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The size of collision solutions in orbital elements space

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Abstract. In the framework of the analytical theory of close encounters, and under suitable assumptions, we compute the size of the region in orbital elements space containing collisions solutions. In the linearized approximation in the semimajor axis/eccentricity plane the collision region is the interior of an ellipse. Examples are given from past cases of Near Earth Asteroids having the possibility of impacting our planet.

Keywords. Collisions, Near-Earth asteroids, planetary close encounters

1. Introduction

Impact monitoring programs like CLOMON2 and Sentry (Milani *et al.* 2004) routinely find collision possibilities of Near-Earth Asteroids (NEAs) with the Earth at specific dates, and characterize them with values of quantities like the stretching and the impact probability (Milani *et al.* 1999, Milani *et al.* 2004, Milani *et al.* 2002). These quantities depend, among other things, on the amount of observational data available for the given NEA, and on assumptions on the statistical features of the data; thus, they do not depend *only* on the orbital parameters of the NEA.

The question we address here is the following: leaving aside observations and statistics, and considering only the celestial mechanics side of the problem, can we meaningfully speak of the "size" of a collision solution and, if yes, how does it vary for different NEA collisions? We look for an answer in the framework of the recent extension (Valsecchi *et al.* 2003, Valsecchi 2004) of the analytical theory of close encounters (Öpik 1976).

2. Collision solutions

Let us start by considering the points in the space of orbital elements leading to an *exact* collision with a point-mass Earth. We consider the Earth to be on a circular orbit of radius 1 AU. The conditions on the orbital elements of the NEA for an exact collision at a given time t are:

$$\frac{a(1-e^2)}{1 \pm e\cos\omega} = 1 \tag{2.1}$$

$$\Omega - \lambda_{\oplus} = \frac{\pi}{2} \mp \frac{\pi}{2} \tag{2.2}$$

$$\omega + f = \frac{\pi}{2} \mp \frac{\pi}{2}, \tag{2.3}$$

where the upper sign is for collisions at the ascending node and $\lambda_\oplus$ is the longitude of the Earth at the time t. Thus the collision solutions for a given date t lie on a 3-dimensional manifold in the 6-dimensional space of orbital elements.

Note that for a collision at another date equations (2.1)–(2.3) would be the same, with only $\lambda_\oplus$ changed. Thus, if we look for the set of collision solutions at an arbitrary date, equation (2.2) would not introduce any constraint on the orbital elements (a, e, i, ω, f), since Ω can be fixed at the suitable value as a function of $\lambda_\oplus$.

In the b-plane of Öpik's theory (Valsecchi *et al.* 2003), the exact collision condition is the point of coordinates $\xi = \zeta = 0$. The impact radius of the Earth on that plane is:

$$b_\oplus = \sqrt{r_\oplus^2 + 2cr_\oplus};$$

where $c = m/U^2$, m is the ratio of the mass of the Earth to that of the Sun,

$$U = \sqrt{3 - \frac{1}{a} - 2\sqrt{a(1-e^2)}\cos i}$$

is the unperturbed geocentric speed of the NEA in units of the heliocentric velocity of the Earth, and $r_\oplus$ is the actual radius of the Earth; the unit of length used here is the AU. In the following, we use $b_\oplus$ to compute the region in elements space corresponding to a *physical* collision.

2.1. *The correspondence between orbital elements and b-plane coordinates*

If at $t = t_0$ the small body is near the Earth, at the point of geocentric cartesian coordinates (X_0, Y_0, Z_0), we have

$$\begin{aligned} \xi &= X_0 \cos\phi - Z_0 \sin\phi \\ \zeta &= (X_0 \sin\phi + Z_0 \cos\phi)\cos\theta - Y_0 \sin\theta, \end{aligned}$$

where $\theta = \theta(a, e, i)$ and $\phi = \phi(a, e, i)$ are the usual angles of Öpik's theory, given by:

$$\begin{aligned} \cos\theta &= \frac{\sqrt{a(1-e^2)}\cos i - 1}{\sqrt{3 - \frac{1}{a} - 2\sqrt{a(1-e^2)}\cos i}} \\ \sin\theta &= \frac{\sqrt{2 - \frac{1}{a} - a(1-e^2)\cos^2 i}}{\sqrt{3 - \frac{1}{a} - 2\sqrt{a(1-e^2)}\cos i}} \\ \sin\phi &= \pm\frac{\sqrt{2 - \frac{1}{a} - a(1-e^2)}}{\sqrt{2 - \frac{1}{a} - a(1-e^2)\cos^2 i}} \\ \cos\phi &= \pm\frac{\sqrt{a(1-e^2)}\sin i}{\sqrt{2 - \frac{1}{a} - a(1-e^2)\cos^2 i}} \,. \end{aligned}$$

In the expression for $\sin\phi$ the upper sign applies to encounters in the post-perihelion branch of the orbit, and in that for $\cos\phi$ to encounters close to the ascending node.

By using equations (2.1)–(2.3) we can express X_0, Y_0, Z_0 as functions of the orbital parameters (to first order in X_0, Y_0, Z_0):

$$\begin{aligned} X_0 &= \frac{a(1-e^2)}{1 + e\cos f_0} - 1 \\ Y_0 &= \Omega + \frac{\pi}{2} \mp \left\{\frac{\pi}{2} - \arctan[\cos i \tan(\omega + f_0)]\right\} - \lambda_\oplus \end{aligned}$$

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$$Z_0 = \sin i \sin(\omega + f_0),$$

where the upper sign is for collisions at the ascending node.

Let us now consider a NEA that, at a generic epoch t^*, has orbital parameters a, e, i, ω, Ω, f, and at $t = t_0$, when its true anomaly is f_0, has a collision with the Earth, located (at $t = t_0$) at longitude $\lambda_\oplus$. Note that

$$t_0 = t^* + 2h\pi a^{3/2},$$

where h is the non-integer number of heliocentric revolutions made by the NEA between t^* and t_0.

To understand what happens to ξ and ζ, when we apply small changes to the orbital elements, we can compute

$$d\xi = \sum_{i=1,6} \frac{\partial \xi}{\partial \mathcal{E}_i} d\mathcal{E}_i$$

$$d\zeta = \sum_{i=1,6} \frac{\partial \zeta}{\partial \mathcal{E}_i} d\mathcal{E}_i,$$

where $\mathcal{E}_1 = a, \mathcal{E}_2 = e, \mathcal{E}_3 = i, \mathcal{E}_4 = \omega, \mathcal{E}_5 = \Omega, \mathcal{E}_6 = f$.

2.2. *The derivatives of ξ, ζ with respect to the elements*

At the collision, the derivatives $\partial\xi/\partial\mathcal{E}_i$, $\partial\zeta/\partial\mathcal{E}_i$ have the form:

$$\frac{\partial \xi}{\partial \mathcal{E}_i} = \frac{\partial X_0}{\partial \mathcal{E}_i} \cos\phi - \frac{\partial Z_0}{\partial \mathcal{E}_i} \sin\phi$$

$$\frac{\partial \zeta}{\partial \mathcal{E}_i} = \frac{\partial X_0}{\partial \mathcal{E}_i} \cos\theta \sin\phi + \frac{\partial Z_0}{\partial \mathcal{E}_i} \cos\theta \cos\phi - \frac{\partial Y_0}{\partial \mathcal{E}_i} \sin\theta,$$

where we have dropped the terms with either X_0, or Y_0, or Z_0 as factor (since $X_0 = Y_0 = Z_0 = 0$). Note that, among the derivatives, only $\partial\xi/\partial a$ and $\partial\zeta/\partial a$ depend on the elapsed time $t_0 - t^*$; their expressions contain h, the non-integer number of heliocentric revolutions made by the NEA between t^* and t_0.

We can then write:

$$\begin{pmatrix} \delta\xi \\ \delta\zeta \end{pmatrix} = \begin{pmatrix} \frac{\partial\xi}{\partial a} & \frac{\partial\xi}{\partial e} & \frac{\partial\xi}{\partial i} & \frac{\partial\xi}{\partial \omega} & \frac{\partial\xi}{\partial \Omega} & \frac{\partial\xi}{\partial f} \\ \frac{\partial\zeta}{\partial a} & \frac{\partial\zeta}{\partial e} & \frac{\partial\zeta}{\partial i} & \frac{\partial\zeta}{\partial \omega} & \frac{\partial\zeta}{\partial \Omega} & \frac{\partial\zeta}{\partial f} \end{pmatrix} \begin{pmatrix} \delta a \\ \delta e \\ \delta i \\ \delta\omega \\ \delta\Omega \\ \delta f \end{pmatrix}. \tag{2.4}$$

Actually, since an explicit computation shows that $\partial\xi/\partial i = \partial\xi/\partial\Omega = \partial\zeta/\partial i = 0$, we have that:

$$\begin{pmatrix} \delta\xi \\ \delta\zeta \end{pmatrix} = \begin{pmatrix} \frac{\partial\xi}{\partial a} & \frac{\partial\xi}{\partial e} & \frac{\partial\xi}{\partial \omega} & 0 & \frac{\partial\xi}{\partial f} \\ \frac{\partial\zeta}{\partial a} & \frac{\partial\zeta}{\partial e} & \frac{\partial\zeta}{\partial \omega} & \frac{\partial\zeta}{\partial \Omega} & \frac{\partial\zeta}{\partial f} \end{pmatrix} \begin{pmatrix} \delta a \\ \delta e \\ \delta\omega \\ \delta\Omega \\ \delta f \end{pmatrix}.$$

Note that, to first order, the changes in (ξ, ζ) do not depend upon the inclination i; however, the partial derivatives do depend upon i, as also evidenced by the examples given in the next Section.

3. Collision solution size in a, e

As we have seen, the 2-dimensional vector giving a small displacement on the b-plane as function of the orbital elements is given by the product of a 2 rows by 6 columns matrix times a 6-dimensional vector in elements space (see equation 2.4). This corresponds to the fact that each point on the b-plane, if the time of close approach is variable, has as preimage a 4-dimensional manifold. However, if we fix four elements and leave only two as variables, the inversion becomes possible. To select the two elements we want to use as parameters, we have to take into account that the coordinates (i, Ω) play a different role: the partial derivatives with respect to i are zero, and Ω is only a function of $\lambda_\oplus$, that is of the close approach time t_0.

If, for example, we set $\delta\omega = \delta\Omega = \delta f = \delta i = 0$, we can find the size of a collision solution in the a-e plane. In this case, in fact:

$$\begin{pmatrix} \delta\xi \\ \delta\zeta \end{pmatrix} = \begin{pmatrix} \frac{\partial\xi}{\partial a} & \frac{\partial\xi}{\partial e} \\ \frac{\partial\zeta}{\partial a} & \frac{\partial\zeta}{\partial e} \end{pmatrix} \begin{pmatrix} \delta a \\ \delta e \end{pmatrix};$$

this can be inverted to give:

$$\begin{pmatrix} \delta a \\ \delta e \end{pmatrix} = \frac{1}{\begin{vmatrix} \frac{\partial\xi}{\partial a} & \frac{\partial\xi}{\partial e} \\ \frac{\partial\zeta}{\partial a} & \frac{\partial\zeta}{\partial e} \end{vmatrix}} \begin{pmatrix} \frac{\partial\zeta}{\partial e} & -\frac{\partial\xi}{\partial e} \\ -\frac{\partial\zeta}{\partial a} & \frac{\partial\xi}{\partial a} \end{pmatrix} \begin{pmatrix} \delta\xi \\ \delta\zeta \end{pmatrix}.$$

Using the above expressions we can find the values of δa and δe corresponding to a circle of radius $b_\oplus$ in the b-plane. The shape of the collision region, in this linear approximation, is an ellipse; it does not need to have the major and minor axes aligned with the coordinate axes.

3.1. *Application: the 2019 collision of 2002 NT$_7$ and the 2008 collision of 2003 EE$_{16}$*

We now compute the sizes in the a-e plane of the collision solutions of two NEAs that have been of some interest in the recent past.

In the summer of 2002 the confidence region of 2002 NT$_7$ in orbital elements space contained a collision taking place about $h = 7.3$ revolutions of the asteroid later, in early 2019. The value of the Palermo scale (Chesley *et al.* 2002) for this collision became, for a short time, larger than 0, before coming down quickly with the accumulation of new observations.

Substituting the numeric values of the orbital elements of 2002 NT$_7$ ($a = 1.74$, $e = 0.53$, $i = 42\overset{\circ}{.}3$) in the expressions for δa and δe we get:

$$\delta a = \frac{-0.19\,\delta\xi - 1.7\,\delta\zeta}{3.4h}$$

$$\delta e = \frac{0.09\,\delta\xi + 0.79\,\delta\zeta}{-4.9h} - 0.62\,\delta\xi.$$

The left panel of Fig. 1 shows this collision in the a-e plane; to highlight the dependence on h (the non-integer number of heliocentric revolutions made by 2002 NT$_7$ before the collision), in the right panel we show the size of two collisions characterized by $h = 1$ and $h = 50$. The shrinking of the region along a, as h grows, is quite noticeable, as is the change in the overall slope of the collision region.

In March 2003 the confidence region of 2003 EE$_{16}$ in orbital elements space contained a collision taking place about $h = 2.9$ revolutions of the asteroid later, in early 2008. This collision solution did not go above the 0 of the Palermo scale at any time. Proceeding as

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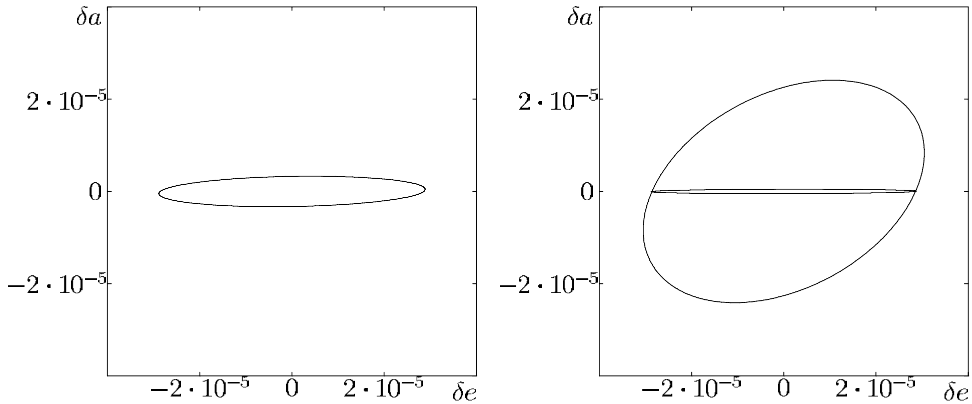

Figure 1. Left: the size of the collision solution of 2002 NT_7 in $\delta e, \delta a$ space for $h = 7.3$. Right: the size of collision solutions, of the same NEA, for $h = 1$ (larger ellipse) and $h = 50$.

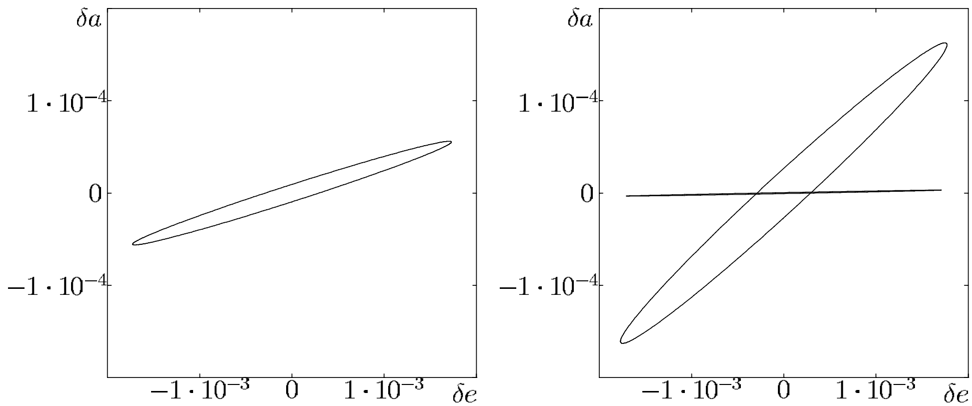

Figure 2. Left: the size of the collision solution of 2003 EE_{16} in $\delta e, \delta a$ space for $h = 2.9$. Right: the size of collision solutions, of the same NEA, for $h = 1$ (larger ellipse) and $h = 50$. Note that the scale for δa is much smaller than for δe, that is, the ellipse is much wider in the δe direction.

before, for $a = 1.42$, $e = 0.62$, $i = 0.^\circ 6$ (the orbital elements of 2003 EE_{16}), we get:

$$\delta a = \frac{-0.19\,\delta\xi - 0.03\,\delta\zeta}{0.059h}$$
$$\delta e = \frac{0.09\,\delta\xi + 0.02\,\delta\zeta}{-0.071h} - 35\,\delta\xi.$$

Figure 2 shows the same information as Fig. 1 for 2003 EE_{16}. The most important difference between the two cases is the scale of the Figures: the collision solution of 2003 EE_{16} is more than an order of magnitude larger in both coordinates. Of course this is due to the much lower inclination of 2003 EE_{16}.

4. Conclusions

We have shown that, under suitable assumptions, it is possible to deduce analytical expressions giving the size of the region in orbital elements space containing orbits colliding with the Earth at a given date. When computed in the a-e plane, the size depends on the number of heliocentric revolutions h made by the NEA before hitting the Earth, and is smaller for larger h.

Applying the above mentioned expressions to two NEAs, we find that the size of the collision region in the a-e plane differs significantly in the two cases.

We plan to continue this research, with the goal of providing an easy to compute first approximation of the location of a Virtual Impactor. This could be a useful tool for the impact monitoring systems.

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