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Zero Cycles on a Twisted Cayley Plane

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Abstract. Let k be a field of characteristic not $2, 3$. Let G be an exceptional simple algebraic group over k of type F_4, ${}^1\mathrm{E}_6$ or E_7 with trivial Tits algebras. Let X be a projective G-homogeneous variety. If G is of type E_7, we assume in addition that the respective parabolic subgroup is of type P_7. The main result of the paper says that the degree map on the group of zero cycles of X is injective.

1 Introduction

Let k be a field and G a simple algebraic group over k. Consider a projective G-homogeneous variety X over k. Any such variety over the separable closure k_s of k becomes isomorphic to the quotient G_s/P, where P is a parabolic subgroup of the split group $G_s = G \times_k k_s$. It is known that conjugacy classes of parabolic subgroups of G_s are in one-to-one correspondence with subsets of the vertices Π of the Dynkin diagram of G_s: we say a parabolic subgroup is of type $\theta \subset \Pi$ and denote it by P_θ if it is conjugate to a standard parabolic subgroup generated by the Borel subgroup and all unipotent subgroups corresponding to roots in the span of Π with no θ terms (see [TW02, 42.3.1]).

In the present paper we assume the field k has characteristic not $2, 3$, G is an exceptional simple algebraic group over k of type F_4, ${}^1\mathrm{E}_6$, or E_7 with trivial Tits algebras and X is a projective G-homogeneous variety over k. The goal of the paper is to compute the group of zero-cycles $\mathrm{CH}_0(X)$ which is an important geometric invariant of a variety. Namely, we prove the following theorem.

Theorem 1.1 *Let k be a field of characteristic not* $2, 3$. *Let G be an exceptional simple algebraic group over k of type* F_4, ${}^1\mathrm{E}_6$, *or* E_7 *with trivial Tits algebras and X a projective G-homogeneous variety over k. If G is of type* E_7, *we assume in addition that X corresponds to the parabolic subgroup of type P_7. Then the degree map* $\mathrm{CH}_0(X) \to \mathbb{Z}$ *is injective.*

The history of the question starts with the work of I. Panin [Pa84] who proved the injectivity of the degree map for Severi–Brauer varieties. For quadrics this was proved by R. Swan [Sw89]. The case of involution varieties was considered by A. Merkurjev [Me95]. For varieties of type F_4 it was announced by M. Rost.

Our work was mostly motivated by the paper of D. Krashen [Kr05], where he reformulated the question in terms of R-triviality of certain symmetric powers and

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proved the injectivity for a wide class of generalized Severi–Brauer varieties and some involutive varieties, hence, generalizing the previously known results by Panin and Merkurjev. Another motivating point is the result of V. Popov [Po05], which gives a full classification of generically n-transitive actions of a split linear algebraic group G on a projective homogeneous variety G/P. For instance, the case of a Cayley plane $X = G/P_1$, where G is split of type E_6 (see [IM05]), provides an example of such an action for $n = 3$. As a consequence, one can identify the open orbit $(S^3X)^0$ of the induced action on the third symmetric power with a homogeneous variety $G/(T \cdot \mathrm{Spin}_8)$, where T is the torus which is complementary to Spin_8. Then by the result of Krashen one reduces the question of injectivity to the question of R-triviality of a twisted form of $(S^3X)^0/S_3$.

Apart from the main result concerning exceptional varieties we provide shortened proofs of the injectivity of the degree map in cases of quadrics and Severi–Brauer varieties.

Recently V. Chernousov and A. Merkurjev have obtained an independent proof of the same results in any characteristic using Rost's Invariant and Chain Lemma (see [CM]). Our proof does not use those tools, but only the geometry of, and some basic facts about, projective homogeneous varieties.

The paper is organized as follows. In the first section we provide several facts about zero-cycles and symmetric powers. Then we prove the theorem for twisted forms of a Cayley plane (here the prime 3 plays the crucial role). In the next section we prove the injectivity in the case of a twisted form of a homogeneous variety of type E_7 (this deals with the prime 2). Combining these two results together with certain facts about rational correspondences we finish the proof of the theorem.

2 Zero Cycles and Symmetric Powers

In the present section we recall several results and definitions [Kr05] concerning the group of zero-cycles of a projective variety X and the group of R-equivalence classes of certain symmetric powers of X.

We shall systematically use the language of Galois descent; we identify a quasi-projective variety X over k with the variety $X_s = X \times_k k_s$ over the separable closure k_s equipped with an obvious action of the absolute Galois group $\Gamma = \mathrm{Gal}(k_s/k)$. By means of this identification the set of k-rational points of X is the set of k_s-rational points of X_s invariant under the action of Γ.

Let X be a variety over k. Two rational points $p, q \in X(k)$ are called *elementary linked* if there exists a rational morphism $\varphi\colon \mathbb{P}^1_k \dashrightarrow X$ such that $p, q \in \mathrm{Im}(\varphi(k))$. The R-equivalence is the equivalence relation generated by this relation. A variety X is called *R-trivial* if the set of rational points is non-empty and any two rational points are R-equivalent. A variety X is called *algebraically R-trivial* if $X_K = X \times_k K$ is R-trivial for any finite field extension K/k.

The n-th symmetric power of X is defined to be the quotient $S^nX = X^n/S_n$, where S_n is the symmetric group acting on the product

$$X^n = \underbrace{X \times \cdots \times X}_{n}$$

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by permuting the factors.

Let p be a prime number. A field k is called *prime-to-p closed* if there is no proper finite field extension K/k of degree prime to p. For any field k, we denote by k_p a prime-to-p closed algebraic extension of k.

Let X be a projective variety over k. By $\widetilde{\mathrm{CH}}_0(X)$ we denote the kernel of the degree map

$$\widetilde{\mathrm{CH}}_0(X) = \mathrm{Ker}(\deg\colon\ \mathrm{CH}_0(X) \to \mathbb{Z}).$$

The following results will be extensively used in the sequel.

Lemma 2.1 ([Kr05, Lemma 2.2]) *Assume $\widetilde{\mathrm{CH}}_0(X_{k_p}) = 0$ for each prime p. Then $\widetilde{\mathrm{CH}}_0(X) = 0$.*

Proposition 2.2 ([Kr05, Theorem 3.12]) *Suppose that k is prime-to-p closed and the following conditions are satisfied:*

(i) *for some integer $n \geq 0$, the p^n-th symmetric power $S^{p^n}X$ is algebraically R-trivial;*
(ii) *for any field extension K/k such that $X(K) \neq \varnothing$, the variety X_K is R-trivial.*

Then $\widetilde{\mathrm{CH}}_0(X) = 0$.

As an immediate consequence of these results, we obtain the proof of the fact that $\widetilde{\mathrm{CH}}_0(\mathrm{SB}(A)) = 0$, where A is a central simple algebra over k and $\mathrm{SB}(A)$ is the respective Severi–Brauer variety [Pa84, Theorem 2.3.7].

For simplicity we may assume $\deg A = p$ is prime. By Lemma 2.1, we may assume the base field k is prime-to-p closed (for a prime q different from p, the algebra A splits over k_q). According to Proposition 2.2 it suffices to show that the p-th symmetric power $S^p\,\mathrm{SB}(A_K)$ is R-trivial for every finite field extension K/k (the second hypothesis of Proposition 2.2 holds for any twisted flag variety). Changing the base, we may assume $K = k$. If A is split, the assertion is trivial, so we may assume A is a division algebra.

According to our conventions, the Severi–Brauer variety $\mathrm{SB}(A)$ is the variety of all parabolic subgroups P of type P_1 in the group $\mathrm{PGL}_1(A \otimes_k k_s)$ with the action of Γ coming from its action on k_s. Therefore, S^pX is the variety of all unordered p-tuples $[P^{(1)}, \dots, P^{(p)}]$ of parabolic subgroups of type P_1 of $\mathrm{PGL}_1(A \otimes_k k_s)$. Let U be an open subset of S^pX defined by the condition that the intersection $P^{(1)} \cap \cdots \cap P^{(p)}$ is a maximal torus in $\mathrm{PGL}_1(A\otimes_k k_s)$. Every maximal torus T in $\mathrm{PGL}_1(A\otimes_k k_s)$ is contained in precisely p parabolic subgroups of type P_1 whose intersection is T. Therefore, U is isomorphic to the variety of all maximal tori in $\mathrm{PGL}_1(A)$. This variety is known to be rational and, hence, R-trivial (since it is homogeneous). To finish the proof, observe that the open embedding $U \to S^pX$ is surjective on k-points. So S^pX is R-trivial.

The same method can be applied to prove that $\widetilde{\mathrm{CH}}_0(Q) = 0$ for a nonsingular projective quadric Q over a field of characteristic not 2 (the result of Swan [Sw89]).

As above, we may assume that $p = 2$ and Q is anisotropic. It suffices to prove that S^2Q is R-trivial. Let q be the corresponding quadratic form on a vector space V. The quadric Q can be viewed as the variety of lines $\langle v\rangle$, where $v \in V \otimes_k k_s$ satisfies $q(v) = 0$, with an obvious action of Γ. Its second symmetric power S^2Q can be

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identified with the variety of pairs $[\langle v_1\rangle, \langle v_2\rangle]$ of lines (satisfying the same property), with the induced action of Γ. Consider the open subset U defined by the condition $b_q(v_1, v_2) \neq 0$ (b_q stands for the polarization of q). Clearly, the embedding $U \to S^2Q$ is surjective on k-points (otherwise the subspace $\langle v_1, v_2\rangle$ defines a totally isotropic subspace over k). So it is enough to check that U is R-trivial.

Consider the open subvariety W of $\mathrm{Gr}(2, V)$ consisting of planes $H \subset V \otimes_k k_s$ such that $q|_H$ is nonsingular. For every such plane there exists (up to scalar factors) exactly one hyperbolic basis $\{v_1, v_2\}$ over k_s. Therefore, the map from U to W sending $[\langle v_1\rangle, \langle v_2\rangle]$ to $\langle v_1, v_2\rangle$ is an isomorphism. But any open subvariety of $\mathrm{Gr}(2, V)$ is R-trivial, and we are done.

We shall use the following observation in the sequel.

Lemma 2.3 *Let $H \subset K \subset G$ be algebraic groups over k. Suppose that the map $\mathrm{H}^1(k, H) \to \mathrm{H}^1(k, K)$ is surjective. Then the morphism $G/H \to G/K$ is surjective on k-points.*

Proof An element x of $G/K(k)$ is represented by an element $g \in G(k_s)$ satisfying the condition that $\gamma(\sigma) = g^{-1} \cdot {}^{\sigma}g$ lies in $K(k_s)$ for all $\sigma \in \Gamma$. But γ is clearly a 1-cocycle with coefficients in K. Therefore by assumption, there exists some $h \in K$ such that $h^{-1}\gamma(\sigma) \cdot {}^{\sigma}h = (gh)^{-1} \cdot {}^{\sigma}(gh)$ is a 1-cocycle with coefficients in H. But then gh represents an element of $G/H(k)$ which goes to x under the morphism $G/H \to G/K$. ■

3 Twisted Forms of a Cayley Plane

In the present section we prove the injectivity of the degree map in the case when X is a twisted form of a Cayley plane.

Let J denote a simple exceptional 27-dimensional Jordan algebra over k, and N_J its norm (which is a cubic form on J). An invertible linear map $f\colon J \to J$ is called a *similitude* if there exists some $\alpha \in k^*$ (called the *multiplier* of f) such that $N_J(f(v)) = \alpha N_J(v)$ for all $v \in J$. The group $G = \mathrm{Sim}(J)$ of all similitudes is a reductive group whose semisimple part has type ${}^1\mathrm{E}_6$, and every group of type ${}^1\mathrm{E}_6$ with trivial Tits algebras can be obtained in this way up to isogeny (see [Ga01, Theorem 1.4]). The (*twisted*) *Cayley plane* $\mathbb{OP}^2(J)$ is the variety of all parabolic subgroups of type P_1 in $\mathrm{Sim}(J)$. Over the separable closure k_s, this variety can be identified with the variety of all lines $\langle e\rangle$ spanned by elements $e \in J_s = J \otimes_k k_s$ satisfying the condition $e \times e = 0$ (see [Ga01, Theorem 7.2]).

The goal of the present section is to prove the following.

Theorem 3.1 $\widetilde{\mathrm{CH}}_0(\mathbb{OP}^2(J)) = 0.$

We start the proof with the following easy reduction.

By Proposition 2.2 it is enough to prove that $(S^p\mathbb{OP}^2(J)) \times_k K$ is R-trivial for any prime p and any finite field extension K/k_p. Changing the base, we may assume $K = k_p$. Moreover, we may assume that the algebra J is not reduced (otherwise $\mathbb{OP}^2(J)$ is a rational homogeneous variety and, hence, is R-trivial).

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Assume $p \neq 3$, then $\mathbb{OP}^2(J)(k_p) \neq \varnothing$ and, hence, is R-trivial. Indeed, choose any cubic étale subalgebra L of J (see [Inv, Proposition 39.20]). It splits over k_p, and therefore, $L \otimes_k k_p$ contains a primitive idempotent e. As an element of $J \otimes_k k_p$, it satisfies the condition $e \times e = 0$ (see [SV, Lemma 5.2.1(i)]). So we may assume $p = 3$.

From now on $p = 3$ and the field k is prime-to-p closed. By definition $S^3(\mathbb{OP}^2(J))$ is the variety of all unordered triples $[\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle]$, where e_i are the elements of $J_s = J \otimes_k k_s$ satisfying the conditions $e_i \times e_i = 0$, with the natural action of Γ. Denote by U the open subvariety of $\mathbb{OP}^2(J)$ defined by the condition

$$N_{J_s}(e_1, e_2, e_3) \neq 0,$$

where N is the polarization of the norm.

The embedding $U \to S^3(\mathbb{OP}^2(J))$ is surjective on k-points. For, if $[\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle]$ is stable under the action of Γ and $N_{J_s}(e_1, e_2, e_3) = 0$, then $\langle e_1, e_2, e_3\rangle$ gives by descent a k-defined subspace V of J such that $N|_V = 0$. But then J is reduced by [SV, Theorem 5.5.2], which leads to a contradiction. So it is enough to show that U is R-trivial.

Choose a cubic étale subalgebra L in J. Over the separable closure this algebra can be represented as $L \otimes_k k_s = k_s e_1 \oplus k_s e_2 \oplus k_s e_3$, where $e_1, e_2, e_3 \in J_s$ are primitive idempotents. We have $e_i \times e_i = 0$, $i = 1, 2, 3$; the norm $N_{J_s}(e_1, e_2, e_3) = N_{L\otimes_k k_s}(e_1, e_2, e_3)$ is non-trivial and the triple $[e_1, e_2, e_3]$ is invariant under the action of Γ (so is L). Hence, the triple $[\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle]$ is a k-rational point of U.

By [SV68, Proposition 3.12], the group G acts transitively on U. Therefore, we have $U \simeq G/\operatorname{Stab}_G([\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle])$. The stabilizer is defined over k, since it is invariant under the action of Γ. Moreover, it coincides with $\operatorname{Stab}_G(L)$. Indeed, one inclusion is obvious, and the other one follows from the fact that e_1, e_2, e_3 are the only elements e of $L \otimes_k k_s$ satisfying the condition $e \times e = 0$ up to scalar factors (see [SV, Theorem 5.5.1]).

Remark 3.2 Consider the Springer decomposition $J = L \oplus V$ of J with respect to L. The pair (L, V) has a natural structure of a twisted composition, and there is a monomorphism $\operatorname{Aut}(L, V) \to \operatorname{Aut}(J)$ sending a pair (φ, t) (where $\varphi\colon L \to L$, $t\colon V \to V$) to $\varphi \oplus t\colon J \to J$ (see [Inv, § 38.A]). Note that $\operatorname{Aut}(L, V)$ coincides with the stabilizer of L in $\operatorname{Aut}(J)$.

Lemma 3.3 *The following sequence of algebraic groups is exact*

$$1 \to \operatorname{Aut}(L, V) \to \operatorname{Stab}_G(L) \to R_{L/k}(\mathbb{G}_m) \to 1,$$
$$f \mapsto f(1)$$

where $R_{L/k}$ stands for the Weil restriction.

Proof Exactness at the middle term follows from Remark 3.2 and the fact that the stabilizer of 1 in G coincides with $\operatorname{Aut}(J)$ (see [SV, Proposition 5.9.4]). To prove the

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exactness at the last term observe that a k_s-point of $R_{L/k}(\mathbb{G}_m)$ is a triple of scalars $(\alpha_0, \alpha_1, \alpha_2) \in k_s^* \times k_s^* \times k_s^*$. We have to find $f \in \mathrm{Stab}_G(L)(k_s)$ which sends 1 to $\mathrm{diag}(\alpha_0, \alpha_1, \alpha_2)$.

Assume first that $\alpha_0\alpha_1\alpha_2 = 1$. Choose a *related triple* (t_0, t_1, t_2) of elements of $\mathrm{GO}^+(\mathbb{O}_d, N_{\mathbb{O}_d})$ ($\mathbb{O}_d$ is the split Cayley algebra) such that $\mu(t_i) = \alpha_i$, $i = 0, 1, 2$ (see [Inv, Corollary 35.5]). Now the transformation f of J defined by

$$\begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_0\varepsilon_0 & t_2(c_2) & \cdot \\ \cdot & \alpha_1\varepsilon_1 & t_0(c_0) \\ t_1(c_1) & \cdot & \alpha_2\varepsilon_2 \end{pmatrix}$$

lies in $\mathrm{Sim}(J)$ by [Ga01, (7.3)], stabilizes $L \otimes_k k_s = \mathrm{diag}(k_s, k_s, k_s)$ and sends $1 \in J_s$ to $\mathrm{diag}(\alpha_0, \alpha_1, \alpha_2)$.

In the general case set $\alpha_i' = \alpha_i(\alpha_0\alpha_1\alpha_2)^{-\frac{1}{3}}$ $(i = 1, 2, 3)$, find f' such that $f'(1) = \mathrm{diag}(\alpha_1', \alpha_2', \alpha_3')$, and define f to be the product of f' and the scalar transformation of J_s with the coefficient $(\alpha_0\alpha_1\alpha_2)^{\frac{1}{3}}$ (which is an element of $\mathrm{Stab}_G(L)(k_s)$). ■

Since $\mathrm{H}^1(k, L^*) = 1$ (by Hilbert '90), the map $\mathrm{H}^1(k, \mathrm{Aut}(L, V)) \to \mathrm{H}^1(k, \mathrm{Stab}_G(L))$ is surjective. By Lemma 2.3 the morphism $G/\mathrm{Aut}(L, V) \to G/\mathrm{Stab}_G(L) \simeq U$ is surjective on k-points. Therefore, it suffices to show that $G/\mathrm{Aut}(L, V)$ is R-trivial.

Consider the morphism $\psi\colon G/\mathrm{Aut}(L, V) \to G/\mathrm{Aut}(J)$. By [Kr05, Corollary 3.18] it suffices to show that

(i) ψ is surjective on k-points;
(ii) $G/\mathrm{Aut}(J)$ is R-trivial;
(iii) The fibers of ψ (which are isomorphic to $\mathrm{Aut}(J)/\mathrm{Aut}(L, V)$) are unirational and R-trivial.

In order to prove surjectivity of ψ on k-points, by Lemma 2.3 it is enough to prove surjectivity of the map $\mathrm{H}^1(k, \mathrm{Aut}(L, V)) \to \mathrm{H}^1(k, \mathrm{Aut}(J))$. The set $\mathrm{H}^1(k, \mathrm{Aut}(L, V))$ classifies all twisted compositions (L', V') which become isomorphic to (L, V) over k_s and $\mathrm{H}^1(k, \mathrm{Aut}(J))$ classifies all (exceptional 27-dimensional) Jordan algebras J'. It is easy to verify that the morphism sends (L', V') to the Jordan algebra $L' \oplus V'$, and hence, the surjectivity follows from the fact that any Jordan algebra admits a Springer decomposition (*cf.* [Inv, Proposition 38.7]).

Let W be the open subvariety of J consisting of elements v with $N_J(v) \neq 0$. Then G acts transitively on W (see [SV, Proposition 5.9.3]) and the stabilizer of the point 1 coincides with $\mathrm{Aut}(J)$. So $G/\mathrm{Aut}(J) \simeq W$ is clearly R-trivial.

Consider the variety Y of all étale cubic subalgebras of J. By [Inv, Proposition 39.20(1)], there is a map from an open subvariety J_0 of regular elements in J to Y (sending a to $k[a]$), surjective on k-points. Therefore Y is unirational and R-trivial 24-dimensional irreducible variety.

The group $\mathrm{Aut}(J)$ acts on Y naturally. Let L' be any k-point of Y. The stabilizer of L' in $\mathrm{Aut}(J)$ is equal to $\mathrm{Aut}(L', V')$ ($J = L' \oplus V'$ is the Springer decomposition). So the orbit of L' is isomorphic to $\mathrm{Aut}(J)/\mathrm{Aut}(L', V')$, and in particular, has dimension 24. Therefore, it is open and, since L' is arbitrary, the action is transitive. So $\mathrm{Aut}(J)/\mathrm{Aut}(L, V) \simeq Y$ is unirational and R-trivial and the proof of the theorem is complete.

4 The Case of E_7/P_7

In the present section we prove the injectivity of the degree map for twisted forms of a projective homogeneous variety corresponding to an exceptional group of type E_7 and a parabolic subgroup of type P_7.

Let $\mathcal{B}$ denote a 56-dimensional Brown algebra over k. It defines (up to a scalar factor) a skew-symmetric form b on $\mathcal{B}$ and a trilinear map t from $\mathcal{B} \times \mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$ such that $(\mathcal{B}, t, b)$ is a Freudenthal triple system (see [Ga01, Definition 3.1] and [Ga01, §4]). An invertible linear map $f\colon \mathcal{B} \to \mathcal{B}$ is called a *similitude* if there exists some $\alpha \in k^*$ (called the *multiplier* of f) such that $b(f(u), f(v)) = \alpha b(u, v)$ and $t(f(u), f(v), f(w)) = \alpha f(t(u, v, w))$ for all $u, v, w \in \mathcal{B}$. The group $G = \mathrm{Sim}(\mathcal{B})$ of all similitudes is a reductive group whose semisimple part has type E_7 and every group of type E_7 with trivial Tits algebras can be obtained in this way up to isogeny (*cf.* [Ga01, Theorem 4.16]).

An element e is called *singular* (or *strictly regular* following [Fe72]) if $t(e, e, \mathcal{B}) \subseteq \langle e\rangle$. In this case $t(e, e, v) = 2b(v, e)e$ for every $v \in V$. An equivalent definition is that $t(e, e, e) = 0$ and $e \in t(e, e, \mathcal{B})$ (see [Fe72, Lemma 3.1]). An algebra $\mathcal{B}$ is called *reduced* if it contains singular elements. There do exist anisotropic groups of type E_7 with trivial Tits algebras over certain fields (see [Ti90]).

Let $X(\mathcal{B})$ be the variety obtained by Galois descent from the variety of all parabolic subgroups of type P_7 in $\mathrm{Sim}(\mathcal{B} \otimes_k k_s)$. It can be identified with the variety of lines $\langle e\rangle$ spanned by singular elements $e \in \mathcal{B} \otimes_k k_s$ (see [Ga01, Theorem 7.6]).

The goal of this section is to prove the following.

Theorem 4.1 $\widetilde{\mathrm{CH}}_0(X(\mathcal{B})) = 0.$

We start with the similar reduction as in the case of E_6.

4.1 Assume first that G has Tits index $E^{66}_{7,1}$ (see [Ti66, Table II]). Its anisotropic kernel is of type D_6 and, since G has trivial Tits algebras, the anisotropic kernel corresponds to a 12-dimensional nondegenerate quadratic form q with split simple factors of its Clifford algebra. A straightforward computation (see [Br05, Thm. 7.4]) shows that $\mathcal{M}(X(\mathcal{B})) \simeq \mathcal{M}(Q) \oplus \mathcal{M}(Y)(6) \oplus \mathcal{M}(Q)(17)$, where Q is the projective quadric corresponding to q, Y is a twisted form of the maximal orthogonal grassmanian of a split 12-dimensional quadric, and $\mathcal{M}$ denotes Chow motive. Therefore, $\widetilde{\mathrm{CH}}_0(X(\mathcal{B})) = \widetilde{\mathrm{CH}}_0(Q) = 0$, where the last equality is due to Swan.

4.2 By Proposition 2.2, it is enough to prove that $(S^p X(\mathcal{B})) \times_k K$ is R-trivial for any prime p and any finite field extension K/k_p. After the base change it suffices to prove it for $K = k_p$. Moreover, we may assume $\mathcal{B}$ is not reduced (otherwise $X(\mathcal{B})$ is rational and, hence, R-trivial).

Assume $p \neq 2$, then $\mathcal{B} \otimes k_p$ is reduced by [Fe72, Corollary 3.4] and therefore, $X(\mathcal{B})(k_p) \neq \varnothing$. So we may assume $p = 2$.

From now on $p = 2$ and $k = k_p$. Since $\mathcal{B}$ is not reduced, the group G has Tits index either $E^{133}_{7,0}$ or $E^{66}_{7,1}$ (see [Ti71, 6.5.5] and [Ti66, Table II]). By 4.1, we may assume

G is anisotropic (has index $E_{7,0}^{133}$).

By definition, $S^2(X(\mathcal{B}))$ is the variety of all unordered pairs $[\langle e_1\rangle, \langle e_2\rangle]$, where e_i are singular elements of $\mathcal{B} \otimes_k k_s$, with the natural action of Γ. Denote by U the open subvariety of $X(\mathcal{B})$ defined by the condition $b(e_1, e_2) \neq 0$.

Lemma 4.2 *The embedding $U \to S^2(X(\mathcal{B}))$ is surjective on k-points.*

Proof Consider the diagonal action of G on $X(\mathcal{B}) \times X(\mathcal{B})$ (we may assume in this proof that G is simple). Over k_s, this action has four orbits: the minimal orbit which is the diagonal and, hence, is isomorphic to G_s/P_7, the open dense orbit which is isomorphic to the quotient $G_s/L(P_7)$, where $L(P_7)$ denotes the Levi part of P_7, and two locally closed orbits. Indeed, there is a one-to-one correspondence between the orbits of the G_s-action and double coset classes $P_7\backslash G_s/P_7$ given by mutually inverse maps $G_s \cdot (x, y) \mapsto P_7x^{-1}yP_7$ and $P_7wP_7 \mapsto G_s \cdot (1, w)$. Observe that the minimal orbit corresponds to the class of the identity and the open dense orbit to the class of the longest element w_0 of the Weyl group of G_s.

Consider the diagonal action of G on $S^2(X(\mathcal{B}))$. Over k_s the subset U is the open dense orbit in $S^2(X(\mathcal{B}))$. Assume that there exists a k-rational point on $S^2(X(\mathcal{B}))\setminus U$. Then the stabilizer H of this point is a subgroup of G defined over k. Observe that over k_s the connected component of the identity H^0 is the stabilizer of one of the non-open orbits for the action of G on $X(\mathcal{B}) \times X(\mathcal{B})$ considered above, *i.e.,* it can be identified with the intersection of two parabolic subgroups $H_s^0 = P_7 \cap wP_7w^{-1}$, where w is the double coset representative corresponding to the orbit. By [DG, Exposé XXVI, Theorem 4.3.2], H_s^0 is reductive if and only if H_s^0 is the Levi subgroup of P_7, *i.e.,* if and only if $P_7wP_7 = P_7w_0P_7$. Therefore, H_s^0 is non-reductive and so is H. The latter implies that G must have a unipotent element over k. But according to [Ti86, p. 265], if G is anisotropic and char $k \neq 2, 3$ this is impossible, a contradiction. ■

According to the lemma it suffices to show that U is R-trivial.

The Brown algebra $\mathcal{B} \otimes_k k_s$ is split, that is, isomorphic to the Brown algebra of matrices of the form $\left(\begin{smallmatrix} F & J_d \\ J_d & F \end{smallmatrix}\right)$, where J_d is the split Jordan algebra. Set

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The pair $[\langle e_1\rangle, \langle e_2\rangle]$ is stable under an arbitrary semiautomorphism of $\mathcal{B} \otimes_k k_s$ (see [Ga01, Proof of Theorem 2.9]) and, in particular, under the action of Γ. Therefore, $[\langle e_1\rangle, \langle e_2\rangle]$ is a k-rational point of U. Moreover, $\langle e_1, e_2\rangle$ defines by descent the (k-defined) étale quadratic subalgebra L of $\mathcal{B}$.

By [Fe72, Proposition 7.6], G acts transitively on U. Therefore,

$$U \simeq G/\operatorname{Stab}_G([\langle e_1\rangle, \langle e_2\rangle]).$$

This stabilizer clearly coincides with $\operatorname{Stab}_G(L)$ (one inclusion is obvious, and another one follows from the fact that e_1, e_2 are the only singular elements of $L \otimes_k k_s$ up to scalar factors).

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Lemma 4.3 *There is an exact sequence of algebraic groups*

$$1 \to \mathrm{Aut}(\mathcal{B}) \to \mathrm{Stab}_G(L) \to R_{L/k}(\mathbb{G}_m) \to 1.$$
$$f \mapsto f(1)$$

Proof This follows from the fact that the stabilizer of 1 in G coincides with $\mathrm{Aut}(\mathcal{B})$. Indeed, we have an obvious injection $\mathrm{Aut}(\mathcal{B}) \to \mathrm{Stab}_G(1)$. To prove the surjectivity we can assume that k is separably closed. Let f be an element of G preserving 1. Since a decomposition into a sum of two non-orthogonal singular elements is unique by [Fe72, Lemma 3.6] and $1 = e_1 + e_2$, the element f must preserve the pair $[e_1, e_2]$. By [Fe72, Lemma 7.5], f has a form η_π^λ, where η is a similitude of J with a multiplier ρ, π is a permutation on $\{1, 2\}$, $\lambda \in k^*$, and η_π^λ acts on $\mathcal{B}$ by formulae [Fe72, (15)]. Now it follows that $\lambda^{-1}\rho^{-1} = 1$ and $\lambda^2\rho = 1$ and therefore $\lambda = \rho = 1$. So f is an automorphism of $\mathcal{B}$, as claimed. The surjectivity of the last map also follows from [Fe72, Lemma 7.5]. ∎

Since $\mathrm{H}^1(k, L^*) = 1$, the map $\mathrm{H}^1(k, \mathrm{Aut}(\mathcal{B})) \to \mathrm{H}^1(k, \mathrm{Stab}_G(L))$ is surjective. By Lemma 2.3 the morphism $G/\mathrm{Aut}(\mathcal{B}) \to G/\mathrm{Stab}_G(L) \simeq U$ is surjective on k-points. Therefore, it suffices to show that $G/\mathrm{Aut}(\mathcal{B})$ is R-trivial.

Let W be an open subvariety of $\mathcal{B}$ consisting of all elements v such that $b(v, t(v, v, v)) \neq 0$. Then G acts transitively on W (it follows easily from [Fe72, Theorem 7.10] or [SK77, p. 140]) and the stabilizer of the point 1 coincides with $\mathrm{Aut}(\mathcal{B})$. So $G/\mathrm{Aut}(\mathcal{B}) \simeq W$ is R-trivial, and we have finished the proof of Theorem 4.1.

5 Other Homogeneous Varieties

In this section using the results of [Me, Ti66], we finish the proof of Theorem 1.1. We start with the following lemma.

Lemma 5.1 *Let X and Y denote projective homogeneous varieties over a field k. Assume X is isotropic over the function field of Y and Y is isotropic over the function field of X. Then the groups of zero-cycles of X and Y are isomorphic.*

Proof The fact that X is isotropic over $k(Y)$ is equivalent to the existence of a rational map $Y \dashrightarrow X$. Hence, we have two composable rational maps $f\colon Y \dashrightarrow X$ and $g\colon X \dashrightarrow Y$, and the compositions $f \circ g$ and $g \circ f$ correspond to taking a $k(X)$-point on X and a $k(Y)$-point on Y, respectively.

Consider the category of rational correspondences $\mathrm{RatCor}(k)$ introduced in [Me]. The objects of this category are smooth projective varieties over k and morphisms $\mathrm{Mor}(X, Y) = \mathrm{CH}_0(Y_{k(X)})$. The key property of this category is that the CH_0-functor factors through it. Namely, CH_0 is a composition of two functors: the first is given by taking a graph of a rational map (any rational map gives rise to a morphism in $\mathrm{RatCor}(k)$), the second is the realization functor (see [Me, Theorem 3.2]).

The maps f and g give rise to the morphisms $[f]$ and $[g]$ in $\mathrm{RatCor}(k)$. By definition the compositions $[f \circ g]$ and $[g \circ f]$ give the identity maps in the category

RatCor(k). Hence, the realizations $[f]_*$ and $[g]_*$ give the respective mutually inverse isomorphisms between $\mathrm{CH}_0(X)$ and $\mathrm{CH}_0(Y)$. ■

The next lemma finishes the proof of Theorem 1.1.

Lemma 5.2 *Let X be an anisotropic projective G-homogeneous variety, where G is a group of type* F_4 *or* ${}^1\mathrm{E}_6$ *with trivial Tits algebras. Then* $\widetilde{\mathrm{CH}}_0(X) = 0$.

Proof According to Lemma 2.1, it is enough to prove the lemma over fields k_p, where $p = 2$ or 3.

Assume $p = 2$ and $k = k_2$. Let G be a group of type ${}^1\mathrm{E}_6$. Consider a Jordan algebra J corresponding to the group G. Since the base field k is prime-to-2 closed, the algebra J is reduced [Inv, Theorem 40.8] and, hence, comes from an octonion algebra $\mathbb{O}$. Consider the variety Y of norm zero elements of $\mathbb{O}$ (which is an anisotropic Pfister quadric). Since G has trivial Tits algebras, there are only two Tits diagrams allowed for G and its scalar extensions, namely, ${}^1\mathrm{E}^0_{6,6}$ and ${}^1\mathrm{E}^{28}_{6,2}$ (see [Ti71, 6.4.5]). Since X is anisotropic (by the hypothesis), extending the scalars to $k(X)$ adds additional circles to the respective Tits diagram and, hence, changes it. Therefore, $G_{k(X)}$ (equivalently $J_{k(X)}$) must be split. The fact that $G_{k(Y)}$ (equiv. $J_{k(Y)}$) is split is obvious (see [Inv, Corollary 37.18]). All this means that the varieties $X_{k(Y)}$ and $Y_{k(X)}$ are isotropic. By Lemma 5.1 we obtain $\widetilde{\mathrm{CH}}_0(X) = \widetilde{\mathrm{CH}}_0(Y) = 0$, where the last equality holds by [Sw89].

In the case G is a group of type F_4, there are three possible Tits diagrams, namely, $\mathrm{F}^{52}_{4,0}$ (anisotropic), $\mathrm{F}^{21}_{4,1}$ (isotropic) and $\mathrm{F}^0_{4,4}$ (split case). Consider the first case, *i.e.,* G is an anisotropic group of type F_4 and X is a variety of parabolic subgroups of type P_4. Let Z be the Pfister form corresponding to the invariant f_5. We claim that $X_{k(Z)}$ and $Z_{k(X)}$ are isotropic. Obviously, $k(X)$ splits Z. The invariants g_3 and f_5 are trivial for the respective Jordan algebra $J_{k(Z)}$. By [PR94, p. 205], this implies that the group $G_{k(Z)}$ is isotropic, *i.e.,* corresponds to the diagram $\mathrm{F}^{21}_{4,1}$. Then the variety $X_{k(Z)}$ is isotropic as well. Again by Lemma 5.1 and [Sw89] we conclude that $\widetilde{\mathrm{CH}}_0(X) = \widetilde{\mathrm{CH}}_0(Z) = 0$.

In case X is the variety of parabolic subgroups of type different from P_4, one can prove that $\widetilde{\mathrm{CH}}_0(X) = 0$ following the same arguments as for the group of type ${}^1\mathrm{E}_6$.

Assume $p = 3$ and $k = k_3$. In this case there are two Tits diagrams allowed for G, namely, ${}^1\mathrm{E}^{78}_{6,0}$ and ${}^1\mathrm{E}^0_{6,6}$ (resp. $\mathrm{F}^{52}_{4,0}$ and $\mathrm{F}^0_{4,4}$). Consider the pair X and $Y = \mathbb{OP}^2(J)$. Again the obvious arguments with Tits diagrams show that $X_{k(Y)}$ and $Y_{k(X)}$ are isotropic. We obtain $\widetilde{\mathrm{CH}}_0(X) = \widetilde{\mathrm{CH}}_0(Y) = 0$, where the last equality holds by Theorem 3.1. ■

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