < \epsilon$ implies $D(a, x) \leq ([d]+1)D(c, x)$. Let c_{k_0} be the first element of $\{c_j\}$ that does not exceed ϵ. Then

$$D(a, c_{k_0}) \leq ([d]+1)D(c, c_{k_0}). \tag{1}$$

If strict inequality holds in (1), alter $\{a_j\}$ as follows. Let A_{n_0} be the set of the first $([d]+1)D(c, c_{k_0}) = n_0$ elements of $\{a_j\}$. If $a_j \in A_{n_0}$, and $a_j \geq c_{k_0}$, do not alter

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a_{ij} if $a_j < c_{k_0}$ replace a_j by c_{k_0}. This procedure gives a new sequence $\{a_j^0\}$ that differs from $\{a_j\}$ in at most finitely many places. (If equality holds in (1), then no alterations take place and $\{a_j^0\} = \{a_j\}$).

It is easy to verify the following facts concerning $\{a_j\}$:

$$\sum_{j=1}^{n_0} a_j \le \sum_{j=1}^{n_0} a_j^0 \le \sum_{a_j > c_{k_0}} a_j + n_0 c_{k_0} \tag{2}$$

$$D(a^0, c_{k_0}) = ([d]+1) D(c, c_{k_0}) \tag{3}$$

$$a_j^0 = a_j < c_{k_0} \qquad (j > n_0) \tag{4}$$

$$D(a^0, x) \le ([d]+1) D(c, x) \qquad (0 < x < c_{k_0}) \tag{5}$$

We now describe an induction step. Assume that for non-negative integer r we have produced a sequence $\{a_j^{(r)}\}$ and three sequences of non-negative integers $0 < k_0 < k_1 < \cdots < k_r$; $0 < n_0 < n_1 < n_2 < \cdots < n_r$ and $0 = m_0, \ m_1, \ldots, m_r$ so that, analogous to (2)–(5) we have

$$\sum_{j=1}^{n_r} a_j \le \sum_{j=1}^{n_r} a_j^{(r)} \le \sum_{a_j > c_{k_0}} a_j + n_0 c_{k_0} + ([d]+1) \sum_{j=0}^{r} m_j c_{k_j} \tag{6}$$

$$D(a^{(r)}, c_{k_r}) = ([d]+1) D(c, c_{k_r}) \tag{7}$$

$$a_j^{(r)} = a_j < c_{k_r} \qquad (j > n_r) \tag{8}$$

$$D(a^{(r)}, x) \le ([d]+1) D(c, x) \qquad (0 < x < c_{k_r}) \tag{9}$$

We take $c_{k_{r+1}}$ to be the first element in $\{c_j\}$ that is less than c_{k_r}. By (9),

$$D(a^{(r)}, c_{k_{r+1}}) \le ([d]+1) D(c, c_{k_{r+1}})$$

Since $D(c, c_{k_{r+1}}) - D(c, c_{k_r}) = m_{r+1} =$ number of occurrences of $c_{k_{r+1}}$ in $\{c_j\}$, we have by (7) and (9)

$$D(a^{(r)}, c_{k_{r+1}}) - D(a^{(r)}, c_{k_r}) \le ([d]+1) m_{r+1} \tag{10}$$

By (7), (8) and (9), the only terms of $\{a_j^{(r)}\}$ counted in (10) are those, if any, with $a_j^{(r)} = c_{k_{r+1}}$. We form $\{a_j^{(r+1)}\}$ by altering $\{a_j^{(r)}\}$. If $j \le n_r$ or $j > n_{r+1} = n_r + ([d]+1) m_{r+1}$, then $a_j^{(r+1)} = a_j^{(r)}$. If $n_r < j \le n_{r+1}$, then $a_j^{(r+1)} = c_{k_{r+1}}$. The result is that (6)–(9) now hold with r replaced by $r+1$.

Now since m_j is the number of occurrences of c_{k_j} in $\{c_j\}$, (6) implies that for each positive integer r,

$$\sum_{j=1}^{n_r} a_j \le \sum_{a_j > c_{k_0}} a_j + n_0 c_{k_0} + ([d]+1) \sum_{j=1}^{n_r} c_j.$$

Thus $\sum a_j$ converges.

The hypotheses of Theorem 2(i) say there is an $\epsilon > 0$ so that for $0 < x < \epsilon$, "$\{a_j\}$ has about d times as many terms as $\{c_j\}$" in $[x, +\infty)$. The theorem is not

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valid if this relation between the distribution of the terms of the sequences holds only on a sequence $\{[x_k, +\infty)\}$ of intervals, with $x_k \to 0^+$. In particular, we can have $\underline{D}(a, c) < +\infty$ with $\sum a_j = +\infty$. For example, take $c_j = j^{-2}$ $(j = 1, 2, \ldots)$ and define $\{a_j\}$ as follows. Let $a_1 = 1$, $a_2 = a_3 = a_4 = a_5 = \frac{1}{2^2}$, $a_6 = a_7 = \cdots = a_{41} = \frac{1}{6^2}$; having defined $a_n = a_{n+1} = \cdots = a_{n^2+n-1} = \frac{1}{n^2}$, we then define $a_{n^2+n} = a_{n^2+n+1} = \cdots = a_{(n^2+n)^2+n^2+n-1} = \frac{1}{(n+n^2)^2}$. We then have $\sum c_j = \frac{\pi^2}{6}$ $\sum a_j = +\infty$ and $\underline{D}(a, c) = 1$.

Reference

1. K. Knopp, *Theory and Application of Infinite Series*, Hafner Publishing Company, New York, 1947.

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