THE ACCURACY OF THE DETERMINATION OF TERRESTRIAL REFRACTION FROM RECIPROCAL ZENITH ANGLES

K. Ramsayer
Geodetic Institute of the University of Stuttgart

To investigate the accuracy of the determination of terrestrial refraction from reciprocal zenith angles and astronomical latitudes and longitudes at both ends of a line a test net with lines from 4 km to 23 km was observed and three dimensionally adjusted. As the measurements of the zenith angles were repeated every hour 40 times in an average the adjusted values were taken as a substitute for the true values. It is shown, that the mean refraction coefficient k, which is changing from k = 0.10 at day up to k = 0.34 at night, and the corresponding refraction angle can be determined very accurately, if both angles are measured simultaneously. Observations with day light are better than observations in the night. For observations with day light the mean difference between the true refraction angle at the observation station and the mean refraction angle of the observed line was smaller than ± 1" independent of the length of the line. That means that the mean deviation of the true effective refraction coefficient in the observation station and the mean refraction coefficient of the observed line was inverse proportional to the distance.

1. INTRODUCTION

In three dimensional networks and traverses the accuracy of the heights is mainly depending on the accuracy with which the influence of refraction to measured zenith angles can be determined. A well known means to determine this influence is to measure the zenith angles and the directions of the verticals at both ends of a line P_1P_2, Fig. 1. Then we have the following relations

E. Tengström and G. Teleki (eds.), Refractional Influences in Astrometry and Geodesy, 203–212.

$$z_1 = z_1' + \delta_1 = z_1' + k_1(d/2r_m)\rho,$$
$$z_2 = z_2' + \delta_2 = z_2' + k_2(d/2r_m)\rho, \qquad (1)$$

z_1, z_2 = true zenith distances of the line P_1P_2,
z_1', z_2' = measured zenith distances,
δ_1, δ_2 = refraction angles,
k_1, k_2 = refraction coefficients,
r_m = mean radius of the earth,
d = distance between P_1 and P_2,
ρ = 206265".

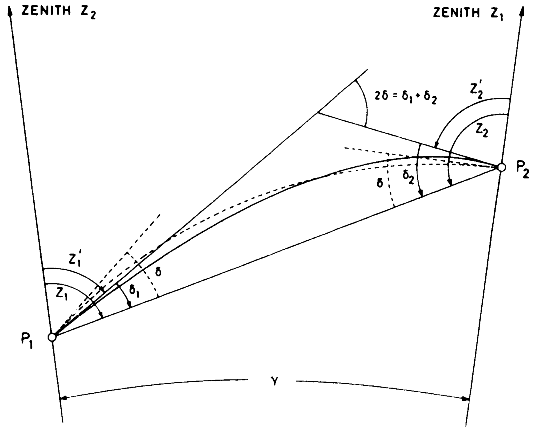

Fig. 1

If we introduce the mean values

$$\delta = (\delta_1 + \delta_2)/2 \quad \text{and} \quad k = (k_1 + k_2)/2 \qquad (2)$$

then we get with

$$\Delta\delta = (\delta_1 - \delta_2)/2 \text{ and } \Delta k = (k_1 - k_2)/2 \qquad (3)$$

$$\begin{aligned} z_1 &= z_1' + \delta + \Delta\delta \\ &= z_1' + k(d/2r_m)\rho + \Delta k(d/2r_m)\rho, \\ z_2 &= z_2' + \delta - \Delta\delta \\ &= z_2' + k(d/2r_m)\rho - \Delta k(d/2r_m)\rho. \end{aligned} \qquad (4)$$

Between the reciprocal zenith angles we have the relation

$$\begin{aligned} z_1 + z_2 &= z_1' + z_2' + 2\delta \\ &= z_1' + z_2' + 2k(d/2r_m)\rho = 180^o + \gamma . \end{aligned} \tag{5}$$

γ is the angle between the verticals in P_1 and P_2, neglecting the small influence that both verticals are not exactly in the same plane. From (5) we can compute the mean values

$$\delta = (\delta_1 + \delta_2)/2 = \left(180^o + \gamma - (z_1'+z_2')\right)/2, \tag{6}$$

$$k = (k_1 + k_2)/2 = (\delta/\rho)(2r_m/d). \tag{7}$$

If we set

$$\delta_1 \simeq \delta_2 \simeq \delta \quad \text{resp.} \quad k_1 \simeq k_2 \simeq k \tag{8}$$

then we get according to (4) the errors

$$\begin{aligned} \varepsilon_1 &= z_1 - (z_1' + \delta) = + \Delta\delta = + \Delta k(d/2r_m)\rho , \\ \varepsilon_2 &= z_2 - (z_2' + \delta) = - \Delta\delta = - \Delta k(d/2r_m)\rho . \end{aligned} \tag{9}$$

If we measure both zenith angles simultaneously, we can hope that the influence of refraction to both angles is approximately the same. In this case we can expect that $\Delta\delta$ resp. Δk and hence ε_1 and ε_2 are small.

2. THE TEST-NET

For the investigation of the errors ε and Δk the network shown in Fig. 2 was observed. For each line both zenith angles were measured simultaneously, each by 6 sets with a standard deviation of ± 0.6". These measurements were repeated each hour, partly during the whole day and at different days. The number of repetitions changes from 12 (line 1)

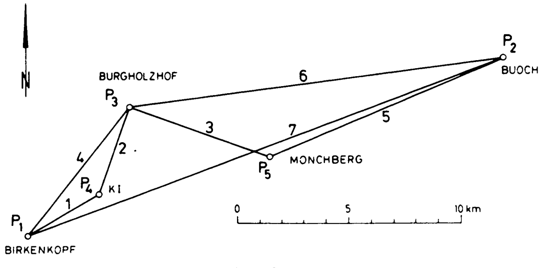

Fig. 2

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and 60 (line 7). The mean value per line is 40. Besides the zenith angles the lengths of the lines were measured with Tellurometer CA 1000. Furtheron in all points of the network astronomical latitude and longitude were determined with Zeiss Ni2-Astrolab by one set with 20 stars in an average. The orientation in azimuth was taken from another three dimensional network.

The network was adjusted rigorously three dimensional in an ellipsoidal reference system. The coordinates of point P_1 were given and kept fix. For each line a special refraction coefficient was determined from all zenith angles measured at both ends of the line. The standard deviations of the adjusted zenith angles referred to the adjusted directions of the verticals change from ± 0.7" (line 1) to ± 1.4" (line 7). The root mean square is ± 1". These errors are relatively large, if we consider the great number of zenith angle measurements. They are mainly caused by the moderate accuracy of the astronomical observations and the small redundancy of the network. In the following the adjusted values of the zenith angles were taken as a substitute of the true values.

3. RESULTS

In Fig. 3 to Fig. 6 some results of the investigations are demonstrated. Fig. 3 shows the influence of refraction for

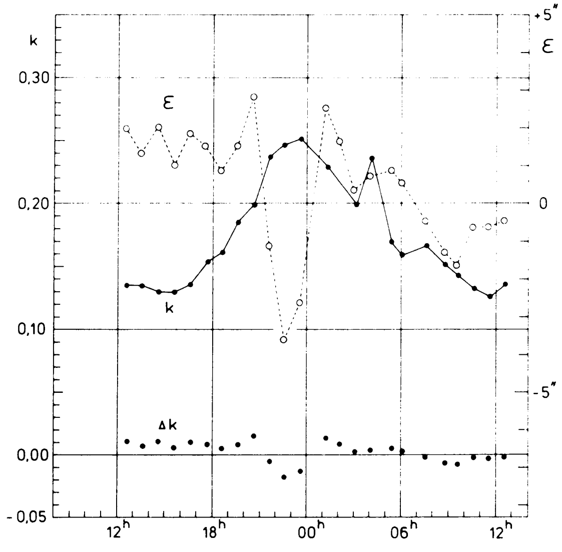

Fig. 3 : Line 5 (P_5P_2), d = 11.641 km, Aug. 25/26

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line 5 from point P_5 to point P_2 between 12 h at 25. August and 12 h at 26. August. The mean refraction coefficient k is changing from the standard value 0.13 at mid-day and 0.25 at mid-night. The difference $\Delta k = k_5 - k$ between the true refraction coefficient k_5 in point P_5 and the mean value k varies between + 0.015 and - 0.019. The quadratic mean of Δk is $\pm$ 0.009. The error ε resulting from the refraction error $\overline{\Delta}k$ changes between + 2.8" and - 3.7". Fig. 4 shows some results of line 6 from P_2 to P_3. We see again that k

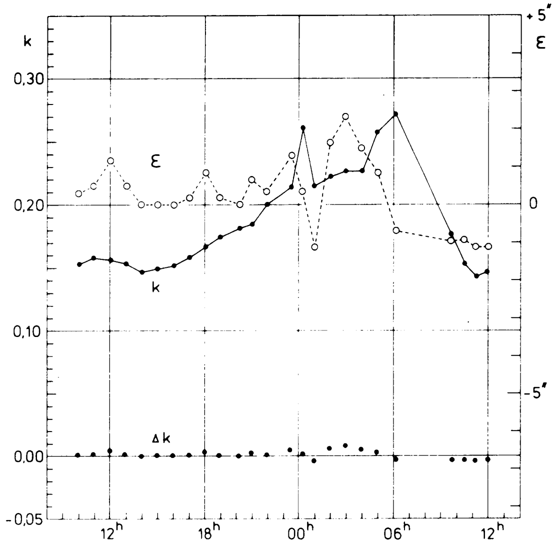

Fig. 4 : Line 6 (P_2P_3), d = 16.908 km, Oct. 3/4

has its minimum at mid-day and its maximum at mid-night and in the early morning. Δk varies between - 0.004 and + 0.008 and ε changes between - 1.2" and + 2.3". Fig. 5 shows the influence of refraction for the longest line from 10 h in the morning to 10 h at the following day. Here the change of k and the errors Δk are relatively small. The errors ε are small too, although the length of the line is more than 20 km. From Fig. 6 follows again that k is changing very much and very rapidly during night. Nevertheless Δk does not exceed 0.022, but by the large distance of the two observation stations the resulting errors ε are enlarged up to 8.2".

Table 1 shows the minimum and maximum of Δk and ε for all lines seperated for day and night. Further the quadratic means m_k and m_ε of Δk and ε and the number n of reciprocal zenith angle measurements are tabulated. We see that for observations at day m_k is decreasing with distance d from

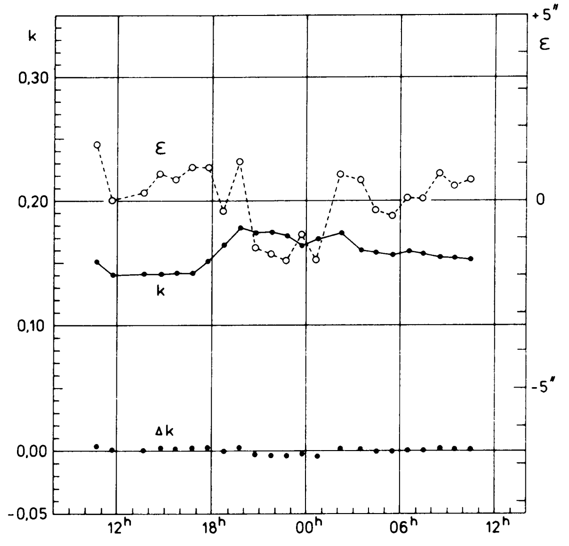

Fig. 5 : Line 7 (P_1P_2) , d = 22.848 km , Sept. 12 / 13

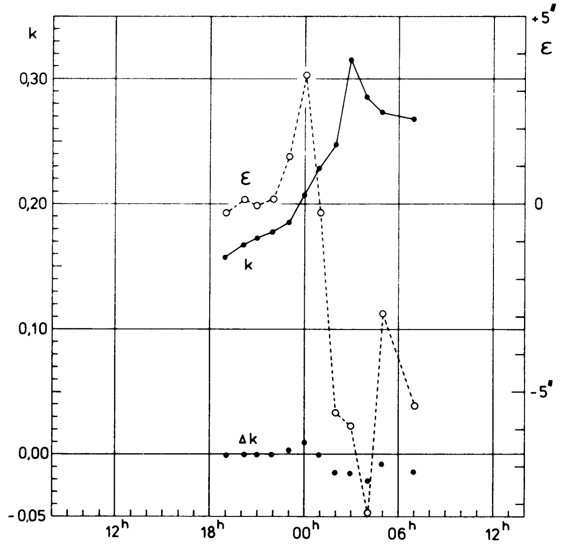

Fig. 6 : Line 7 (P_1P_2) , d = 22.848 km , Sept. 16 / 17

LINE		d	DAY (07^h - 19^h)							NIGHT (19^h - 07^h)						
			Δk			ε			n	Δk			ε			n
NO.	FROM-TO	[km]	MIN	MAX	m_k	MIN	MAX	m_ε		MIN	MAX	m_k	MIN	MAX	m_ε	
			-	+	±	-	+	±		-	+	±	-	+	±	
1	P_1 - P_4	3.7	.030	.026	.018	1.9"	1.6"	1.1"	12							
2	P_4 - P_3	4.3	.017	.013	.015	1.2	0.9	1.1	9	.013	.007	.011	1.3"	0.5"	0.8"	5
3	P_5 - P_3	6.5	.012	.006	.006	1.3	0.7	0.7	28	.018	.016	.003	2.0	1.7	0.9	26
4	P_1 - P_3	7.6	.013	.011	.005	1.6	1.3	0.7	26	.006	.011	.006	0.8	1.4	0.7	13
5	P_5 - P_2	11.6	.010	.011	.006	1.9	2.0	1.1	25	.019	.015	.008	3.7	2.9	1.5	24
6	P_2 - P_3	16.9	.006	.004	.003	1.7	1.2	0.9	21	.007	.008	.004	2.0	2.3	1.2	24
7	P_1 - P_2	22.8	.001	.004	.002	0.4	1.6	0.6	25	.022	.013	.007	8.2	4.9	2.6	35

Table 1

$\pm$ 0.018 to $\pm$ 0.002, whilst m_ε differs only slightly from the mean value of $\pm$ 1". At night m_k and m_ε are somewhat larger than at day with exception of line 2. The decrease of m_k with distance is not so marked as at day time and m_ε is increasing with distance.

Fig. 7 shows m_k and m_ε for day observations depending on

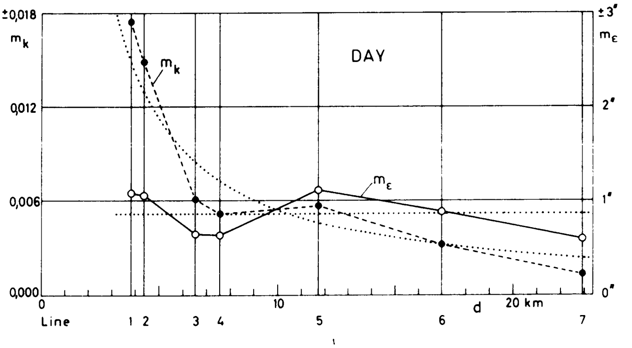

Fig. 7

distance d. In a rough approximation m_ε has the constant value

$$\bar{m}_\varepsilon = \pm\ 0.9''. \tag{10}$$

The corresponding error of the determination of the refraction coefficient is

$$\bar{m}_k = \pm\ (\bar{m}_\varepsilon/\rho)(2r_m/d) = \pm\ 0.055\,[km]/d\,[km]. \tag{11}$$

The obvious decrease of m_k with distance may be explained by the fact, that in the test-field the light path runs the more through the free atmosphere the longer the line.

Fig. 8 shows m_k and m_ε for night-observations. Here m_ε increases with distance from $\pm$ 0.8" up $\pm$ 2.7", whilst the decrease of m_k with distance is somewhat smaller as for day-observations.

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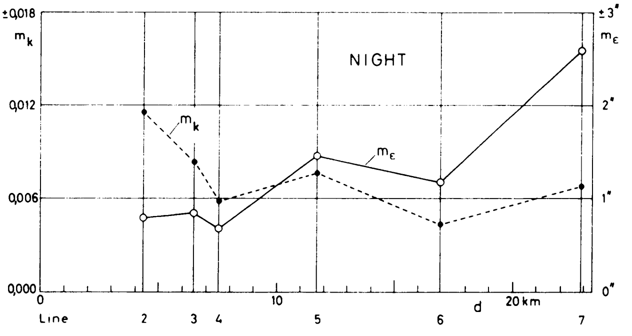

Fig. 8

4. SUMMARY AND CONCLUSIONS

1. The influence of refraction to zenith angles changes very strongly with day time especially during the night. The standard value 0.13 for the refraction coefficient is a rough approximation from 10 h to 15 h. At night k was increasing up to 0.34.

2. The mean refraction coefficient can be determined with good accuracy by measuring the reciprocal zenith angles and the astronomical latitudes and longitudes at both ends of a line.

3. The measurements of the reciprocal zenith angles should be made simultaneously. In this case the differences between the true refraction coefficients at the observation stations and the mean refraction coefficient are small. Observations with day light are better than observations in the night.

4. For observations with day light the standard deviation of a single set of a zenith angle which is reduced for mean refraction was approximately ± 1" independent of the length of the line.

5. For observations with day light the mean deviation of the true effective refraction coefficient in the observation station and the mean refraction coefficient of the observed line was inverse proportional to the distance.

All together we can say that the determination of refraction by reciprocal zenith angle measurements and astronomical latitude and longitude observations is a surprisingly accurate method.

DISCUSSION

K. Poder: Thank you professor Ramsayer. I must say that apart from professor Hradilek and you a lot of the geodetic community should actually be very ashamed because we have known how to determine zenith distances for more than one hundred years, but obviously nobody has really considered it as seriously as you and professor Hradilek have done. And I think it is of very much interest. We have professionally in Greenland practically only heights by zenith distances, but we have never looked so carefully into the matter, I'm afraid to say. So, I think this is a very interesting paper.

J.A. Hughes: There seems to be an inverse proportion between error and distance. The further you look the better it is. Could you explain that effect to me, a non-geodesist?

K. Ramsayer: You wonder why the deviation of the true refraction coefficient from the mean refraction coefficient of the observed line is inverse proportional to the distance. I think that the reason is, that the longer the distance is the more we come into the free atmosphere. I explain it so. I was also surprised. It is however not always the case. If the lightpath goes very near to the earth's surface you have very uncertain relations.

K. Poder: I can add that this is really going to change the weightning function, which we use for trigonometric levelling. I have assumed that it was independent of the distance. It seems that most observation equations are rather friendly. And the first equation I had was a very unfriendly one, but using your law I think I will get a friendly equation.

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