BULL. AUSTRAL. MATH. SOC. 26D15
VOL. 51 (1995) [417–420]

# A SIMPLE PROOF OF THE BECKENBACH-LORENTZ INEQUALITY

B. MOND AND J.E. PEČARIĆ

One of the well-known generalisations of the Hölder inequality was given by Beckenbach. An inverse to this inequality for the discrete case has appeared in the literature. Here we give a simple proof of the inverse to the Beckenbach inequality that is applicable to both the integral and discrete cases.

## 1. INTRODUCTION

Let $(X, \Sigma, \mu)$ be a finite measure space and $L_p = L_p(X, \Sigma, \mu)$ be the space of all $p^{th}$ power nonnegative integrable functions over $(X, \Sigma, \mu)$. If $p > 1$, $1/p + 1/q = 1$, and $f \in L_p$, $g \in L_q$, then $fg \in L_1$ and the Hölder inequality

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q \tag{1}$$

holds, where $\|f\|_p = \left(\int_X f^p d\mu\right)^{1/p}$, et cetera. Equality holds in (1) if and only if $\alpha f^p = \beta g^q$ almost everywhere for some nonzero constants $\alpha$ and $\beta$.

As is well-known, there are several generalisations of the Hölder inequality. One of them is the well-known Beckenbach inequality (see, for example [4]):

**THEOREM A.** *Suppose $(X, \Sigma, \mu)$, $L_p$, $p$ and $q$ are defined as above, $p > 1$. Then, for any $f \in L_p$, $g \in L_q$, and positive numbers $a, b, c$, the inequality*

$$\frac{\left(a + c\int_X f^p d\mu\right)^{1/p}}{b + c\int_X fg d\mu} \geqslant \frac{\left(a + c\int_X h^p d\mu\right)^{1/p}}{b + c\int_X hg d\mu} \tag{2}$$

*holds, where $h = (ag/b)^{q/p}$. Equality holds in (2) if and only if $f = h$ almost everywhere. The sign of the inequality in (2) is reversed if $0 < p < 1$.*

An inverse inequality for (2) in the discrete case is proved in [3] by a functional equation approach. Here we give a simple proof of an inverse inequality for (2), which, in the discrete case, provides a simple proof of a result from [3].

Received 28th June, 1994.

## 2. The Beckenbach-Lorentz inequality

**Theorem 1.** *Suppose* $(X,\Sigma,\mu)$, $L_p$, $p>1$, $q$, $f$, $g$, $h$, $a$, $b$, $c$ *are as in Theorem A. Further, let*

$$a-c\int_X f^p d\mu>0 \qquad \text{and} \qquad a-c\int_X h^p d\mu>0.$$

*Then*

$$\frac{\left(a-c\int_X f^p d\mu\right)^{1/p}}{b-c\int_X fg d\mu}\leqslant\frac{\left(a-c\int_X h^p d\mu\right)^{1/p}}{b-c\int_X hg d\mu} \tag{3}$$

*with equality if and only if* $f=h$ *almost everywhere.*

Proof: Obviously, $h\in L_p$. Noting that $1+q/p=q$, the right-hand side of (3) becomes

$$\frac{\left(a-c\int_X (ag/b)^q d\mu\right)^{1/p}}{b-c\int_X (ag/b)^{q/p} g d\mu}=\frac{(a/b)^{q/p}\left(a(b/a)^q-c\int_X g^q d\mu\right)^{1/p}}{(a/b)^{q/p}\left(b(b/a)^{q/p}-c\int_X g^q d\mu\right)}$$
$$=\left(a^{-q/p}b^q-c\int_X g^q d\mu\right)^{-1/q}.$$

∎

We need the following lemma [**2**], [**1**, pp. 118–119]:

**Lemma 1.** *Let* $A=(a_1,a_2,\dots,a_n)$ *and* $B=(b_1,b_2,\dots,b_n)$ *be* $n$*-tuples of non-negative numbers such that*

$$a_1^p-a_2^p-\cdots-a_n^p>0 \quad \text{and} \quad b_1^q-b_2^q-\cdots-b_n^q>0 \tag{5}$$

*where* $p>1$, $1/p+1/q=1$. *Then*

$$(a_1^p-a_2^p-\cdots-a_n^p)^{1/p}(b_1^q-b_2^q-\cdots-b_n^q)^{1/q}\leqslant a_1b_1-a_2b_2-\cdots-a_nb_n \tag{6}$$

*with equality if and only if* $a_1^p/b_1^q=\cdots=a_n^p/b_n^q$.

*For* $n=2$, $a_1=a^{1/p}$, $a_2=c^{1/p}\left(\int_X f^p d\mu\right)^{1/p}$, $b_1=a^{-1/p}b$, $b_2=c^{1/q}\left(\int_X g^q d\mu\right)^{1/q}$, *(6) becomes*

$$\left(a-c\int_X f^p d\mu\right)^{1/p}\left(a^{-q/p}b^q-c\int_X g^q d\mu\right)^{1/q}\leqslant b-c\left(\int_X f^p d\mu\right)^{1/p}\left(\int_X g^q d\mu\right)^{1/q}$$
$$\leqslant b-c\int_X fg d\mu,$$

*where in the last step, we have used the Hölder inequality (1).*

https://doi.org/10.1017/S0004972700014234 Published online by Cambridge University Press

REMARK. Note that by using the fact that for $p > 1$, we have the reverse inequality in (1) and (6), we can give a corresponding reverse inequality for (3).

If $X = \{m+1, m+2, \ldots, n\}$ and $\mu$ is chosen to be the counting measure on $X$, then $L_p = \ell_p$ and $f \in L_p$ is a finite sequence $X = (x_{m+1}, x_{m+2}, \ldots, x_n)$, where $x_{m+1}, x_{m+2}, \ldots, x_n$ are nonnegative. In this case, we obtain a discrete analogue of Theorem 1, which contains a result from [3] as follows:

THEOREM 2. *Suppose that* $a, b, c > 0$, $x_i, y_i \geqslant 0$, $z_i = (ay_i/b)^{q/p}$ $(i = m+1, m+2, \ldots, n)$, $p > 1$, $1/p + 1/q = 1$ *and*

$$a - c\sum_{i=m+1}^{n} x_i^p > 0, \quad a - c\sum_{i=m+1}^{n} z_i^p > 0. \tag{7}$$

*Then*

$$\frac{\left(a - c\sum_{i=m+1}^{n} x_i^p\right)^{1/p}}{b - c\sum_{i=m+1}^{n} x_i y_i} \leqslant \frac{\left(a - c\sum_{i=m+1}^{n} z_i^p\right)^{1/p}}{b - c\sum_{i=m+1}^{n} y_i z_i}. \tag{8}$$

*Equality holds if and only if* $x_i = z_i$ $(i = m+1, m+2, \ldots, n)$.

Of course (8) can also be proved directly from Popoviciu's inequality (that is, the Hölder-Lorentz inequality) (6):

$$\begin{aligned} b - c\sum_{i=m+1}^{n} x_i y_i &= a^{1/p}\left(ba^{-1/p}\right) - \sum_{i=m+1}^{n}\left(c^{1/p}x_i\right)\left(c^{1/q}y_i\right) \\ &\geqslant \left(a - c\sum_{i=m+1}^{n} x_i^p\right)^{1/p}\left(a^{-q/p}b^q - c\sum_{i=m+1}^{n} y_i^q\right)^{1/q} \\ &= \left(a - c\sum_{i=m+1}^{n} x_i^p\right)^{1/p}\frac{b - c\sum_{i=m+1}^{n} y_i z_i}{\left(a - c\sum_{i=m+1}^{n} z_i^p\right)^{1/p}}. \end{aligned}$$

REMARK. In [3], we have the special case of (8): $c = 1$, $a = x_1^p - \sum_{i=2}^{m} x_i^p$, $b = x_1 y_1 - \sum_{i=2}^{m} x_i y_i$.

## References


[1] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and new inequalities in analysis* (Kluwer Acad. Publ., Dordrecht, Boston, London, 1993).

[2] T. Popiviciu, 'On an inequality', (Romanian), *Gaz. Mat. Fiz. A.* **11** (1959), 451–461.

[3] C.-L. Wang, 'Functional equation approach to inequalities, VI', *J. Math. Anal. Appl.* **104** (1984), 95–102.

[4] Y.-D. Zhuang, 'The Beckenbach Inequality and its Inverse', *J. Math. Anal. Appl.* **175** (1993), 118–125.



Department of Mathematics
La Trobe University
Bundoora Vic 3083
Australia

Faculty of Textile Technology
University of Zagreb
Zagreb
Croatia
and
Department of Mathematics
La Trobe University
Bundoora Vic 3083
Australia