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SAMPLING AND BIRKHOFF REGULAR PROBLEMS

M. H. ANNABY✉**, S. A. BUTERIN and G. FREILING**

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Abstract

We establish new sampling representations for linear integral transforms associated with arbitrary general Birkhoff regular boundary value problems. The new approach is developed in connection with the analytical properties of Green's function, and does not require the root functions to be a basis or complete. Unlike most of the known sampling expansions associated with eigenvalue problems, the results obtained are, generally speaking, of Hermite interpolation type.

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1. Introduction

Let $\Lambda := \{t_k\}_{k=-\infty}^{\infty}$ be a sequence of distinct real numbers for which $\lim_{k\to\pm\infty} t_k = \pm\infty$. Suppose that the infinite product

$$G(\lambda) := (\lambda - t_0)\prod_{k=1}^{\infty}\left(1-\frac{\lambda}{t_k}\right)\left(1-\frac{\lambda}{t_{-k}}\right) \tag{1.1}$$

converges uniformly on bounded subsets of the complex plane. For $k \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ define the interpolating functions of Lagrange and Hermite types

$$\mathcal{L}_k(\lambda;\,\Lambda) := \frac{G(\lambda)}{(\lambda - t_k)\,G'(t_k)},$$

where $\mathcal{L}_k(t_k;\,\Lambda) = 1$; and

$$\mathcal{H}_{m,k,\mu}(\lambda;\,\Lambda) := \sum_{j=0}^{m-\mu}\frac{(G(\lambda))^{m+1}}{(\lambda-t_k)^{j+1}}\,\frac{\{((\lambda-t_k)/G(\lambda))^{m+1}\}^{(m-\mu-j)}(t_k)}{\mu!\,(m-\mu-j)!},$$

where $m \in \mathbb{Z}^+$, $\mu = 0, 1, \ldots, m$.

The sampling theory of signal analysis is concerned with the representation of certain entire functions (signals) in terms of their values at a discrete sequence of points. These points (nodes) are called the sampling points. Therefore the following question is extensively studied. Under what conditions on an entire function f and on the nodes Λ does any of the series

$$f(\lambda)=\sum_{k=-\infty}^{\infty} f(t_k)\mathcal{L}_k(\lambda;\,\Lambda), \tag{1.2}$$

or

$$f(\lambda)=\sum_{k=-\infty}^{\infty}\sum_{\mu=0}^{m} f^{(\mu)}(t_k)\mathcal{H}_{m,k,\mu}(\lambda;\,\Lambda),\qquad \lambda\in\mathbb{C}, \tag{1.3}$$

converge to f? Moreover, what is the type of convergence? As is indicated in [24, 31, 32], further conditions must be imposed on the nodes as well as on the function f to guarantee a convergence of (1.2) or (1.3). In general, the sampling points Λ are assumed to be close to the integers in the sense that there is a positive constant δ such that

$$\sup_{k\in\mathbb{Z}}|t_k-k|<\delta, \tag{1.4}$$

and the sampled (interpolated) functions are assumed to be in an appropriate Paley–Wiener space, PW_σ^p, $p\geq 1$, $\sigma>0$, the latter being the space of all $L^p(\mathbb{R})$ entire functions of exponential type σ. In [38] it is proved that if $\delta=1/4$, then expansion (1.2) holds uniformly on compact subsets of $\mathbb{R}$ for functions of PW_π^2 (see also [34, 42]). More generally, Hinsen [31, 32] has proved that if $\delta<1/4$, $1\leq p\leq 2$ or $\delta<1/2p$, $2<p\leq\infty$, then the sampling series in (1.2) converges uniformly on bounded sets of $\mathbb{C}$ to $f\in PW_\pi^p$ (see also [33]). Hinsen also proved in [32] that if $f\in PW_{\pi(m+1)}^p$, $\delta<1/(4(m+1))$ when $1\leq p\leq 2$ and $\delta<1/(2p(m+1))$ when $2<p<\infty$, then expansion (1.3) converges uniformly on bounded sets of $\mathbb{C}$ to f. Grozev and Rahman have investigated the absolute convergence of (1.3) in [24]. Expansion (1.2) is called a Lagrange interpolation expansion and (1.3) is called a Hermite interpolation expansion. Other related references are [25, 30, 46, 47].

There are several papers that connect sampling theorems of Lagrange interpolation type with eigenvalue problems for differential operators (see, for example, [1, 4, 14, 17, 18, 23, 49, 50]). To the best of our knowledge, except for the example given by Higgins in [28], all known sampling theorems associated with eigenvalue problems are of Lagrange interpolation type. So studies in sampling theory associated with eigenvalue problems that lead to Hermite interpolation are rare. This paper is devoted to this task. We derive sampling theorems associated with Birkhoff regular eigenvalue problems. The resulting sampling formulae turn out to be of Hermite type interpolation when the problem is neither selfadjoint nor strongly regular. If the problem is strongly regular with simple eigenvalues, the associated sampling series becomes a Lagrange interpolation (see, for example, [1, 8]). Basic definitions and a brief account of sampling and strongly regular problems will be given in the next section. Section 3

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contains some auxiliary results. In particular, an expansion of Green's function of regular problems is given with uniform and absolute convergence. This expansion plays a major role in the derivation of the sampling theorem of Section 4. These sampling theorems are sampling theorems of Hermite-type interpolation for integral transforms whose kernels are basically Green's function of Birkhoff-regular problems. The last section contains some illustrative examples.

2. Birkhoff regularity and sampling

In the previous notation, letting $\Lambda = \mathbb{Z}$, expansion (1.2) turns out to be

$$f(\lambda) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(\lambda - k)}{\pi(\lambda - k)}, \tag{2.1}$$

with uniform convergence on both $\mathbb{R}$ and bounded subsets of $\mathbb{C}$ and with absolute convergence on $\mathbb{C}$ for PW_π^2 functions. This result is the celebrated classical sampling theorem of Whittaker, Kotel'nikov and Shannon (see, for example, [12, 13, 27]). The space PW_π^2 is the space of all entire $L^2(\mathbb{R})$ functions with exponential type π. Equivalently (see, for example, [11, 42]), PW_π^2 is the space of all $L^2(\mathbb{R})$ functions whose Fourier transforms vanish outside $[-\pi, \pi]$, thus $f \in PW_\pi^2$ if and only if f is of the form

$$f(\lambda) = \int_{-\pi}^{\pi} g(x) \exp(ix\lambda)\, dx, \quad g(x) \in L^2(-\pi, \pi), \quad \lambda \in \mathbb{C}. \tag{2.2}$$

Notice that the kernel of the Fourier transform (2.2) generates the orthogonal $L^2(-\pi, \pi)$ basis $\{\exp(ikx)\}_{k\in\mathbb{Z}}$ consisting of eigenfunctions of the first order boundary value problem

$$-iy' = \lambda y, \quad y(-\pi) = y(\pi). \tag{2.3}$$

See [15, 20, 26] for the connection between the Whittaker–Kotel'nikov–Shannon sampling theorem and first order eigenvalue problems. Kramer in [36] has given a general sampling lemma where the Fourier kernel of the sampled integral transform is replaced by a general one that generates an orthogonal L^2 basis. Since we are interested in problems that are not necessarily selfadjoint, where the eigenfunctions are not necessarily orthogonal, we will state the biorthogonal sampling result of Higgins [27]. In the next sections the role of sampling points is always played by eigenvalues and for convenience we will denote them by λ_k, $k \in \mathbb{N}$.

THEOREM 2.1 (Kramer's lemma).*Let $I \subset \mathbb{R}$ be a bounded interval and $K(\cdot, \lambda)$, $K^*(\cdot, \lambda) : I \times \mathbb{C} \longrightarrow \mathbb{C}$ be $L^2(I)$ functions for all $\lambda \in \mathbb{C}$. Suppose that there are sequences $\{\lambda_k\}_{k\in\mathbb{N}}$, $\{\lambda_k^*\}_{k\in\mathbb{N}} \subset \mathbb{C}$ such that $\{K(x, \lambda_k)\}_{k\in\mathbb{N}}$ is a basis of $L^2(I)$ and $\{K^*(x, \lambda_k^*)\}_{k\in\mathbb{N}}$ is a biorthogonal basis. Set*

$$\nu_k := \int_I K(x, \lambda_k)\overline{K^*(x, \lambda_k^*)}\, dx.$$

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Then the integral transform

$$f(\lambda) = \int_I g(x) K(x, \lambda)\, dx, \quad \lambda \in \mathbb{C},\ g \in L^2(I),$$

has the sampling representation

$$f(\lambda) = \sum_{k=1}^{\infty} f(\lambda_k) S_k(\lambda), \quad S_k(\lambda) = \frac{1}{\nu_k} \int_I K(x, \lambda) \overline{K^*(x, \lambda_k^*)}\, dx. \tag{2.4}$$

Dually the transform

$$f^*(\lambda) = \int_I g(x) K^*(x, \lambda)\, dx, \quad \lambda \in \mathbb{C},\ g \in L^2(I),$$

has the expansion

$$f^*(\lambda) = \sum_{k=1}^{\infty} f^*(\lambda_k^*) S_k^*(\lambda), \quad S_k^*(\lambda) = \frac{1}{\overline{\nu_k}} \int_I K^*(x, \lambda) \overline{K(x, \lambda_k)}\, dx. \tag{2.5}$$

The sampling series (2.4) and (2.5) converge absolutely on $\mathbb{C}$ and uniformly on the subsets of $\mathbb{C}$ if $\|K(\cdot, \lambda)\|_{L^2(I)}$ and $\|K^*(\cdot, \lambda)\|_{L^2(I)}$ are bounded, respectively.

A more general Kramer-type sampling principle is given in [29]. The point now is in what setting one might find the kernels and the sampling points for which the lemma is applicable and, furthermore, whether it is possible that Kramer expansions are of Lagrange or Hermite interpolation type. Weiss, even before Kramer, in [48] showed that kernels and sampling points can be extracted from selfadjoint second order Sturm–Liouville problems with separate type boundary conditions. In this case we have only one expansion. This is extended in [2, 3, 5, 19, 21, 50], where it is also proved that the resulting Kramer expansions associated with regular or singular second order boundary value problems turn out to be of Lagrange interpolation type. Kramer also indicated that his orthogonal lemma could be applied using general nth order selfadjoint problems without studying the nature of the expansion. This is investigated and applied in [1, 14, 49] where Kramer expansions associated with nth order selfadjoint boundary value problems are shown to be nothing but Lagrange interpolation ones. As for the biorthogonal version, the prototype situation where kernels and sampling points are found is nonselfadjoint problems. Since the eigenfunctions of nonselfadjoint problems are not necessarily orthogonal or L^2 bases, further conditions are imposed on the problems in this case as in [1, 4, 6, 7], where the boundary value problem is assumed to be strongly regular with simple eigenvalues. In other words, consider the boundary value problem that consists of the differential equation

$$\ell(y) := i^n y^{(n)} + \sum_{j=0}^{n-2} p_j(x) y^{(j)} = \lambda y =: \rho^n y, \quad 0 \le x \le 1, \tag{2.6}$$

and the boundary conditions

$$U_\nu(y) := U_{\nu 0}(y) + U_{\nu 1}(y) = 0, \quad 1 \le \nu \le n, \tag{2.7}$$

where $p_j(x) \in L(0,1)$ and

$$U_{\nu 0}(y) = \alpha_\nu y^{(k_\nu)}(0) + \sum_{j=0}^{k_\nu - 1} \alpha_{\nu j} y^{(j)}(0),$$
$$U_{\nu 1}(y) = \beta_\nu y^{(k_\nu)}(1) + \sum_{j=0}^{k_\nu - 1} \beta_{\nu j} y^{(j)}(1).$$

Without loss of generality (see [41]), we assume that the boundary conditions are normalized, that is,

$$n-1 \geq k_1 \geq k_2 \geq \cdots \geq k_n, \quad k_\nu > k_{\nu+2}, \quad |\alpha_j| + |\beta_j| > 0, \quad 1 \leq j \leq n.$$

We recall that boundary conditions (2.7) are called *regular* (or, more precisely, *Birkhoff regular*) [10, 41, 44, 45], if certain numbers $\theta_0,\ \theta_1$ for odd n and $\theta_{-1},\ \theta_1$ for even n do not vanish. For the reader's convenience we give the definition of the θ_k. Let ω_ν, $1 \leq \nu \leq n$, be all nth roots of unity such that

$$\operatorname{Re}(\rho\omega_1) < \operatorname{Re}(\rho\omega_2) < \cdots < \operatorname{Re}(\rho\omega_n)$$

for a certain fixed $\rho \in \mathbb{C}$. Then the numbers θ_k are defined via the identity

$$\theta_0 + \theta_1 s = \begin{vmatrix} \alpha_1\omega_1^{k_1} & \ldots & \alpha_1\omega_{\mu-1}^{k_1} & (\alpha_1 + s\beta_1)\omega_\mu^{k_1} & \beta_1\omega_{\mu+1}^{k_1} & \ldots & \beta_1\omega_n^{k_1} \\ \alpha_2\omega_1^{k_2} & \ldots & \alpha_2\omega_{\mu-1}^{k_2} & (\alpha_2 + s\beta_2)\omega_\mu^{k_2} & \beta_2\omega_{\mu+1}^{k_2} & \ldots & \beta_2\omega_n^{k_2} \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ \alpha_n\omega_1^{k_n} & \ldots & \alpha_n\omega_{\mu-1}^{k_n} & (\alpha_n + s\beta_n)\omega_\mu^{k_n} & \beta_n\omega_{\mu+1}^{k_n} & \ldots & \beta_n\omega_n^{k_n} \end{vmatrix}$$

for odd $n = 2\mu - 1$, and by the identity

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s$$
$$= \begin{vmatrix} \alpha_1\omega_1^{k_1} & \ldots & \alpha_1\omega_{\mu-1}^{k_1} & (\alpha_1 + s\beta_1)\omega_\mu^{k_1} & \left(\alpha_1 + \frac{1}{s}\beta_1\right)\omega_{\mu+1}^{k_1} & \beta_1\omega_{\mu+2}^{k_1} & \ldots & \beta_1\omega_n^{k_1} \\ \alpha_2\omega_1^{k_2} & \ldots & \alpha_2\omega_{\mu-1}^{k_2} & (\alpha_2 + s\beta_2)\omega_\mu^{k_2} & \left(\alpha_2 + \frac{1}{s}\beta_2\right)\omega_{\mu+1}^{k_2} & \beta_2\omega_{\mu+2}^{k_2} & \ldots & \beta_2\omega_n^{k_2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_n\omega_1^{k_n} & \ldots & \alpha_n\omega_{\mu-1}^{k_n} & (\alpha_n + s\beta_n)\omega_\mu^{k_n} & \left(\alpha_n + \frac{1}{s}\beta_n\right)\omega_{\mu+1}^{k_n} & \beta_n\omega_{\mu+2}^{k_n} & \ldots & \beta_n\omega_n^{k_n} \end{vmatrix}$$

for even $n = 2\mu$. The boundary conditions (2.7) are called *strongly regular* if they are regular and if additionally $\theta_0^2 \neq 4\theta_1\theta_{-1}$ for even n. Neither regularity nor strong regularity of (2.7) depend on the choice of ρ. The eigenvalue problem (2.6), (2.7) is called *regular* (*strongly regular*) if its boundary conditions (2.7) are regular (strongly regular). We mention that, in particular, according to [22, 40, 43] all selfadjoint boundary value problems of the form (2.6), (2.7) are regular. The adjoint problem to a (strongly) regular problem is also (strongly) regular (see, for example, [39]). It is well known (see, for example, [9, 35, 39]) that the eigen- and associated functions of strongly regular problems as well as those of the adjoint one are unconditional (Riesz) biorthogonal bases of $L^2(a, b)$. Based on these facts, several sampling expansions associated with strongly regular problems with simple eigenvalues were given. Moreover, the sampling expansions are of Lagrange interpolation type (see, for example, [1, 8] and the more the general settings studied in [4, 6, 7]). In the following we investigate the general case. So it is assumed below that the eigenvalue problem (2.6), (2.7) is regular and that $n > 1$. As will be seen in the next sections the sampling theorems associated with regular nonselfadjoint problems are in general of Hermite interpolation type. This is the first treatment in sampling theory.

3. An expansion of Green's function

In this section we derive an expansion of Green's function of regular problems and study its convergence properties. Let $y_1(x, \lambda), \dots, y_n(x, \lambda)$ be the fundamental system of solutions of differential equation (2.6) satisfying the initial conditions

$$y_k^{(j-1)}(0, \lambda) = \delta_{jk}, \quad 1 \le j, k \le n.$$

Then for every fixed $x \in [0, 1]$ the functions $y_1(x, \lambda), \dots, y_n(x, \lambda)$ are entire with respect to λ and the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ $(|\lambda_k| \le |\lambda_{k+1}|,\ \lambda_k \ne \lambda_j$ for $k \ne j)$ of (2.6), (2.7) are the zeros of the characteristic determinant

$$\Delta(\lambda) := \det \|U_j(y_k)\|_{j,k=1}^n,$$

which is also an entire function of λ. Moreover, one can split the sequence $\{\lambda_k\}$ of all eigenvalues (counted with multiplicities) into two sequences $\{\lambda'_k\}_{k\in\mathbb{N}}$, $\{\lambda''_k\}_{k\in\mathbb{N}}$ with the following asymptotics (see [41]):

$$\lambda'_k = (2k\pi)^n\left(1 + \frac{A'}{k} + O\left(\frac{1}{k^\alpha}\right)\right), \quad \lambda''_k = (-2k\pi)^n\left(1 + \frac{A''}{k} + O\left(\frac{1}{k^\alpha}\right)\right), \tag{3.1}$$

where for the strongly regular case $\alpha = 2$, and otherwise, generally speaking, $\alpha = 3/2$ and A', A'' are constants. For the strongly regular case almost all (that is, all except some finite number) eigenvalues are algebraically simple, while in the general regular problems almost all eigenvalues have algebraic multiplicity $m_a(\lambda_k) \le 2$ (see [41]). The geometric multiplicity $m_g(\lambda_k)$ of λ_k is determined as the number of linearly independent eigenfunctions corresponding to λ_k. Let $y_{k,j}$, $1 \le j \le m_g(\lambda_k)$, be one of these eigenfunctions. Then the system of functions

$$y_{k,j,1}, \dots, y_{k,j,p}$$

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is called a chain of functions associated with the eigenfunction $y_{k,j}$, if

$$\ell(y_{k,j,l}) = \lambda_k y_{k,j,l} + y_{k,j,l-1}, \quad U_\nu(y_{k,j,l}) = 0,\ 1 \le \nu \le n,\ 1 \le l \le p,$$

where $y_{k,j,0} = y_{k,j}$. The value m_{kj} is called a multiplicity of the eigenfunction $y_{k,j}(x)$, if the latter possesses a chain of associated functions consisting of $m_{kj} - 1$ functions and no such chain of m_{kj} functions. It is known (see [41]) that

$$\sum_{j=1}^{m_g(\lambda_k)} m_{kj} = m_a(\lambda_k).$$

Now we define the class of integral transforms, for which sampling formulae will be derived. As a kernel of the sampled transforms we use Green's function multiplied by the characteristic determinant, which gives a rich variety of such kernels. If λ is not an eigenvalue of (2.6), (2.7), then for any function $f \in L^2(0, 1)$ the solution of the boundary value problem

$$\ell(y) = \lambda y + f, \quad U_\nu(y) = 0,\ 1 \le \nu \le n,$$

has the form (see [41])

$$y(x) = \int_0^1 G(x, \xi, \lambda) f(\xi)\, d\xi, \quad 0 \le x \le 1,$$

where $G(x, \xi, \lambda)$ is Green's function of (2.6), (2.7). It is known that $G(x, \xi, \lambda)$ is a meromorphic function with respect to λ for every fixed x, ξ with poles at the eigenvalues λ_k and that the order of the pole λ_k does not exceed $m_a(\lambda_k)$. Moreover, the principal part of the power expansion of $G(x, \xi, \lambda)$ with respect to λ in a vicinity of λ_k has the form (see [41])

$$\operatorname*{res}_{\zeta=\lambda_k} \frac{G(x, \xi, \zeta)}{\lambda - \zeta} = \sum_{j=1}^{m_g(\lambda_k)} \sum_{\nu=1}^{m_{kj}} \frac{1}{(\lambda - \lambda_k)^\nu} \sum_{l=1}^{m_{kj}+1-\nu} \overline{z_{k,j,l-1}(\xi)}\, y_{k,j,m_{kj}+1-\nu-l}(x), \tag{3.2}$$

where

$$z_{k,j,0},\ z_{k,j,1},\ \ldots,\ z_{k,j,m_{kj}-1}, \quad 1 \le j \le m_g(\lambda_k),$$

is a system of eigen- and associated functions of the adjoint problem corresponding to the eigenvalue $\overline{\lambda_k}$, which is appropriately normalized via the biorthogonality relation

$$\int_0^1 y_{k,j,q}(x)\overline{z_{k_1,j_1,q_1}(x)}\, dx = -\delta_{k,k_1}\delta_{j,j_1}\delta_{q+q_1,m_{kj}-1}.$$

If the problem is selfadjoint, then $m_g(\lambda_k) = m_a(\lambda_k)$ for all $k \in \mathbb{N}$, thus $m_{kj} = 1$ for all k, j and $G(x, \xi, \lambda)$ has only simple poles (see [16, 41]).

Let $\rho_k^n = \lambda_k$. Then the following assertion is valid (see [41]).

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LEMMA 3.1. *Fix $\delta > 0$. In the domain $\mathbb{C}_\delta := \{\lambda : |\rho - \rho_k| \geq \delta, k \in \mathbb{N}\}$ the estimate*

$$|G(x, \xi, \lambda)| \leq \frac{M}{|\rho|^{n-1}} \tag{3.3}$$

holds.

Using the contour integral method and taking into account Lemma 3.1 together with the analytical nature of Green's function, one can easily prove the following known result.

LEMMA 3.2. *The following representation holds (for $\lambda \neq \lambda_k$):*

$$G(x, \xi, \lambda) = \sum_{k=1}^{\infty} \sum_{j=1}^{m_g(\lambda_k)} \sum_{\nu=1}^{m_{kj}} \frac{1}{(\lambda - \lambda_k)^\nu} \sum_{l=1}^{m_{kj}+1-\nu} \overline{z_{k,j,l-1}(\xi)} y_{k,j,m_{kj}+1-\nu-l}(x). \tag{3.4}$$

The series in (3.4) and all its termwise derivatives with respect to λ converge uniformly for $x, \xi \in [0, 1]$ and for λ on bounded subsets of $\mathbb{C}$.

PROOF. Let Γ_N, $N \in \mathbb{N}$, be concentric circles in the λ-plane with center the origin and radii $R_N \to \infty$. Fix $\delta > 0$. According to (3.1), one can assume that $\Gamma_N \subset \mathbb{C}_\delta$. Consider the contour integral

$$I_N(\lambda) := \frac{1}{2\pi i} \int_{\Gamma_N} \frac{G(x, \xi, \zeta)}{\lambda - \zeta} \, d\zeta, \quad \lambda \in \operatorname{int} \Gamma_N.$$

The estimate (3.3) implies that $I_N^{(\nu)}(\lambda) \to 0$ for every $\nu \geq 0$ as $N \to \infty$ uniformly for $x, \xi \in [0, 1]$ and for λ on bounded subsets of $\mathbb{C}$. On the other hand, the residue theorem yields

$$I_N(\lambda) = -G(x, \xi, \lambda) + \sum_{k=1}^{q_N} \operatorname*{res}_{\zeta=\lambda_k} \frac{G(x, \xi, \zeta)}{\lambda - \zeta}, \quad \lambda \in \operatorname{int} \Gamma_N \setminus \{\lambda_k\}_{k \in \mathbb{N}},$$

where q_N is the number of eigenvalues $\lambda_1, \ldots, \lambda_{q_N}$ lying in $\operatorname{int} \Gamma_N$. Thus, we obtain

$$G(x, \xi, \lambda) = \sum_{k=1}^{\infty} \operatorname*{res}_{\zeta=\lambda_k} \frac{G(x, \xi, \zeta)}{\lambda - \zeta}$$

and, applying the representation (3.2), we arrive at (3.4). □

The previous lemma plays the major role in proving uniform convergence of the sampling results obtained in this paper. Since we are also concerned with absolute convergence, which in general needs special treatment and estimates (contrast with [24]), we state and prove the following absolute convergence property of the Green's function expansion.

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LEMMA 3.3. *The series in (3.4) and all its termwise derivatives with respect to* λ *converge absolutely. Moreover,*

$$|R_{k,1}(x, \xi)| \le C, \quad |R_{k,2}(x, \xi)| \le Ck^{n-1}, \tag{3.5}$$

where $R_{k,\nu}(x, \xi)$ *is the coefficient of* $(\lambda - \lambda_k)^{-\nu}$ *in expansion (3.4) and* C *does not depend on* k, x, ξ.

PROOF. For sufficiently large k we have $m_a(\lambda_k) \le 2$. Thus, it is sufficient to prove the estimates (3.5). Put $\gamma_k := \{\rho : |\rho - \rho_k| = \delta\}$. Then estimate (3.3) implies the inequality

$$|R_{k,1}(x, \xi)| = \left|\frac{1}{2\pi}\int_{\gamma_k} n\rho^{n-1} G(x, \xi, \rho^n)\, d\rho\right| \le n\delta M.$$

Furthermore,

$$|R_{k,2}(x, \xi)| = \left|\frac{1}{2\pi}\int_{\gamma_k} n\rho^{n-1}(\rho^n - \lambda_k) G(x, \xi, \rho^n)\, d\rho\right| \le n\delta M \max_{\rho\in\gamma_k} |\rho^n - \lambda_k|.$$

Using asymptotics (3.1) we obtain, for $\rho \in \gamma_k$,

$$|\rho^n - \lambda_k| = O(k^{n-1})$$

uniformly with respect to $\arg(\rho - \rho_k)$, which together with the previous estimate gives the second estimate in (3.5). □

4. The main results

This section contains the main results of the paper. We introduce sampling theorems of Hermite interpolation type associated with problem (2.6), (2.7). First we define the class of integral transforms for which the sampling formulae will be established. Let us fix an arbitrary $\xi_0 \in [0, 1]$ and consider the function

$$\varphi(x, \lambda) := \Delta(\lambda) G(x, \xi_0, \lambda). \tag{4.1}$$

After removing the singularities the function $\varphi(x, \lambda)$ is entire with respect to λ for every fixed $x \in [0, 1]$. Consider the set $\mathbb{F}$ of linear transforms $F(\lambda)$ of the form

$$F(\lambda) = \int_0^1 f(x)\varphi(x, \lambda)\, dx, \quad f(x) \in L^2(0, 1). \tag{4.2}$$

REMARKS 4.1. Denote by $\mathbb{S}$ the set of all zeros of the eigenfunctions $z_{k,j,0}(x)$ of the adjoint problem. Obviously $\mathbb{S}$ is at most countable. If $m_g(\lambda_k) = 1$ for all k and $\xi_0 \in [0, 1] \setminus \mathbb{S}$, then the transformation (4.2) is a bijection from $L^2(0, 1)$ to $\mathbb{F}$.

Our goal is to derive sampling representations for functions from $\mathbb{F}$. We first study the case when all eigenvalues have algebraic multiplicity not exceeding 2. Thus we

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assume that $m_a(\lambda_k) \le 2$ and consequently $m_g(\lambda_k) \le 2$ for all $k \in \mathbb{N}$. Denote

$$\mathbb{N}_1 := \{k \mid k \in \mathbb{N},\, m_a(\lambda_k) = 1\},$$
$$\mathbb{N}_2 := \{k \mid k \in \mathbb{N},\, m_a(\lambda_k) = 2,\, m_g(\lambda_k) = 1\},$$
$$\mathbb{N}_3 := \{k \mid k \in \mathbb{N},\, m_a(\lambda_k) = 2,\, m_g(\lambda_k) = 2\}.$$

Then our standing assumption means that $\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3 = \mathbb{N}$. Moreover, (3.4) takes the form

$$G(x, \xi, \lambda) = \sum_{k=1}^{\infty} \sum_{j=1}^{2} \frac{R_{k,j}(x, \xi)}{(\lambda - \lambda_k)^j}, \tag{4.3}$$

where

$$R_{k,1}(x, \xi) = \overline{z_{k,1,0}(\xi)}\, y_{k,1,0}(x), \qquad R_{k,2}(x, \xi) = 0, \qquad k \in \mathbb{N}_1,$$

$$\left.\begin{aligned} R_{k,1}(x, \xi) &= \overline{z_{k,1,0}(\xi)}\, y_{k,1,1}(x) + \overline{z_{k,1,1}(\xi)}\, y_{k,1,0}(x), \\ R_{k,2}(x, \xi) &= \overline{z_{k,1,0}(\xi)}\, y_{k,1,0}(x), \end{aligned}\right\} \quad k \in \mathbb{N}_2,$$

$$\left.\begin{aligned} R_{k,1}(x, \xi) &= \overline{z_{k,1,0}(\xi)}\, y_{k,1,0}(x) + \overline{z_{k,2,0}(\xi)}\, y_{k,2,0}(x), \\ R_{k,2}(x, \xi) &= 0, \end{aligned}\right\} \quad k \in \mathbb{N}_3.$$

THEOREM 4.2. *For any function $F(\lambda)$ of the form (4.2) the following sampling representation holds:*

$$\begin{aligned} F(\lambda) = {} & \sum_{k \in \mathbb{N}_1} F(\lambda_k) \frac{\Delta(\lambda)}{(\lambda - \lambda_k)\Delta'(\lambda_k)} \\ & + \sum_{k \in \mathbb{N}_2} \left(F(\lambda_k) \left\{ \frac{2\Delta(\lambda)}{(\lambda - \lambda_k)^2 \Delta''(\lambda_k)} - \frac{2\Delta'''(\lambda_k)\Delta(\lambda)}{3(\lambda - \lambda_k)(\Delta''(\lambda_k))^2} \right\} \right. \\ & \left. + F'(\lambda_k) \frac{2\Delta(\lambda)}{(\lambda - \lambda_k)\Delta''(\lambda_k)} \right) + \sum_{k \in \mathbb{N}_3} F'(\lambda_k) \frac{2\Delta(\lambda)}{(\lambda - \lambda_k)\Delta''(\lambda_k)}. \end{aligned} \tag{4.4}$$

The series on the right-hand side of (4.4) and all its termwise derivatives converge absolutely and uniformly on every bounded subset of $\mathbb{C}$. Moreover,

$$\begin{aligned} & \left| \frac{F(\lambda_k)}{\Delta'(\lambda_k)} \right| \le C, && k \in \mathbb{N}_1, \\ & \left| \frac{F(\lambda_k)}{\Delta''(\lambda_k)} \right| \le C k^{n-1}, \quad \left| \frac{F'(\lambda_k)}{\Delta''(\lambda_k)} - \frac{F(\lambda_k)\Delta'''(\lambda_k)}{3(\Delta''(\lambda_k))^2} \right| \le C, && k \in \mathbb{N}_2, \\ & \left| \frac{F'(\lambda_k)}{\Delta''(\lambda_k)} \right| \le C, && k \in \mathbb{N}_3, \end{aligned} \tag{4.5}$$

where C is a positive constant which does not depend on k.

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PROOF. Substituting (4.1) and then (4.3) with $\xi = \xi_0$ into (4.2) gives

$$F(\lambda) = \sum_{k=1}^{\infty} \sum_{j=1}^{2} \frac{\Delta(\lambda)}{(\lambda - \lambda_k)^j} \int_0^1 f(x) R_{k,j}(x, \xi_0)\, dx. \tag{4.6}$$

Lemma 3.2 implies that the series on the right-hand side of (4.6) and all its termwise derivatives converge absolutely and uniformly on every bounded subset of $\mathbb{C}$. Let $k \in \mathbb{N}_1$. Then taking $\lambda = \lambda_k$ in (4.6), we obtain

$$\int_0^1 f(x) R_{k,1}(x, \xi_0)\, dx = \frac{F(\lambda_k)}{\Delta'(\lambda_k)}, \qquad \int_0^1 f(x) R_{k,2}(x, \xi_0)\, dx = 0, \qquad k \in \mathbb{N}_1. \tag{4.7}$$

Differentiation in (4.6) with respect to λ gives

$$F'(\lambda) = \sum_{k=1}^{\infty} \sum_{j=1}^{2} \frac{\Delta'(\lambda)(\lambda - \lambda_k)^j - j(\lambda - \lambda_k)^{j-1} \Delta(\lambda)}{(\lambda - \lambda_k)^{2j}} \int_0^1 f(x) R_{k,j}(x, \xi_0)\, dx. \tag{4.8}$$

Further, let $k \in \mathbb{N}_2$. For $\lambda = \lambda_k$ formula (4.6) takes the form

$$F(\lambda_k) = \frac{\Delta''(\lambda_k)}{2} \int_0^1 f(x) R_{k,2}(x, \xi_0)\, dx, \qquad k \in \mathbb{N}_2. \tag{4.9}$$

Taking $\lambda = \lambda_k$ in (4.8) we obtain, for $k \in \mathbb{N}_2$,

$$\begin{aligned} F'(\lambda_k) = \frac{\Delta''(\lambda_k)}{2} & \int_0^1 f(x) R_{k,1}(x, \xi_0)\, dx \\ & + \frac{\Delta'''(\lambda_k)}{6} \int_0^1 f(x) R_{k,2}(x, \xi_0)\, dx. \end{aligned} \tag{4.10}$$

Solving the system of linear algebraic equations (4.9), (4.10), we find that

$$\left. \begin{aligned} \int_0^1 f(x) R_{k,1}(x, \xi_0)\, dx &= \frac{2F'(\lambda_k)}{\Delta''(\lambda_k)} - \frac{2\Delta'''(\lambda_k) F(\lambda_k)}{3(\Delta''(\lambda_k))^2}, \\ \int_0^1 f(x) R_{k,2}(x, \xi_0)\, dx &= \frac{2F(\lambda_k)}{\Delta''(\lambda_k)}, \end{aligned} \right\} \quad k \in \mathbb{N}_2. \tag{4.11}$$

For $k \in \mathbb{N}_3$ it is obvious that $F(\lambda_k) = 0$. Moreover, formula (4.8) gives

$$\int_0^1 f(x) R_{k,1}(x, \xi_0)\, dx = \frac{2F'(\lambda_k)}{\Delta''(\lambda_k)}, \qquad \int_0^1 f(x) R_{k,2}(x, \xi_0)\, dx = 0, \qquad k \in \mathbb{N}_3. \tag{4.12}$$

Substituting (4.7), (4.11) and (4.12) into (4.6) leads directly to (4.4). The estimates (4.5) follow directly from (3.5), (4.7), (4.11) and (4.12). □

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Let us compare the sampling representation (4.4) with the expansions (1.2) and (1.3). Obviously the first sum of (4.4) is of Lagrange interpolation type. To make the comparison clearer, let us consider the Hermite interpolation formula (1.3) when $m = 1$:

$$f(\lambda) = \sum_{k=-\infty}^{\infty} \left(f(t_k)\mathcal{H}_{1,k,0}(\lambda;\, \Lambda) + f'(t_k)\mathcal{H}_{1,k,1}(\lambda;\, \Lambda)\right), \tag{4.13}$$

where

$$\mathcal{H}_{1,k,1}(\lambda;\, \Lambda) = \frac{G^2(\lambda)}{(\lambda - t_k)\,(G'(t_k))^2} \tag{4.14}$$

and

$$\mathcal{H}_{1,k,0}(\lambda;\, \Lambda) = \frac{G^2(\lambda)}{(\lambda - t_k)(G'(t_k))^2}\left(\frac{1}{\lambda - t_k} - \frac{G''(t_k)}{G'(t_k)}\right). \tag{4.15}$$

For simplicity we assume that zero is not an eigenvalue. Using Hadamard's factorization theorem (see, for example, [11, 37]), the characteristic determinant $\Delta(\lambda)$ can be written as the product

$$\Delta(\lambda) = C\Delta_1(\lambda)\,\Delta_2^2(\lambda)\Delta_3^2(\lambda), \quad \Delta_j(\lambda) := \prod_{k\in\mathbb{N}_j}\left(1 - \frac{\lambda}{\lambda_k}\right), \quad j = 1, 2, 3, \tag{4.16}$$

where C is a constant. Consequently expansion (4.4) becomes

$$\begin{aligned} F(\lambda) = {} & \sum_{k\in\mathbb{N}_1} F(\lambda_k)\mathcal{L}_k(\lambda) + \sum_{k\in\mathbb{N}_2}\left(F(\lambda_k)\Psi_{1,k,0}(\lambda) + F'(\lambda_k)\Psi_{1,k,1}(\lambda)\right) \\ & + \sum_{k\in\mathbb{N}_3} F'(\lambda_k)\Phi_{1,k,1}(\lambda), \end{aligned} \tag{4.17}$$

where

$$\mathcal{L}_k(\lambda) = \frac{\Delta_2^2(\lambda)\,\Delta_3^2(\lambda)}{\Delta_2^2(\lambda_k)\,\Delta_3^2(\lambda_k)}\frac{\Delta_1(\lambda)}{(\lambda - \lambda_k)\,\Delta_1'(\lambda_k)}, \quad k \in \mathbb{N}_1$$

and

$$\begin{aligned} \Psi_{1,k,1}(\lambda) &= \frac{\Delta_1(\lambda)\Delta_3^2(\lambda)}{\Delta_1(\lambda_k)\Delta_3^2(\lambda_k)}\frac{\Delta_2^2(\lambda)}{(\lambda - \lambda_k)\,(\Delta_2'(\lambda_k))^2}, \quad k \in \mathbb{N}_2, \\ \Psi_{1,k,0}(\lambda) &= \frac{\Delta_1(\lambda)\,\Delta_3^2(\lambda)}{\Delta_1(\lambda_k)\,\Delta_3^2(\lambda_k)}\frac{\Delta_2^2(\lambda)}{(\lambda - \lambda_k)\,(\Delta_2'(\lambda_k))^2} \\ & \qquad \times\left\{\frac{1}{\lambda - \lambda_k} - \frac{\Delta_2''(\lambda_k)}{\Delta_2'(\lambda_k)} - \frac{\Delta_1'(\lambda_k)}{\Delta_1(\lambda_k)} - 2\frac{\Delta_3'(\lambda_k)}{\Delta_3(\lambda_k)}\right\}, \quad k \in \mathbb{N}_2, \\ \Phi_{1,k,1}(\lambda) &= \frac{\Delta_1(\lambda)\,\Delta_2^2(\lambda)}{\Delta_1(\lambda_k)\,\Delta_2^2(\lambda_k)}\,\frac{\Delta_3^2(\lambda)}{(\lambda - \lambda_k)\,(\Delta_3'(\lambda_k))^2}, \quad k \in \mathbb{N}_3. \end{aligned}$$

Therefore the first expansion in (4.4) is of Lagrange interpolation type. Since $\mathcal{L}_k(\lambda)$ is not exactly like $\mathcal{L}_k(\lambda,\ \Lambda)$ in some of the literature, such an expansion is called a quasi-Lagrange interpolation expansion (see, for example, [17]). The other sums in (4.4) lead to what we may call a quasi-Hermite interpolation series.

Now we consider the general case. Denote

$$m_k := \max_{1\le j\le m_g(\lambda_k)} \{m_{kj}\}.$$

Thus, m_k is the maximal multiplicity of an eigenfunction corresponding to the eigenvalue λ_k.

THEOREM 4.3. *For any function $F(\lambda)$ of the form (4.2) the following representation holds:*

$$F(\lambda) = \sum_{k=1}^{\infty} \sum_{\nu=m_a(\lambda_k)-m_k}^{m_a(\lambda_k)-1} F^{(\nu)}(\lambda_k) S_{k,\nu}(\lambda), \tag{4.18}$$

where

$$S_{k,\nu}(\lambda) = \frac{1}{\nu!} \sum_{j=1}^{m_a(\lambda_k)-\nu} C_{k,j} \frac{\Delta(\lambda)}{(\lambda-\lambda_k)^{m_a(\lambda_k)-\nu+1-j}},$$

and the numbers $C_{k,j}$, $1\le j\le m_k$, can be found from the triangular nonsingular system of linear algebraic equations

$$\sum_{j=1}^{s} C_{k,j} \frac{\Delta^{(m_a(\lambda_k)+s-j)}(\lambda_k)}{(m_a(\lambda_k)+s-j)!} = \delta_{s,1}, \qquad 1\le s\le m_k. \tag{4.19}$$

The series in (4.18) and all its termwise derivatives converge absolutely and uniformly on every bounded subset of $\mathbb{C}$. Moreover, the estimates (4.5) remain valid. In particular, they yield

$$\left| \frac{d^l}{d\lambda^l} \sum_{\nu=m_a(\lambda_k)-m_k}^{m_a(\lambda_k)-1} F^{(\nu)}(\lambda_k) S_{k,\nu}(\lambda) \right| \le \frac{C_l}{k^n}, \qquad l\ge 0, \tag{4.20}$$

for λ from bounded subsets of $\mathbb{C}$, where C_l does not depend on k.

PROOF. We notice that formula (4.4) is a particular case of (4.18). Since $m_a(\lambda_k)\le 2$ for almost all k, formula (4.18) differs from (4.4) in at most a finite number of terms. Hence, estimate (4.20) follows from (4.5). Thus, the function

$$\tilde{F}(\lambda) := \sum_{k=1}^{\infty} \sum_{\nu=m_a(\lambda_k)-m_k}^{m_a(\lambda_k)-1} F^{(\nu)}(\lambda_k) S_{k,\nu}(\lambda)$$

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is entire and it remains to prove the equality $F(\lambda) \equiv \tilde{F}(\lambda)$. One can deduce that

$$F^{(\nu)}(\lambda_k) = 0, \quad 0 \le \nu \le m_a(\lambda_k) - m_k - 1, \tag{4.21}$$

for those k with $m_k < m_a(\lambda_k)$. Let us show that

$$S_{k,\nu}^{(l)}(\lambda_p) = \delta_{l,\nu}\delta_{k,p}, \quad 0 \le l \le m_a(\lambda_p) - 1, \quad m_a(\lambda_k) - m_k \le \nu \le m_a(\lambda_k) - 1, \tag{4.22}$$

where $k, p \in \mathbb{N}$. Clearly, if $p \ne k$, then $S_{k,\nu}^{(l)}(\lambda_p) = 0$, $0 \le l \le m_a(\lambda_p) - 1$. We prove (4.22) for $p = k$. Indeed, the lowest power of $(\lambda - \lambda_k)$ in $S_{k,\nu}(\lambda)$ is ν. Thus, for $l = 0, \ldots, \nu - 1$, (4.22) is valid. For $l = \nu, \ldots, m_a(\lambda_k) - 1$, by differentiation we get

$$S_{k,\nu}^{(l)}(\lambda_k) = \frac{l!}{\nu!} \sum_{j=1}^{l-\nu+1} C_{k,j} \frac{\Delta^{(m_a(\lambda_k)+l-\nu+1-j)}(\lambda_k)}{(m_a(\lambda_k) + l - \nu + 1 - j)!}, \quad 1 \le l - \nu + 1 \le m_k.$$

Hence, using (4.19), we obtain $S_{k,\nu}^{(l)}(\lambda_k) = \delta_{l-\nu+1,1} = \delta_{l,\nu}$, and (4.22) is proved. According to (4.21), (4.22), the function

$$\gamma(\lambda) := \frac{F(\lambda) - \tilde{F}(\lambda)}{\Delta(\lambda)}$$

is, after removing the singularities, entire with respect to λ. Moreover, $\gamma(\lambda)$ is bounded in $\mathbb{C}_\delta$ and consequently in $\mathbb{C}$. Therefore Liouville's theorem implies that the function $\gamma(\lambda)$ is constant. On the other hand, $\gamma(\lambda) \to 0$ as $\lambda \to \infty$. Thus, $F(\lambda) \equiv \tilde{F}(\lambda)$ and the theorem is proved. □

REMARKS 4.4. The Hadamard factorization theorem [37] gives

$$\Delta(\lambda) = C\lambda^{\delta_{0,\lambda_1} m_a(\lambda_1)} \prod_{k=1+\delta_{0,\lambda_1}}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right)^{m_a(\lambda_k)},$$

where C is a constant.

REMARKS 4.5. Formula (4.18) can be also derived by applying the contour integral method directly to transform (4.2). According to (3.3), (4.1), (4.2),

$$\left|\frac{F(\lambda)}{\Delta(\lambda)}\right| < \frac{C}{\rho^{n-1}}, \quad \lambda \in \mathbb{C}_\delta. \tag{4.23}$$

The residue theorem gives, for $\lambda \in \operatorname{int} \Gamma_N \setminus \{\lambda_k\}_{k\in\mathbb{N}}$,

$$\frac{1}{2\pi i} \int_{\Gamma_N} \frac{F(\zeta)\, d\zeta}{\Delta(\zeta)(\lambda - \zeta)} = -\frac{F(\lambda)}{\Delta(\lambda)} + \sum_{\lambda_k \in \operatorname{int} \Gamma_N} \operatorname*{res}_{\zeta=\lambda_k} \left\{\frac{F(\zeta)}{\Delta(\zeta)(\lambda - \zeta)}\right\}. \tag{4.24}$$

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Let us calculate

$$\begin{aligned}\operatorname*{res}_{\zeta=\lambda_k}\left\{\frac{F(\zeta)}{\Delta(\zeta)(\lambda-\zeta)}\right\} &= \frac{1}{(m_k-1)!}\lim_{\zeta\to\lambda_k}\frac{d^{m_k-1}}{d\zeta^{m_k-1}}\left\{\frac{(\zeta-\lambda_k)^{m_k}F(\zeta)}{\Delta(\zeta)(\lambda-\zeta)}\right\}\\ &= \frac{1}{(m_k-1)!}\lim_{\zeta\to\lambda_k}\frac{d^{m_k-1}}{d\zeta^{m_k-1}}\left\{\frac{F(\zeta)}{(\zeta-\lambda_k)^{m_a(\lambda_k)-m_k}}\right.\\ &\qquad\left.\times\frac{(\zeta-\lambda_k)^{m_a(\lambda_k)}}{\Delta(\zeta)(\lambda-\zeta)}\right\}.\end{aligned}$$

Differentiating and passing to the limit, we arrive at

$$\begin{aligned}&\operatorname*{res}_{\zeta=\lambda_k}\left\{\frac{F(\zeta)}{\Delta(\zeta)(\lambda-\zeta)}\right\}\\ &\quad=\sum_{\nu=0}^{m_k-1}\frac{F^{(m_a(\lambda_k)-m_k+\nu)}(\lambda_k)}{(m_a(\lambda_k)-m_k+\nu)!}\sum_{j=0}^{m_k-1-\nu}\frac{1}{j!}\lim_{\zeta\to\lambda_k}\frac{d^j}{d\zeta^j}\left\{\frac{(\zeta-\lambda_k)^{m_a(\lambda_k)}}{\Delta(\zeta)}\right\}\\ &\qquad\times\frac{1}{(\lambda-\lambda_k)^{m_k-\nu-j}}\\ &\quad=\sum_{\nu=m_a(\lambda_k)-m_k}^{m_a(\lambda_k)-1}\frac{F^{\nu}(\lambda_k)}{\nu!}\sum_{j=1}^{m_a(\lambda_k)-\nu}\frac{1}{(j-1)!}\lim_{\zeta\to\lambda_k}\frac{d^{j-1}}{d\zeta^{j-1}}\left\{\frac{(\zeta-\lambda_k)^{m_a(\lambda_k)}}{\Delta(\zeta)}\right\}\\ &\qquad\times\frac{1}{(\lambda-\lambda_k)^{m_a(\lambda_k)-\nu+1-j}}.\end{aligned}$$

Substituting this into (4.24) and taking the limit as $N\to\infty$, taking account of (4.23), we obtain

$$F(\lambda)=\sum_{k=1}^{\infty}\sum_{\nu=m_a(\lambda_k)-m_k}^{m_a(\lambda_k)-1}F^{\nu}(\lambda_k)\frac{1}{\nu!}\sum_{j=1}^{m_a(\lambda_k)-\nu}A_{k,j}\frac{\Delta(\lambda)}{(\lambda-\lambda_k)^{m_a(\lambda_k)-\nu+1-j}},\tag{4.25}$$

where

$$A_{k,j}=\frac{1}{(j-1)!}\lim_{\zeta\to\lambda_k}\frac{d^{j-1}}{d\zeta^{j-1}}\left\{\frac{(\zeta-\lambda_k)^{m_a(\lambda_k)}}{\Delta(\zeta)}\right\}.$$

Comparing (4.18) with (4.25), it is easy to conclude that $C_{k,j}=A_{k,j}$.

5. Examples and open questions

In this section we give some examples illustrating the application of the theorems derived. We discuss the relationship between the sampling expansions obtained and the interpolation series stated in Section 2.

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EXAMPLE 5.1. Consider the regular boundary value problem

$$-y'' = \lambda y, \quad 0 \le x \le 1, \quad y(0) = y'(0) + y'(1) = 0. \tag{5.1}$$

Then

$$y_1(x,\lambda) = \cos\sqrt{\lambda}x, \quad y_2(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$\Delta(\lambda) = -1 - \cos\sqrt{\lambda} = -2\prod_{k=1}^{\infty}\Big(1 - \frac{\lambda}{\lambda_k}\Big)^2, \quad \lambda_k = \pi^2(2k-1)^2.$$

Obviously, $m_a(\lambda_k) = 2$, $m_g(\lambda_k) = 1$ for all $k \in \mathbb{N}$. Hence, $\mathbb{N}_2 = \mathbb{N}$. It is easy to show that Green's function of (5.1) has the form

$$G(x,\xi,\lambda) = \begin{cases} \dfrac{\sin\sqrt{\lambda}x\cos\sqrt{\lambda}(1-\xi)}{\sqrt{\lambda}(1+\cos\sqrt{\lambda})} - \dfrac{\sin\sqrt{\lambda}(x-\xi)}{\sqrt{\lambda}}, & \xi \le x, \\[2ex] \dfrac{\sin\sqrt{\lambda}x\cos\sqrt{\lambda}(1-\xi)}{\sqrt{\lambda}(1+\cos\sqrt{\lambda})}, & \xi \ge x. \end{cases}$$

For simplicity we choose $\xi_0 = 1$; then

$$\varphi(x,\lambda) = \Delta(\lambda)G(x,1,\lambda) = -\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.$$

Furthermore, it is easy to calculate

$$\Delta''(\lambda_k) = -\frac{1}{4\lambda_k}, \quad \Delta'''(\lambda_k) = \frac{3}{8\lambda_k^2}.$$

Thus, according to Theorem 4.2, for any function $F(\lambda)$ of the form

$$F(\lambda) = -\int_0^1 f(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\,dx, \quad f(x) \in L^2(0,1),$$

the representation

$$F(\lambda) = \sum_{k=1}^{\infty}\left(F(\lambda_k)\frac{4(\lambda+\lambda_k)(1+\cos\sqrt{\lambda})}{(\lambda-\lambda_k)^2} + F'(\lambda_k)\frac{8\lambda_k(1+\cos\sqrt{\lambda})}{\lambda-\lambda_k}\right)$$

holds, where $\lambda_k = \pi^2(2k-1)^2$. We notice that in the notation of the previous section

$$C = -2, \quad \Delta_1(\lambda) = \Delta_3(\lambda) = 1, \quad \Delta_2(\lambda) = \prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_k}\right) = \cos\frac{\sqrt{\lambda}}{2}.$$

Therefore the previously obtained expansion is of type (1.3) when $m = 1$.

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EXAMPLE 5.2. Now let us consider the regular eigenvalue problem

$$i^n y^{(n)} = \lambda y, \quad n > 1, \tag{5.2}$$

$$y^{(\nu)}(0) + a y^{(\nu)}(1) = 0, \quad a \neq 0, 0 \leq \nu \leq n-1. \tag{5.3}$$

It is easy to verify that its Green's function has the form

$$G(x, \xi, \lambda) = \frac{1}{in\rho^{n-1}} \times \begin{cases} -a \displaystyle\sum_{k=1}^{n} \omega_k \frac{\exp(-i\rho\omega_k(1 + x - \xi))}{1 + a\exp(-i\rho\omega_k)}, & x \leq \xi, \\ \displaystyle\sum_{k=1}^{n} \omega_k \frac{\exp(-i\rho\omega_k(x - \xi))}{1 + a\exp(-i\rho\omega_k)}, & x \geq \xi. \end{cases}$$

Problem (5.2), (5.3) is selfadjoint if and only if $|a| = 1$. One can easily prove that it is strongly regular if and only if $a^2 \neq 1$ or n is odd. Moreover, $\lambda = 0$ is an eigenvalue if and only if the boundary conditions are periodic ($a = -1$), and for the periodic case $m_a(0) = 1$ holds.

For $\rho \neq 0$ the functions $\{\exp(-i\omega_k\rho x)\}_{k=1}^{n}$ form a fundamental system of solutions of equation (5.2) and, for $\lambda \neq 0$,

$$\Delta(\lambda) = C(\rho)\Delta_1(\rho), \quad C(\rho) \neq 0,$$

where

$$\Delta_1(\rho) = \det \|(-i\omega_k\rho)^{\nu-1}(1 + a\exp(-i\omega_k\rho))\|_{\nu,k=1}^{n}$$

$$= (-i\rho)^{n(n-1)/2} W \prod_{\nu=1}^{n} (1 + a\exp(-i\omega_k\rho)), \quad W = \det \|\omega_k^{\nu-1}\|_{\nu,k=1}^{n}.$$

Hence, the eigenvalue problem (5.2), (5.3) has two sequences of eigenvalues (counted with multiplicities)

$$\lambda_k' = (2\pi k + \alpha)^n, \quad \lambda_k'' = (2\pi(1-k) + \alpha)^n, \quad k \in \mathbb{N},$$

where $\alpha = -i\ln(-a)$. Therefore,

$$\Delta(\lambda) = C\widetilde{\Delta}(\lambda), \quad \widetilde{\Delta}(\lambda) = \lambda^{\delta_{a,-1}} \prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_k'}\right) \prod_{k=1+\delta_{a,-1}}^{\infty}\left(1 - \frac{\lambda}{\lambda_k''}\right),$$

where C is a constant. Let us choose $\xi_0 \in [0, 1]$ and put

$$F(\lambda) = \frac{\widetilde{\Delta}(\lambda)}{in\rho^{n-1}} \sum_{k=1}^{n} \frac{\omega_k \exp(i\rho\omega_k\xi_0)}{1 + a\exp(-i\rho\omega_k)} \left(\int_{\xi_0}^{1} f(x)\exp(-i\rho\omega_k x)\,dx - a\exp(-i\rho\omega_k)\int_0^{\xi_0} f(x)\exp(-i\rho\omega_k x)\,dx\right), \quad f(x) \in L^2(0, 1). \tag{5.4}$$

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Denote λ_k, $k \in \mathbb{N}$, the eigenvalues ($\lambda_k \neq \lambda_\nu$ for $k \neq \nu$) enumerated so that $|\lambda_k| \leq |\lambda_{k+1}|$. Consider now all possible cases.

Case. Let n be even and let the boundary conditions (5.3) be periodic. Then $\lambda_k = (2(k-1)\pi)^n$ and $m_a(\lambda_k) = 2 - \delta_{k1}$, $k \in \mathbb{N}$. Moreover, since the functions $\exp(\pm 2i(k-1)\pi x)$ are eigenfunctions corresponding to λ_k, then $m_g(\lambda_k) = 2 - \delta_{k1}$. According to Theorem 4.2, for any function $F(\lambda)$ of the form (5.4) the following representation holds:

$$F(\lambda) = F(0)\frac{\Delta(\lambda)}{\lambda\Delta'(0)} + \sum_{k=2}^{\infty} F'(\lambda_k)\frac{2\Delta(\lambda)}{(\lambda-\lambda_k)\Delta''(\lambda_k)}.$$

Case. Let n be even and let the boundary conditions (5.3) be anti-periodic ($a = 1$). Then $\lambda_k = (2k-1)^n\pi^n$ and $m_a(\lambda_k) = m_g(\lambda_k) = 2$ for all k. Thus, we come to the expansion

$$F(\lambda) = \sum_{k=1}^{\infty} F'(\lambda_k)\frac{2\Delta(\lambda)}{(\lambda-\lambda_k)\Delta''(\lambda_k)}.$$

Case. If $a \neq \pm 1$ or if n is odd, then there exist $\nu \in \{0, \ldots, n-1\}$ and positive integers $k_1, \ldots, k_\nu$ such that $m_a(\lambda_{k_j}) = m_g(\lambda_{k_j}) = 2$, $1 \leq j \leq \nu$, and $m_a(\lambda_k) = 1$ for $k \in \mathbb{N} \setminus \{k_1, \ldots, k_\nu\}$. Thus, we obtain the representation

$$F(\lambda) = \sum_{k\in\mathbb{N}\setminus\{k_j\}_{j=\overline{1,\nu}}} F(\lambda_k)\frac{\Delta(\lambda)}{(\lambda-\lambda_k)\Delta'(\lambda_k)} + \sum_{j=1}^{\nu} F'(\lambda_{k_j})\frac{2\Delta(\lambda)}{(\lambda-\lambda_{k_j})\Delta''(\lambda_{k_j})}.$$

We now treat the example of Tamarkin [45] on the interval $[0, 1]$.

EXAMPLE 5.3. Consider the regular eigenvalue problem

$$-y'' = \lambda y, \quad 0 \leq x \leq 1, \quad 2y(0) - y(1) = y'(0) - y'(1) = 0. \tag{5.5}$$

Thus, we have the fundamental system of solutions

$$y_1(x, \lambda) = \cos\sqrt{\lambda}x, \quad y_2(x, \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

and the characteristic determinant

$$\Delta(\lambda) = 3(1 - \cos\sqrt{\lambda}) = \frac{3}{2}\lambda\prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{4\pi^2k^2}\right)^2. \tag{5.6}$$

Thus, the eigenvalues have the form $\lambda_k = 4\pi^2k^2$, where $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Here $m_a(\lambda_0) = 1$ and for $k \in \mathbb{N}$ we have $m_a(\lambda_k) = 2$, while $m_g(\lambda_k) = 1$. We calculate

$$\Delta'(\lambda_0) = \frac{3}{2}, \quad \Delta''(\lambda_k) = \frac{3}{4\lambda_k}, \quad \Delta'''(\lambda_k) = -\frac{9}{8\lambda_k^2}, \quad k \in \mathbb{N}. \tag{5.7}$$

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Green's function $G(x, \xi, \lambda)$ of (5.5) is equal to

$$\frac{1}{2\sqrt{\lambda}\,\Delta(\lambda)}\begin{cases}(\cos\sqrt{\lambda}\xi+\cos\sqrt{\lambda}(\xi-1))(\sin\sqrt{\lambda}(x-1)-2\sin\sqrt{\lambda}x)\\ +(2\sin\sqrt{\lambda}\xi+\sin\sqrt{\lambda}(\xi-1))(\cos\sqrt{\lambda}x-\cos\sqrt{\lambda}(x+1))\\ +3(1-\cos\sqrt{\lambda})\sin\sqrt{\lambda}(\xi-x), \qquad \xi\le x,\\ \\ (\cos\sqrt{\lambda}\xi+\cos\sqrt{\lambda}(\xi-1))(\sin\sqrt{\lambda}(x-1)-2\sin\sqrt{\lambda}x)\\ +(2\sin\sqrt{\lambda}\xi+\sin\sqrt{\lambda}(\xi-1))(\cos\sqrt{\lambda}x-\cos\sqrt{\lambda}(x+1))\\ -3(1-\cos\sqrt{\lambda})\sin\sqrt{\lambda}(\xi-x), \qquad \xi\ge x.\end{cases}$$

Fix an arbitrary $\xi_0 \in [0, 1]$. Then according to (5.6), (5.7) and Theorem 4.2, the integral transform

$$F(\lambda)=\int_0^1 f(x)\varphi(x,\lambda)\,dx, \qquad f(x)\in L^2(0,1),$$

has the Hermite-type sampling expansion

$$F(\lambda)=\frac{2(1-\cos\sqrt{\lambda})}{\lambda}F(0)$$
$$+\sum_{k=1}^{\infty}\left(F(\lambda_k)\frac{4(\lambda+\lambda_k)(1-\cos\sqrt{\lambda})}{(\lambda-\lambda_k)^2}+F'(\lambda_k)\frac{8\lambda_k(1-\cos\sqrt{\lambda})}{\lambda-\lambda_k}\right).$$

Now we discuss some open questions concerning the above mentioned results.

1. The boundary value problem (5.5) of the last example was introduced by Tamarkin in [45]. The only change is that he considered the interval $[-\pi, \pi]$ instead of $[0, 1]$. Let us consider his interval and the sampling treatment associated with this problem introduced by Higgins in [28] in a different manner. Thus the eigenvalues will in this case be $\lambda_k = k^2$, $k \in \mathbb{N}_0$. The algebraic multiplicity of all eigenvalues is 2 and the geometric multiplicity is 1. Therefore every eigenfunction has one and only one associated function. In the above notation [45],

$$y_{0,0,0}(x)=3x+\pi, \qquad y_{0,k,0}(x)=\sin kx,\ k\in\mathbb{N}, \qquad y_{0,k,1}(x)=(3x+\pi)\cos nx.$$

Also the adjoint eigenvalue problem will have the system of eigen- and associated functions

$$z_{0,0,0}(x)=1, \qquad z_{0,k,0}(x)=\cos kx,\ k\in\mathbb{N}, \qquad z_{0,k,1}(x)=-(3x-\pi)\sin nx.$$

Higgins's treatment depends on the fact that we can concretely find functions (kernels) that generate all eigen- and associated functions. Therefore there is a possibility to

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define kernels $K(x, \lambda)$ and $K^*(x, \lambda)$ such that the biorthogonal version of Kramer's lemma is applicable, since the eigen- and associated functions of the problem and its adjoint are biorthogonal Riesz bases. These kernels are

$$\begin{aligned} K(x, \lambda) &= \cos 2(\pi\sqrt{\lambda}/2)\,(3x+\pi)\, \cos(\sqrt{\lambda}x/2) \\ &\quad + \sin 2(\pi\sqrt{\lambda}/2)\, \sin((\sqrt{\lambda}+1)x/2), \\ K^*(x, \lambda) &= \sin 2(\pi\sqrt{\lambda}/2)\,(\pi-3x)\, \sin((\sqrt{\lambda}+1)x/2) \\ &\quad + \cos 2(\pi\sqrt{\lambda}/2)\, \cos(\sqrt{\lambda}x/2). \end{aligned}$$

The question now concerns the possibility of finding kernels in general nth ($n \geq 2$) order regular problems which generate all eigen- and associated functions and for which Kramer's biorthogonal lemma is applicable. Even for strongly regular problems this question has not been answered [1].

2. Our sampling nodes (the eigenvalues) do not satisfy relations like (1.4) above. Instead, we have the asymptotics (3.1). Our second question is twofold. First assume we have nodes which satisfy (3.1). Can we explicitly construct a regular eigenvalue problem with these nodes as eigenvalues and derive the associated sampling results? This part is connected to the inverse problem for differential operators. The second part in concerned with the same point, but when the nodes satisfy (1.4) instead of (3.1).

3. Finally, is it possible to derive a sampling theorem associated with irregular problems, and what type of sampling representations would be obtained?

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M. H. ANNABY, Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
e-mail: mannaby@qu.edu.qa, mhannaby@yahoo.com
and
Current address: Department of Mathematics and Physics, Qatar University,
PO Box 2713 Doha, Qatar

S. A. BUTERIN, Department of Mathematics, Saratov State University,
Astrakhanskaya str. 83, 410012 Saratov, Russia
e-mail: buterinsa@info.sgu.ru

G. FREILING, Fachbereich Mathematik, Universität Duisburg-Essen,
D-47057 Duisburg, Germany
e-mail: freiling@math.uni-duisburg.de