9 Supersymmetry

In a standard advanced field theory course, one learns about a number of symmetries: Poincaré invariance, global continuous symmetries, discrete symmetries, gauge symmetries, approximate and exact symmetries. These latter symmetries all have the property that they commute with Lorentz transformations and in particular with rotations. So, the multiplets of the symmetries always contain particles of the same spin; in particular, they always consist of either bosons or fermions.

For a long time, it was believed that these were the only allowed types of symmetry; this statement was even embodied in a theorem, known as the Coleman–Mandula theorem. However, physicists studying theories based on strings stumbled on a symmetry which related fields of different spin. Others quickly worked out simple field theories with this new symmetry, called *supersymmetry*.

Supersymmetric field theories can be formulated in dimensions up to eleven. These higher-dimensional theories will be important when we consider string theory. In this chapter we consider theories in four dimensions. The supersymmetry charges, because they change spin, must themselves carry spin – they are spin-$1/2$ operators. They transform as doublets under the Lorentz group, just like the two-component spinors χ and χ^*. (The theory of two-component spinors is reviewed in Appendix A, where our notation, which is essentially that of the text by Wess and Bagger (1992), is explained.) There can be 1, 2, 4 or 8 such spinors; correspondingly, the symmetry is said to be $N = 1, 2, 4$ or 8 supersymmetry. Like the generators of an ordinary group, the supersymmetry generators obey an algebra; unlike an ordinary bosonic group, however, the algebra involves anticommutators as well as commutators (it is said to be "graded").

There are at least four reasons to think that supersymmetry might have something to do with TeV-scale physics. The first is the hierarchy problem: as we will see, supersymmetry can both explain how hierarchies arise, and why there are no large radiative corrections. The second is the unification of couplings. We have seen that while the gauge group of the Standard Model can in a rather natural way be unified in a larger group, the couplings do not unify properly. In the minimal supersymmetric extension of the Standard Model (the minimal supersymmetric Standard Model, or MSSM) the couplings unify nicely if the scale of supersymmetry breaking is about 1 TeV. Third, the assumption of TeV-scale supersymmetry almost automatically yields a suitable candidate for dark matter, with a density in the required range. Finally, low-energy supersymmetry is strongly suggested by string theory, though at present one cannot assert that this is an actual prediction.

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9.1 The supersymmetry algebra and its representations

Because the supersymmetry generators are spinors, they do not commute with the Lorentz generators. Perhaps, then, it is not surprising that a supersymmetry algebra involves translation generators Q, ($\bar{Q}_{\dot{\alpha}} = Q^*_{\dot{\alpha}}$)[1] with anticommutators

$$\left\{Q^A_\alpha, \bar{Q}^B_{\dot{\beta}}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}}\delta^{AB}P_\mu, \tag{9.1}$$

$$\left\{Q^A_\alpha, \bar{Q}^B_\beta\right\} = \epsilon_{\alpha\beta}X^{AB}; \tag{9.2}$$

here $A,\ B = 1, \dots, N$, where the integer N labels a particular algebra. The X^{AB}s are Lorentz scalars, antisymmetric in A, B, known as *central charges*.

If nature is supersymmetric, it is likely that for the low-energy symmetry $N = 1$, corresponding to only one possible value for the index A above. Only $N = 1$ supersymmetry has chiral representations. Of course, one might imagine that the chiral matter would arise at the point where supersymmetry was broken. As we will see, it is very difficult to break $N > 1$ supersymmetry spontaneously; however, this is not the case for $N = 1$. The smallest irreducible representations of $N = 1$ supersymmetry which can describe massless fields are as follows:

- chiral superfields (ϕ, ψ_α), comprising a complex scalar and a chiral fermion;
- vector superfields (λ, A_μ), comprising a chiral fermion and a vector meson, both, in general, in the adjoint representation of the gauge group;
- the gravity supermultiplet $(\psi_{\mu,\alpha}, g_{\mu\nu})$, compressing a spin-$3/2$ particle, the *gravitino*, and a spin-2 particle, the *graviton*.

One can work in terms of these fields, writing down supersymmetry transformation laws and constructing invariants. This turns out to be rather complicated; one must use the equations of motion to realize the full algebra. Great simplification is achieved by enlarging space–time to include commuting and anticommuting variables. The result is called *superspace*.

9.2 Superspace

We may conveniently describe $N = 1$ supersymmetric field theories by using superspace. Superspace allows a simple description of the action of the symmetry on fields and provides an efficient algorithm for the construction of invariant Lagrangians. In addition, calculations of Feynman graphs and other quantities are often greatly simplified using superspace, at least in the limit where supersymmetry is unbroken or nearly so.

[1] The notation with the bar over the Qs and θs is helpful here and conforms with that of the classic text of Wess and Bagger. Note that this differs from our notation in earlier chapters, where we used a bar on left-handed *fields* to distinguish particles transforming in, say, the 3 or $\bar{3}$ representation of $SU(3)$.

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In superspace, in addition to the ordinary coordinates x^μ one has a set of anticommuting, *Grassmann*, coordinates, θ_α and $\theta^*_{\dot\alpha} = \bar\theta_{\dot\alpha}$. The Grassmann coordinates obey

$$\{\theta_\alpha, \theta_\beta\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = \{\theta_\alpha, \bar\theta_{\dot\beta}\} = 0. \tag{9.3}$$

Grassmann coordinates provide a representation of the classical configuration space for fermions; they are familiar from the problem of formulating the fermion functional integral. Note that the square of any θ vanishes. The derivatives also anticommute:

$$\left\{\frac{\partial}{\partial\theta_\alpha}, \frac{\partial}{\partial\bar\theta_{\dot\beta}}\right\} = 0, \quad \text{etc.} \tag{9.4}$$

Crucial in the discussion of Grassmann variables is the problem of integration. In discussing the Poincaré invariance of ordinary field-theory Lagrangians, the property of ordinary integrals that

$$\int_{-\infty}^{\infty} dx\, f(x+a) = \int_{-\infty}^{\infty} dx\, f(x) \tag{9.5}$$

is important. We require that the analogous property hold for Grassmann integration (here for one variable):

$$\int d\theta f(\theta + \epsilon) = \int d\theta f(\theta). \tag{9.6}$$

This is satisfied by the integration rule

$$\int d\theta (1, \theta) = (0, 1). \tag{9.7}$$

For the case of $\theta_\alpha, \bar\theta_{\dot\alpha}$, one can write a simple integral table:

$$\int d^2\theta\, \theta^2 = 1, \quad \int d^2\bar\theta\, \bar\theta^2 = 1, \tag{9.8}$$

all other such integrals vanish.

One can formulate a superspace description for both local and global supersymmetry. The local case is rather complicated, and we will not deal with it here, referring the interested reader to the suggested reading and confining our attention to the global case.

The goal of the superspace formulation is to provide a classical description of the action of the symmetry on fields, just as one describes the action of the Poincaré generators. Consider a function of the superspace variables, $f(x^\mu, \theta, \bar\theta)$. The supersymmetry generators act on such a function as differential operators:

$$Q_\alpha = \frac{\partial}{\partial\theta_\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, \quad \bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu. \tag{9.9}$$

Note that the θs have mass dimension $-1/2$. It is easy to check that the Q_αs obey the algebra. For example,

$$\{Q_\alpha, Q_\beta\} = \left\{\left(\frac{\partial}{\partial\theta_\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\right), \left(\frac{\partial}{\partial\theta_\beta} - i\sigma^\nu_{\beta\dot\beta}\bar\theta^{\dot\beta}\partial_\nu\right)\right\} = 0, \tag{9.10}$$

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since the θs and their derivatives anticommute. With just slightly more effort one can construct the $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}$ anticommutator.

One can think of the Qs as generating infinitesimal transformations in superspace with Grassmann parameter ϵ. One can construct finite transformations as well by exponentiating the Qs; because there are only a finite number of non-vanishing polynomials in the θs, these exponentials contain only a finite number of terms. The result can be expressed elegantly:

$$e^{\epsilon Q + \bar{\epsilon}\bar{Q}} \Phi(x^\mu, \theta, \bar{\theta}) = \Phi(x^\mu - i\epsilon\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\epsilon}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}). \tag{9.11}$$

If one expands Φ in powers of θ, there are only a finite number of terms. These can be decomposed into two irreducible representations of the algebra, corresponding to the chiral and vector superfields described above. To understand these, we need to introduce one more set of objects, the covariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$. These are objects which anticommute with the supersymmetry generators and thus are useful for writing down invariant expressions. They are given by

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \quad \bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \tag{9.12}$$

They satisfy the anticommutation relations

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu, \quad \{D_\alpha, D_\alpha\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0. \tag{9.13}$$

We can use the Ds to construct irreducible representations of the supersymmetry algebra. Because the Ds anticommute with the Qs, the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{9.14}$$

is invariant under supersymmetry transformations. Fields that satisfy this condition are called chiral fields. To construct such fields, we would like to find combinations of x^μ, θ and $\bar{\theta}$ which are annihilated by $\bar{D}_\alpha$. Writing

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}, \tag{9.15}$$

then

$$\Phi = \Phi(y) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \tag{9.16}$$

is a chiral (scalar) superfield. Expanding in θ, we see that the expansion terminates:

$$\Phi = \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi + \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi \tag{9.17}$$
$$+ \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta^2 F.$$

We can work out the transformation laws. Starting with

$$\delta\Phi = \epsilon^\alpha Q_\alpha\Phi + \epsilon^*_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}, \tag{9.18}$$

the components transform as follows:

$$\delta\phi = \sqrt{2}\epsilon\psi, \quad \delta\psi = \sqrt{2}\epsilon F + \sqrt{2}i\sigma^\mu\epsilon^*\partial_\mu\phi, \quad \delta F = i\sqrt{2}\epsilon^*\bar{\sigma}^\mu\partial_\mu\psi. \tag{9.19}$$

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Vector superfields form another irreducible representation of the algebra; they satisfy the condition

$$V = V^{\dagger}. \tag{9.20}$$

Again, it is easy to check that this condition is preserved by supersymmetry transformations. A vector superfield V can be expanded in a power series in the θs:

$$V = i\chi - i\chi^{\dagger} - \theta\sigma^{\mu}\theta^{*}A_{\mu} + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D. \tag{9.21}$$

Here χ is not quite a chiral field. It is a superfield which is a function of θ only, i.e. it has terms with zero, one or two θs; χ^{*} is its conjugate.

If V is to describe a massless field, the presence of A_{μ} indicates that there should be some underlying gauge symmetry, which generalizes the conventional transformation of bosonic theories. In the case of a $U(1)$ theory, gauge transformations act by

$$V \to V + i\Lambda - i\Lambda^{\dagger} \tag{9.22}$$

where Λ is a chiral field. The $\theta\theta^{*}$ term in Λ is precisely a conventional gauge transformation of A_{μ}. In the case of a $U(1)$ theory, one can define a gauge-invariant field strength

$$W_{\alpha} = -\frac{1}{4}\bar{D}^2 D_{\alpha} V. \tag{9.23}$$

By a gauge transformation, we can set $\chi = 0$. The resulting gauge is known as the Wess–Zumino gauge. This gauge is analogous to the Coulomb gauge in electrodynamics:

$$W_{\alpha} = -i\lambda_{\alpha} + \theta_{\alpha}D - \sigma_{\alpha}^{\mu\nu}\beta F_{\mu\nu}\theta_{\beta} + \theta^2\sigma^{\mu}_{\alpha\dot{\beta}}\partial_{\mu}\lambda^{*\dot{\beta}}. \tag{9.24}$$

The gauge transformation of a chiral field of charge q is given by

$$\Phi \to e^{-iq\Lambda}\Phi. \tag{9.25}$$

One can form gauge-invariant combinations using the vector superfield (connection) V:

$$\Phi^{\dagger}e^{+qV}\Phi. \tag{9.26}$$

We can also define a *gauge-covariant derivative* by

$$\mathcal{D}_{\alpha}\Phi = D_{\alpha}\Phi + D_{\alpha}V\Phi. \tag{9.27}$$

This construction has a non-Abelian generalization. It is most easily motivated by first generalizing the transformation of Φ to

$$\Phi \to e^{-i\Lambda}\Phi, \tag{9.28}$$

where Λ is now a matrix-valued chiral field.

Now we want to combine $\phi^{\dagger}$ and ϕ in a gauge-invariant way. By analogy with what we did in the Abelian case, we introduce a matrix-valued field V and require that

$$\Phi^{\dagger}e^{V}\Phi \tag{9.29}$$

be gauge-invariant. So we require that

$$e^{V} \to e^{-i\Lambda^{*}}e^{V}e^{i\Lambda}. \tag{9.30}$$

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From this, we can define a gauge-covariant field strength,

$$W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V. \tag{9.31}$$

This transforms under gauge transformations like a chiral field in the adjoint representation:

$$W_\alpha \to e^{i\Lambda} W_\alpha e^{-i\Lambda}. \tag{9.32}$$

9.3 $N = 1$ Lagrangians

In ordinary field theories we construct Lagrangians that are invariant under translations by integrating densities over all space. The Lagrangian changes by a derivative under translations, so the *action* is invariant. Similarly, if we start with a Lagrangian density in superspace, a supersymmetry transformation acts by differentiation with respect to x or θ. So, integrating the variation over the full superspace gives zero. This is the basic feature of the integration rules that we introduced earlier. In terms of equations we have

$$\delta \int d^4x \int d^4\theta\ h(\Phi, \Phi^\dagger, V) = \int d^4x d^4\theta\ (\epsilon^\alpha Q_\alpha + \epsilon_{\dot{\alpha}} Q^{\dot{\alpha}}) h(\Phi, \Phi^\dagger, V) = 0. \tag{9.33}$$

For chiral fields, integrals over *half* superspace are invariant. If $f(\Phi)$ is a function of chiral fields only, f itself is chiral. As a result,

$$\delta \int d^4x d^2\theta f(\Phi) = \int d^4x d^2\theta (\epsilon^\alpha Q_\alpha + \epsilon_{\dot{\alpha}} Q^{\dot{\alpha}}) f(\Phi). \tag{9.34}$$

The integrals over the Q_α terms vanish when integrated over x with respect to $d^2\theta$. The Q^* terms also give zero. To see this, note that $f(\Phi)$ is itself chiral (check), so that

$$\bar{Q}_{\dot{\alpha}} f \propto \theta^\alpha \sigma^\mu{}_{\alpha\dot{\alpha}} \partial_\mu f. \tag{9.35}$$

We can construct a general Lagrangian for a set of chiral fields Φ_i and gauge group $\mathcal{G}$. The chiral fields have dimension one (again, note that the θs have dimension $-1/2$). The vector superfields V are dimensionless, while W_α has dimension $3/2$. With these ingredients, we can write down the most general renormalizable Lagrangian. First, there are terms involving integration over the full superspace:

$$\mathcal{L}_{\text{kin}} = \int d^4\theta \sum_i \Phi_i^\dagger e^V \Phi_i, \tag{9.36}$$

where the factor e^V is in the representation of the gauge group appropriate to the field Φ_i. We can also write down an integral over half of superspace:

$$\mathcal{L}_W = \int d^2\theta\, W(\Phi_i) + \text{c.c.} \tag{9.37}$$

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Here $W(\Phi)$ is a holomorphic function of the Φ_is (it is a function of Φ_i, not $\Phi_i^\dagger$), called the superpotential. For a renormalizable theory,

$$W = \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}\Gamma_{ijk}\Phi_i\Phi_j\Phi_k. \tag{9.38}$$

Finally, for the gauge fields we can write

$$\mathcal{L}_{\text{gauge}} = \frac{1}{g^{(i)2}}\int d^2\theta\, W_\alpha^{(i)2}. \tag{9.39}$$

The full Lagrangian density is

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_W + \mathcal{L}_{\text{gauge}}. \tag{9.40}$$

The superspace formulation has provided us with a remarkably simple way to write the general Lagrangian. In this form, however, the meaning of these various terms is rather opaque. We would like to express them in terms of the component fields. We can do this by using our expressions for the fields in terms of their components, and our simple integration table. We first consider a single chiral field Φ that is neutral under any gauge symmetries. Then

$$\mathcal{L}_{\text{kin}} = |\partial_\mu\Phi|^2 + i\psi_\Phi\,\partial_\mu\sigma^\mu\psi_\Phi^* + F_\Phi^*F_\Phi. \tag{9.41}$$

The field F is referred to as an *auxiliary field*, as it appears without derivatives in the action. Its equation of motion will be algebraic and can be solved easily. It has no dynamics. For several fields, labeled with an index i, the generalization is immediate:

$$\mathcal{L}_{\text{kin}} = |\partial_\mu\phi_i|^2 + i\psi_i\partial_\mu\sigma^\mu\psi_i^* + F_i^*F_i. \tag{9.42}$$

It is also easy to work out the component form of the superpotential terms. We will write this down for several fields:

$$\mathcal{L}_W = \frac{\partial W}{\partial\Phi_i}F_i + \frac{\partial^2 W}{\partial\Phi_i\Phi_j}\psi_i\psi_j. \tag{9.43}$$

For our special choice of superpotential this becomes

$$\mathcal{L}_W = F_i(m_{ij}\Phi_j + \lambda_{ijk}\Phi_j\Phi_k) + (m_{ij} + \lambda_{ijk}\Phi_k)\psi_i\psi_j + \text{c.c.} \tag{9.44}$$

It is a simple matter to solve for the auxiliary fields:

$$F_i^* = -\frac{\partial W}{\partial\Phi_i}. \tag{9.45}$$

Substituting back into the Lagrangian, we obtain

$$V = |F_i|^2 = \left|\frac{\partial W}{\partial\Phi_i}\right|^2. \tag{9.46}$$

To work out the couplings of the gauge fields, it is convenient to choose the Wess–Zumino gauge. Again, this is analogous to the Coulomb gauge, in that it makes manifest the physical degrees of freedom (the gauge bosons and gauginos) but the

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supersymmetry is not explicit. We will leave performing the integrations over superspace to the exercises, and just quote the full Lagrangian in terms of the component fields:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}g_a^{-2}F_{\mu\nu}^{a2} - i\lambda^a\sigma^\mu D_\mu\lambda^{a*} + |D_\mu\phi_i|^2 - i\psi_i\sigma^\mu D_\mu\psi_i^* \\ &+ \frac{1}{2g^2}(D^a)^2 + D^a\sum_i \phi_i^* T^a\phi_i + F_i^*F_i - F_i\frac{\partial W}{\partial\phi_i} + \text{c.c.} \\ &+ \sum_{ij}\frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j + i\sqrt{2}\sum\lambda^a\psi_i T^a\phi_i^*.\end{aligned} \tag{9.47}$$

The scalar potential is found by solving for the auxiliary D and F fields:

$$V = |F_i|^2 + \frac{1}{2g_a^2}(D^a)^2 \tag{9.48}$$

with

$$F_i = \frac{\partial W}{\partial\phi_i^*}, \quad D^a = \sum_i (g^a\phi_i^* T^a\phi_i). \tag{9.49}$$

In the case where there is a $U(1)$ factor in the gauge group, there is one more term one can include in the Lagrangian, known as the Fayet–Iliopoulos D term. In superspace,

$$\xi\int d^4\theta V \tag{9.50}$$

is supersymmetric and gauge invariant, since the integral $\int d^4\theta\Phi$ vanishes for any chiral field. In components, this is simply a term linear in D, ξD; so, solving for D from its equations of motion, we obtain

$$D = \xi + \sum_i q_i\phi_i^*\phi_i. \tag{9.51}$$

9.4 The supersymmetry currents

We have written down classical expressions for the supersymmetry generators, but for many purposes it is valuable to have expressions for these objects as operators in quantum field theory. We can obtain these by using the Noether procedure. We need to be careful, though, because the Lagrangian is not invariant under supersymmetry transformations but instead transforms by a total derivative. This is similar to the problem of translations in field theory. To see that there is a total derivative in the variation, recall that the Lagrangian has the form, in superspace,

$$\int d^4\theta f(\theta,\bar{\theta}) + \int d^2\theta W(\theta) + \text{c.c.} \tag{9.52}$$

The supersymmetry generators all involve a $\partial/\partial\theta$ term and a $\theta\partial_\mu$ term. The variation of the Lagrangian is proportional to $\int d^4\theta\epsilon Qf + \cdots$. The term involving $\partial/\partial\theta$ integrates to

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zero, but the extra term does not; only in the action, obtained by integrating the Lagrangian density over space–time, does the derivative term drop out.

So, in performing the Noether procedure the variation of the Lagrangian will have the form

$$\delta\mathcal{L} = \epsilon\partial_\mu K^\mu + (\partial_\mu\epsilon)T^\mu. \tag{9.53}$$

Integrating by parts, we have that $K^\mu - T^\mu$ is conserved. Taking this into account, for a theory with a single chiral field,

$$j^\mu_\alpha = \sqrt{2}\sigma^\nu_{\alpha\dot\beta}\bar\sigma^{\mu\dot\beta\gamma}\psi_\gamma\partial_\nu\phi^* + i\sqrt{2}F\sigma^{\mu\alpha\dot\alpha}\psi^*_{\dot\alpha} \tag{9.54}$$

and similarly for $j^\mu_{\dot\alpha}$. The generalization for several chiral fields is obvious: one makes the replacements $\psi \to \psi_i$, $\phi \to \phi_i$, etc. and sums over i. One can check that the (anti)commutators of the Qs (which are integrals over j^0) with the various fields gives the correct transformations laws. One can do the same for the gauge fields. Working with the action written in terms of W there are no derivatives, so the variation of the Lagrangian comes entirely from the $\partial_\mu K^\mu$ term in Eq. (9.53). We have already seen that the variation of $\int d^2\theta$ is a total derivative. The current is worked out in the exercises at the end of this chapter.

9.5 The ground state energy in globally supersymmetric theories

One striking feature of the Lagrangian of Eq. (9.47) is that the potential $V \geq 0$. This fact can be traced back to the supersymmetry algebra. Start with the equation

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2P_\mu\sigma^\mu_{\alpha\dot\beta}, \tag{9.55}$$

multiply by σ^0 and take the trace:

$$E = \frac{1}{4}Q_\alpha\bar{Q}_{\dot\alpha} + \bar{Q}_{\dot\alpha}Q_\alpha. \tag{9.56}$$

Since the left-hand side is positive, the energy is always greater than or equal to zero.

In global supersymmetry, $E = 0$ is very special: the expectation value of the energy is an *order parameter* for supersymmetry breaking. If the supersymmetry is unbroken then $Q_\alpha|0\rangle = 0$, so the ground-state energy vanishes *if and only if* the supersymmetry is unbroken.

Alternatively, consider the supersymmetry transformation laws for λ and ψ. One has, under a supersymmetry transformation with parameter ϵ,

$$\delta\psi = \sqrt{2}\epsilon F + \cdots, \quad \delta\lambda = i\epsilon D + \cdots. \tag{9.57}$$

In quantum theory the supersymmetry transformation laws become operator equations

$$\delta\psi = i\{Q, \psi\}; \tag{9.58}$$

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so, taking the vacuum expectation value of both sides, we see that a non-vanishing field F means broken supersymmetry. Again the vanishing of the energy is an indicator of supersymmetry breaking. So, if either F or D has an expectation value, the supersymmetry is broken.

The signal of ordinary (bosonic) symmetry breakdown is a Goldstone boson. In the case of supersymmetry the signal is the presence of a Goldstone fermion, or *goldstino*. One can prove a goldstino theorem in almost the same way as one proves Goldstone's theorem. We will do this shortly, when we consider simple models of supersymmetry and its breaking.

9.6 Some simple models

In this section we consider some simple models, in order to develop some practice with supersymmetric Lagrangians and to illustrate how supersymmetry is realized in the spectra of these theories.

9.6.1 The Wess–Zumino model

One of the earliest, and simplest, models is the Wess–Zumino model, a theory of a single chiral field (no gauge interactions). For the superpotential we take

$$W = \frac{1}{2}m\phi^2 + \frac{\lambda}{3}\phi^3. \tag{9.59}$$

The scalar potential is (using ϕ for the super-and-scalar field)

$$V = |m\phi + \lambda\phi^2|^2 \tag{9.60}$$

and the ϕ field has mass-squared $|m|^2$. The fermion mass term is

$$\frac{1}{2}m\psi\psi, \tag{9.61}$$

so the fermion also has mass m.

We will now consider the symmetries of the model. First, set $m = 0$. The theory then has a continuous global symmetry. This is perhaps not obvious from the form of the superpotential, $W = (\lambda/3)\phi^3$. But the Lagrangian is an integral over superspace of W, so it is possible for W to transform and for the θs to transform in a compensating fashion. Such a symmetry, which does not commute with supersymmetry, is called an *R symmetry*. If, by convention, we take the θs to carry charge 1 then the $d\theta$s carry charge -1 (think of the integration rules). So the superpotential must carry charge 2. In the present case, this means that ϕ carries charge $2/3$. Note that each component of the superfield transforms differently:

$$\phi \to e^{i(2/3)\alpha}\phi, \quad \psi \to e^{i(2/3-1)\alpha}\psi, \quad F \to e^{i(2/3-2)\alpha}F. \tag{9.62}$$

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Now consider the problem of mass renormalization at one loop in this theory. First suppose again that $m = 0$. From our experience with non-supersymmetric theories we might expect a quadratically divergent correction to the scalar mass. But ϕ^2 carries charge $4/3$, and this forbids a mass term in the superpotential. For the fermion the symmetry does not permit us to draw any diagram which corrects the mass. For the boson, however, there are two diagrams, one with intermediate scalars and one with fermions. We will study these in detail later. Consistently with our argument, these two diagrams are found to cancel.

What if, at tree level, $m \neq 0$? We will see shortly that there are still no corrections to the mass term in the superpotential. In fact, perturbatively, there are no corrections to the superpotential at all. There are, however, wave-function renormalizations; rescaling ϕ corrects the masses. In four dimensions, the wave-function corrections are logarithmically divergent, so there are logarithmically divergent corrections to the masses but no quadratic divergences.

9.6.2 A $U(1)$ gauge theory

Consider a $U(1)$ gauge theory, with two charged chiral fields, ϕ^+ and ϕ^-, having charges ± 1, respectively. First suppose that the superpotential vanishes. Our experience with ordinary field theories would suggest that we start developing a perturbation expansion about the point in field space $\phi^\pm = 0$. But, consider the potential in this theory. In the Wess–Zumino gauge we have

$$V(\phi^\pm) = \frac{1}{2}D^2 = \frac{g^2}{2}(|\phi^+|^2 - |\phi^-|^2)^2. \tag{9.63}$$

Zero-energy supersymmetric minima have $D = 0$. By a gauge choice we can set

$$\phi^+ = v, \quad \phi^- = v' e^{i\alpha}, \tag{9.64}$$

with v, v' parameters with dimensions of mass. Then $D = 0$ if $v = v'$. In field theory, as discussed in Section 2.3, when one has such a continuous degeneracy, just as in the case of global symmetry breaking, one must choose a vacuum. Each vacuum is physically distinct – in this case, the spectra are different – and there are no transitions between vacua.

It is instructive to work out the spectrum in a vacuum with a given v. One has, first, the gauge bosons, with masses

$$m_v^2 = 4g^2v^2. \tag{9.65}$$

This accounts for three degrees of freedom. From the Yukawa couplings of the gaugino λ to the ϕs, one has a term

$$\mathcal{L}_\lambda = \sqrt{2}gv\lambda(\psi_{\phi^+} - \psi_{\phi^-}), \tag{9.66}$$

so we have a Dirac fermion with mass $2gv$. Now we have accounted for three bosonic and two fermionic degrees of freedom. The fourth bosonic degree of freedom is a scalar; one can think of it as the partner of the Higgs, which is eaten in the Higgs phenomenon.

To compute its mass, note that, expanding the scalars as

$$\phi^{\pm} = v + \delta\phi^{\pm}, \tag{9.67}$$

we have

$$D = gv(\delta\phi^{+} + \delta\phi^{+*} - \delta\phi^{-} - \delta\phi^{-*}). \tag{9.68}$$

So D^2 gives a mass to the real part of $\delta\phi^{+} - \delta\phi^{-}$, equal to the mass of the gauge bosons and gauginos. Since the masses differ in states with different v, these states are physically inequivalent.

There is also a massless state: a single chiral field. For the scalars, this follows on physical grounds: the expectation value v is undetermined and one phase is undetermined, so there is a massless complex scalar. For the fermions, the linear combination $\psi_{\phi^+} + \psi_{\phi^-}$ is massless. So we have the correct number of fields to construct a massless chiral multiplet. We can describe this elegantly by introducing the composite chiral superfield or *modulus*

$$\Phi = \phi^{+}\phi^{-} \approx v^2 + v(\delta\phi^{+} + \delta\phi^{-}). \tag{9.69}$$

Its components are precisely the massless complex scalar and the chiral fermion which we identified above.

This is our first encounter with a phenomenon which is nearly ubiquitous in supersymmetric field theories and string theory: there are often continuous sets of vacuum states, at least in some approximation. The set of such physically distinct vacua is known as the *moduli space*. In this example the set of such states is parameterized by the values of the modulus field Φ.

In quantum mechanics, in such a situation we would solve for the wave function of the modulus and the ground state would typically involve a superposition of the different classical ground states. We have seen, though, that in field theory one must choose a value for the modulus field. In the presence of such a degeneracy, for each such value one has, in effect, a different field theory – no physical process leads to transitions between one such state and another. Once the degeneracy is lifted, however, this is no longer the case and transitions, as we will frequently see, are possible.

9.7 Non-renormalization theorems

In ordinary field theories, as we integrate out the physics between one scale and another, we generate every term in the effective action permitted by the symmetries. This is not the case in supersymmetric field theories. This feature gives such theories surprising, and possibly important, properties when we consider questions of naturalness. It also gives us a powerful tool to explore the dynamics of these theories, even at strong coupling. This power comes easily; in this section, we will enumerate these theorems and explain how they arise.

So far, we have restricted our attention to renormalizable field theories. But we have seen that, in considering Beyond the Standard Model physics, we may wish to relax this restriction. It is not hard to write down the most general, globally supersymmetric, theory with at most two derivatives, using the superspace formalism:

$$\mathcal{L} = \int d^4\theta K(\phi_i, \phi_i^\dagger) + \int d^2\theta W(\phi_i) + \text{c.c.} + \int d^2\theta f_\mathrm{a}(\phi)\big(W_\alpha^{(a)}\big)^2 + \text{c.c.} \tag{9.70}$$

The function K is known as the Kahler potential. Its derivatives dictate the form of the kinetic terms for the different fields. The functions W and f_a are holomorphic (what physicists would comfortably call "analytic") functions of the chiral fields. In terms of the component fields (see the exercises) the real part of f couples to $F_{\mu\nu}^2$; the functions W and f_a thus determine the gauge couplings. The imaginary parts couple to the now-familiar operator $F\tilde{F}$. These features of the Lagrangian will be important in much of our discussion of supersymmetric field theories and string theory.

Non-supersymmetric theories have the property that they tend to be generic; any term permitted by symmetries in the theory will appear in the effective action, with an order of magnitude determined by dimensional analysis.[2] Supersymmetric theories are special in that this is not the case. In $N = 1$ theories, there are non-renormalization theorems governing the superpotential and the gauge coupling functions f of Eq. (9.70). These theorems assert that the superpotential is not corrected in perturbation theory beyond its tree level value, while f is at most renormalized at one loop.[3]

Originally, these theorems were proven by the detailed study of Feynman diagrams. Seiberg has pointed out that they can be understood in a much simpler way. Both the superpotential and the functions f are holomorphic functions of the chiral fields, i.e. they are functions of the ϕ_is and not the ϕ_i^*s. This is evident from their construction. Seiberg argued that the coupling constants of a theory may be thought of as *expectation values* of chiral fields and so the superpotential must be a holomorphic function of these as well. For example, consider a theory of a single chiral field Φ with superpotential

$$W = m\Phi^2 + \lambda\Phi^3. \tag{9.71}$$

We can think of λ and m as the expectation values of chiral fields $\lambda(x,\theta)$ and $m(x,\theta)$.

In the Wess–Zumino Lagrangian, if we first set λ to zero then there is an R symmetry under which Φ has R-charge 1 and λ has R-charge -1. Now consider corrections to the effective action in perturbation theory. For example, renormalizations of λ in the superpotential necessarily involve positive powers of λ. But such terms (apart from λ^1) have the wrong R-charge to preserve the symmetry. So there can be no renormalization of this coupling. There can be wave function renormalization, since K is not holomorphic, so $K = K(\lambda^\dagger\lambda)$ is allowed in general.

There are many interesting generalizations of these ideas, and we will not survey them here but will just mention two further examples. First, gauge couplings can be thought of

[2] In some cases, there may be suppression by a few powers of the coupling.

[3] There is an important subtlety connected with these theorems. Both should be interpreted as applying only to a Wilsonian effective action, in which one integrates out the physics above some scale μ. If infrared physics is included, the theorems do not necessarily hold. This is particularly important for the gauge couplings.

in the same way, i.e. we can treat g^{-2} as part of a chiral field. More precisely, we define

$$S = \frac{8\pi^2}{g^2} + ia + \cdots . \tag{9.72}$$

The real part of the scalar field in this multiplet couples to $F_{\mu\nu}^2$ but the imaginary part, a, couples to $F\tilde{F}$. Because $F\tilde{F}$ is a total derivative, in perturbation theory there is a symmetry under constant shifts of a. The effective action should respect this symmetry. Because the gauge coupling function f is holomorphic, this implies that

$$f(g^2) = S + \text{const} = \frac{8\pi^2}{g^2} + \text{const}. \tag{9.73}$$

The first term is just the tree level term. One-loop corrections yield a constant, but there are no higher-order corrections in perturbation theory! This is quite a striking result. It is also paradoxical, since the two-loop beta functions for supersymmetric Yang–Mills theories were computed long ago and are, in general, non-zero. The resolution of this paradox is subtle and interesting. It provides a simple computation of the two-loop beta function. In a particular renormalization scheme, it gives an exact expression for the beta function. This is explained in Appendix D.

Before explaining the resolution of the above paradox, there is one more non-renormalization theorem which we can prove rather trivially here. This is the statement that if there is no Fayet–Iliopoulos D term at tree level, this term can be generated at most at one loop. To prove this, write the D term as

$$\int d^4\theta d(g, \lambda) V. \tag{9.74}$$

Here $d(g, \lambda)$ is some unknown function of the gauge and other couplings in the theory. But, if we think of g and λ as chiral fields then this expression is only gauge invariant if d is a constant, corresponding to a possible one-loop contribution. Such contributions do arise in string theory.

In string theory, all the parameters *are* expectation values of chiral fields. Indeed, non-renormalization theorems in string theory, both for world-sheet and string perturbation theory, were proven by the sort of reasoning we have used above.

9.8 Local supersymmetry: supergravity

If supersymmetry has anything to do with nature, and is not merely an accident, then it must be a local symmetry. There is not space here for a detailed exposition of local supersymmetry. For most purposes, both theoretical and phenomenological, there are fortunately only a few facts we need to know. The field content (in four dimensions) is like that of global supersymmetry, except that now one has a graviton and a gravitino. Note that the number of additional bosonic and fermionic degrees of freedom (a minimal requirement if the theory is to be supersymmetric) is the same. The graviton is described by a traceless symmetric tensor; in $d-2 = 2$ dimensions, this has two independent components. Similarly, the gravitino ψ_μ has both a vector and a spinor index. It satisfies a constraint similar to tracelessness,

$$\gamma^\mu \psi_\mu = 0. \tag{9.75}$$

In $d-2$ dimensions, this amounts to two conditions, leaving two physical degrees of freedom.

As in global supersymmetry (without the restriction of renormalizability), the terms in the effective action with at most two derivatives or four fermions are completely specified by three functions:

1. the Kahler potential $K(\phi, \phi^\dagger)$, a function of the chiral fields;
2. the superpotential $W(\phi)$, a holomorphic function of the chiral fields;
3. the gauge coupling functions $f_a(\phi)$, which are also holomorphic functions of the chiral fields.

The Lagrangian which follows from these is quite complicated, as it includes many two- and four-fermion interactions. It can be found in the suggested reading. Our main concern in this text will be the scalar potential. This is given by

$$V = e^K \left[\left(\frac{\partial W}{\partial \phi_i} + \frac{\partial K}{\partial \phi_i} W \right) g^{i\bar{j}} \left(\frac{\partial W^*}{\partial \phi^*_{\bar{j}}} + \frac{\partial K}{\partial \phi^*_{\bar{j}}} W \right) - 3|W|^2 \right], \tag{9.76}$$

where

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial \phi_i \partial \phi_{\bar{j}}} \tag{9.77}$$

is the (Kahler) metric associated with the Kahler potential. In this equation, we have adopted units in which $M = 1$, where Newton's gravitational constant is given by

$$G_N = \frac{1}{8\pi M^2} \tag{9.78}$$

and $M \approx 2 \times 10^{18}$ GeV is known as the *reduced Planck mass*.

Suggested reading

The text by Wess and Bagger (1992) provides a good introduction to superspace, the fields and Lagrangians of supersymmetric theories in four dimensions and supergravity. Other texts include those by Gates *et al*. (1983) and Mohapatra (2003). Appendix B of Polchinski's (1998) text provides a concise introduction to supersymmetry in higher dimensions. The supergravity Lagrangian is derived and presented in its entirety in Cremmer *et al*. (1979) and Wess and Bagger (1992) and is reviewed in, for example, Nilles (1984). Non-renormalization theorems were first discussed from the viewpoint presented here by Seiberg (1993).

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Exercises

(1) Verify the commutators of the Qs and the Ds.
(2) Check that, given the definition Eq. (9.15), Φ is chiral. Show that any function of chiral fields is a chiral field.
(3) Verify that W_α transforms as in Eq. (9.32) and that $\mathrm{Tr}W_\alpha^2$ is gauge invariant.
(4) Derive the expression (9.47) for the component Lagrangian including gauge interactions and the superpotential, by performing the superspace integrals. For an $SU(2)$ theory with a scalar triplet $\vec{\phi}$ and singlet, X, take $W = \lambda(\vec{\phi}^2 - \mu^2)$. Find the ground state and work out the spectrum.
(5) Derive the supersymmetry current for a theory with several chiral fields. For a single field Φ and $W = (1/2)\, m\Phi^2$, verify, using the canonical commutation relations, that the Qs obey the supersymmetry algebra. Work out the supercurrent for a pure supersymmetric gauge theory.

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