LINEAR SURJECTIVE ISOMETRIES BETWEEN VECTOR-VALUED FUNCTION SPACES

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Abstract

We prove some Banach–Stone type theorems for linear isometries of vector-valued continuous function spaces, by making use of the extreme point method.

1. Introduction and preliminaries

The present paper deals with surjective linear isometries between subspaces of vector-valued continuous function spaces and proves some variants of the classical Banach–Stone theorem. Extensive research has been made in this direction and relevant references include [2, 19–22, 27, 32–34] for scalar-valued function spaces and [1, 3, 5–7, 10, 18, 24, 26, 28, 36] for vector-valued function spaces. The survey in [23] and monographs in [16, 17] are valuable sources of information. For compact Hausdorff spaces X and Y and a real or complex Banach space E, we consider an isometry $T : A \to B$ defined on a subspace A of $C(X, E)$ onto a subspace B of $C(Y, E)$, where $C(X, E)$ denotes the Banach space of all E-valued continuous functions on X with the supremum norm. The Mazur–Ulam theorem [37] implies that T is a real-linear isometry whenever $T(0) = 0$. When A and B have 'enough functions' (see Condition (S1) and Condition (S2) below) and the duals A^* and B^* have enough extreme points (see Condition (ext) below), T is often, but not always, a *generalized weighted composition operator* in the sense that there exists a homeomorphism $\varphi : Y_B \to X_A$ defined on a subset Y_B of Y onto a subset X_A of X and a collection of linear operators $(V_y)_{y \in Y_B}$ on E such that

$$T f(y) = V_y(f(\varphi(y)))$$

for each $f \in A$ and for each $y \in Y_B$. Moreover, the map $y \mapsto V_y$ is continuous on Y_B to the space of linear operators on E when the latter is endowed with the strong operator topology. A natural aim is to find a set of conditions on X, Y, A, B and E which guarantees that every linear isometry $T : A \to B$ is a generalized weighted composition operator.

We give a framework for obtaining such conditions (Theorem 3.1) and, as its applications, we show the following results (undefined terminologies will be explained later). If one of following conditions holds, then every surjective linear isometry $T : A \to B$ is a generalized weighted composition operator for which the subspaces Y_B and X_A are the Choquet boundaries of B and A, respectively (Theorems 3.4, 3.5 and 4.6):

(1) E is strictly convex and reflexive, the dual of E is strictly convex, and A and B satisfy Condition (S1), Condition (S3) and Condition (M); or

(2) E is reflexive, the dual of E is strictly convex, A and B satisfy Condition (S1), Condition (S2) and Condition (ext) and $\mathrm{Ch}_E(A) = X$, $\mathrm{Ch}_E(B) = Y$ and further X is homotopically rigid; or

(3) E is separable and reflexive, $\dim E \geq 3$, the dual of E is strictly convex, A and B satisfy Condition (S1), Condition (S2) and Condition (ext), X is metrizable and its topological dimension is at most one.

The result (1) follows the same line of reasoning as previous results [1, 6, 18, 25, 28]. On the other hand, (2) and (3) seem to exhibit a relatively new variant in that their assumptions are involved in the topology of underlying spaces X and Y (see [33] and a paper by K. Kawamura and Miura, 'Real-linear surjective isometries between function spaces,' which has been submitted for publication). The whole of the paper is based on the extreme point method which is described in [1, 14, 17].

This paper is organized as follows. The rest of this section fixes notation and recalls some basic results. Section 2 collects some results on extreme points of the dual unit balls and the generalized Choquet boundaries. These are applied in Section 3 to study homeomorphisms induced by the adjoint operators of the duals and to obtain a general framework to derive Banach–Stone type theorems. Results (1) and (2) are proved in Section 3. Also a previous result, given in [27], on complex-valued scalar function spaces is reviewed from our viewpoint. Section 4 studies the topological dimension of the underlying space and the result (3) above is proved.

For a Banach space E over a scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the closed unit ball and the unit sphere are denoted by $B(E)$ and $S(E)$, respectively. For an $\mathbb{F}$-subspace M of E, $M_{\mathbb{F}}^*$ denotes the $\mathbb{F}$-dual of M, the space of all $\mathbb{F}$-linear functionals of M with the operator norm. For two $\mathbb{F}$-linear spaces L_1 and L_2, $\mathcal{L}_{\mathbb{F}}(L_1, L_2)$ denotes the $\mathbb{F}$-linear maps of L_1 to L_2. Under this notation, we have $M_{\mathbb{F}}^* = \mathcal{L}_{\mathbb{F}}(M, \mathbb{F})$. For a compact Hausdorff space X and a Banach space E with the norm $\| \|$, $C(X, E)$ denotes the Banach space of all E-valued continuous maps of X to E with the sup norm $\|f\|_\infty = \sup_{x \in X} \|f(x)\|$. An *extreme point* of a convex set C of a linear space L is a point $\xi \in C$ with the property that the equality $\xi = \eta + \zeta/2$ with $\eta, \zeta \in C$ forces $\eta = \zeta = \xi$ (see [12]). The set of all extreme

points of C is denoted by $\text{ext}(C)$. In particular, the extreme points of the closed unit ball $B(E)$ of a Banach space E are simply denoted by $\text{ext}(E)$. Under this notation, a Banach space E is strictly convex if and only if $\text{ext}(E) = S(E)$ (see [31]). For an $\mathbb{F}$-subspace M of $C(X, E)$ and a point $x \in X$, let $\delta_x : M \to E$ be the evaluation map defined by $\delta_x(f) = f(x)$, $f \in M$. Notice that $\delta_x \in \mathcal{L}(M, E)$.

In most of the present paper, a subspace A of $C(X, E)$ is assumed to satisfy the following conditions. For a vector $u \in E$, the constant map $c_u : X \to E$ on a compact Hausdorff space X is defined by $c_u(x) = u$ for each $x \in X$.

(S1) For each $u \in E$, the constant map $c_u : X \to E$ belongs to A: $c_u \in A$.
(S2) For each pair of distinct points x_1 and x_2 of X and for each $u \in E$, there exists a function $f \in A$ such that $f(x_1) = u$ and $f(x_2) = 0$.

In the study of function algebras, it is common to assume the following, often referred to 'the point separation condition.'

(s2) For each pair of distinct points x_1 and x_2 of X, there exists a map $f \in A$ such that $f(x_1) \neq f(x_2)$.

The following condition, or its variants, sometimes replaces Condition (S1) and Condition (S2) (see [1, 18]).

(N) For each $f \in C(X, \mathbb{F})$ and for each $u \in E$, $fc_u \in A$.

Condition (N) above implies Condition (S1) and Condition (S2). A related notion is the one of *a function algebra module*: a subspace A of $C(X, E)$, where E is a Banach space over $\mathbb{F}$, is called a *function algebra module* if, for each $f \in C(X, \mathbb{F})$ and for each $g \in A$, $fg \in A$ (cf. [3]).

It is readily seen from the definition that (S2) implies (s2). Our first observation is that (S2) is equivalent to (s2) for scalar-valued function spaces satisfying (S1).

LEMMA 1.1. *Let X be a compact Hausdorff space and let A be a subspace of $C(X, \mathbb{F})$ satisfying Condition (S1). Then A satisfies Condition (S2) for the scalar $\mathbb{F}$ if and only if A satisfies Condition (s2).*

PROOF. Let A be a subspace satisfying Condition (S1) and Condition (s2). Take an arbitrary $\lambda \in \mathbb{F} \setminus \{0\}$ and take distinct points x_1 and x_2. By Condition (s2), there exists an $f_0 \in A$ such that $f_0(x_1) \neq f_0(x_2)$. Let f_1 be the function defined by

$$f_1(x) = f_0(x) - f_0(x_2), \quad x \in X.$$

By Condition (S1), the function f_1 belongs to A and we see that $f_1(x_1) \neq 0 = f_1(x_2)$. Let

$$f(x) = \lambda(f_1(x_1))^{-1} f_1(x), \quad x \in X,$$

which is again an element of A. We obtain $f(x_1) = \lambda$ and $f(x_2) = 0$, as desired.   □

2. Extreme points of the dual unit balls

Throughout this section, a scalar field $\mathbb{F} = \mathbb{R}$, or $\mathbb{C}$ is fixed and will be omitted in the notation. Let E be a Banach space over $\mathbb{F}$ and let X be a compact Hausdorff space. The $\mathbb{F}$-dual of E is simply denoted by E^*. Observing that $\xi \circ \delta_x$ is an $\mathbb{F}$-linear functional on $C(X, E)$, we define a set $\mathcal{E}_{X,E}$ of $C(X, E)^*$ by

$$\mathcal{E}_{X,E} = \{\xi \circ \delta_x \mid \xi \in \mathrm{ext}(E^*),\ x \in X\},$$

where $\mathrm{ext}(E^*)$ denotes the set of all extreme points of the unit ball of E^*. A natural map $\Phi_{X,E} : \mathrm{ext}(E^*) \times X \to \mathcal{E}_{X,E}$ is defined by $\Phi_{X,E}(\xi, x) = \xi \circ \delta_x$. When $\mathrm{ext}(E^*)$ and $C(X, E)^*$ are endowed with the weak*-topology, the map $\Phi_{X,E}$ is continuous. For an $\mathbb{F}$-subspace A of $C(X, E)$, it is known from [8, Lemma 3.3] that

$$\mathrm{ext}(A^*) \subset \mathcal{E}_{X,E}.$$

Condition (S1) and Condition (S2) imply the injectivity of $\Phi_{X,E}$, as shown in the following lemma.

LEMMA 2.1. *Assume that a subspace A satisfies Condition (S1) and Condition (S2). If $\xi_1 \circ \delta_{x_1}|A = \xi_2 \circ \delta_{x_2}|A$ for $\xi_1, \xi_2 \in E^* \setminus \{0\}$, $x_1, x_2 \in X$, then $(\xi_1, x_1) = (\xi_2, x_2)$.*

PROOF. The hypothesis means that $\xi_1(f(x_1)) = \xi_2(f(x_2))$ for each $f \in A$. By (S1), $\xi_1(u) = \xi_1(c_u(x_1)) = \xi_2(c_u(x_2)) = \xi_2(u)$ for each $u \in E$. Hence $\xi_1 = \xi_2$. Suppose that $x_1 \neq x_2$ and take $u \in E$ such that $\xi_1(u) \neq 0$. By (S2), we may find an $f \in A$ such that $f(x_1) = u, f(x_2) = 0$. This yields a contradiction, as

$$0 \neq \xi_1(u) = \xi_1(f(x_1)) = \xi_2(f(x_2)) = \xi_2(0) = 0. \qquad \square$$

For a subspace A of $C(X, E)$, let

$$\Phi_A = \Phi_{X,E}|(\Phi_{X,E})^{-1}(\mathrm{ext}(A^*)) : (\Phi_{X,E})^{-1}(\mathrm{ext}(A^*)) \to \mathrm{ext}(A^*)$$

be the restriction of $\Phi_{X,E}$ to the set $(\Phi_{X,E})^{-1}(\mathrm{ext}(A^*))$. At the presence of the conditions (S1) and (S2), Φ_A is a bijection, by Lemma 2.1. Also, Φ_A is continuous when the duals are endowed with the weak*-topology. Moreover, the compactness of X implies the following lemma which seems to be well known to the experts. A proof is sketched below for completeness.

LEMMA 2.2. *Assume that a subspace A satisfies Condition (S1) and Condition (S2). Then the map Φ_A is a homeomorphism.*

SKETCH OF PROOF. Take a net $(\xi_\alpha, x_\alpha)_\alpha$ and a point (ξ, x) in $(\Phi_{X,E})^{-1}(\mathrm{ext}(A^*))$ such that $\Phi_{X,E}((\xi_\alpha, x_\alpha)) \to \Phi_{X,E}((\xi, x))$. This implies that

$$\xi_\alpha(f(x_\alpha)) \to \xi(f(x))$$

for each $f \in A$. For each $u \in E$, we see that $\xi_\alpha(u) = \xi_\alpha(c_u(x_\alpha)) \to \xi(c_u(x)) = \xi(u)$ and hence

$$\xi_\alpha \to \xi \tag{2.1}$$

in the weak*-topology. By the compactness of X, the net (x_α) contains a convergent subnet. Let $(x_{\chi(\nu)})$ be an arbitrary such subnet so that $x_{\chi(\nu)} \to x_0$. This, together with (2.1), implies that $\xi_{\chi(\nu)} \circ \delta_{\chi(\nu)} \to \xi \circ \delta_{x_0}$ and hence we obtain the equality

$$\xi \circ \delta_{x_0} = \xi \circ \delta_x.$$

By Lemma 2.1, $x = x_0$ and hence $x_{\chi(\nu)} \to x$. Since $(x_{\chi(\nu)})$ is an arbitrary convergent subnet of (x_α), we see that the net (x_α) itself converges to x.   □

We follow [1] to define the *Choquet boundary* $\mathrm{Ch}_E(A)$ as the set

$$\mathrm{Ch}_E(A) = \{x \in X \mid \xi \circ \delta_x \in \mathrm{ext}(A^*) \text{ for some } \xi \in \mathrm{ext}(E^*)\}.$$

Here the target vector space E is designated in order to be distinguished from the usual Choquet boundary for which $E = \mathbb{C}$. A complexity we encounter when dealing with vector-valued function settings is that, for some $x \in \mathrm{Ch}_E(A)$, there may exist a $\xi' \in \mathrm{ext}(E^*)$ such that $\xi' \circ \delta_x \notin \mathrm{ext}(A^*)$ (see [1, Example 3.2]). This obstacle leads us to consider the following condition on A.

(ext) For each $x \in \mathrm{Ch}_E(A)$, we have $\xi \circ \delta_x \in \mathrm{ext}(A^*)$ for each $\xi \in \mathrm{ext}(E^*)$.

Under the assumption (ext) on A,

$$(\Phi_{X,E})^{-1}(\mathrm{ext}(A^*)) = \mathrm{ext}(E^*) \times \mathrm{Ch}_E(A)$$

and Lemma 2.2 is rephrased as follows.

PROPOSITION 2.3. *Assume that A satisfies Condition (S1), Condition (S2) and Condition (ext). Then the map $\Phi_A : \mathrm{ext}(E^*) \times \mathrm{Ch}_E(A) \to \mathrm{ext}(A^*)$ is a homeomorphism.*

Condition (ext) plays an important role in the proof of Theorem 3.1. A sufficient condition for a subspace A to satisfy the condition (ext) above is given by Proposition 2.4 below. A consequence of [35, Theorem 1.1] and [29] is that for a real Banach space E, a subspace A of $C(X, E)$ satisfies Condition (ext) if it satisfies the condition (N).

PROPOSITION 2.4. *Let A be a subspace of $C(X, E)$ and x be a point of X satisfying the following conditions.*

(a) *A satisfies Condition (S1).*
(b) *For each $f \in A$ with $f(x) = 0$ and for each $\varepsilon > 0$, there exist a neighborhood U of x and $f_\varepsilon \in A$ such that $\|f - f_\varepsilon\|_\infty < \varepsilon$ and $f_\varepsilon|U \equiv 0$.*
(c) *For each $\xi \in S(E^*)$ and for each neighborhood U of x, there exists an $f \in A$ such that $\|f\|_\infty = 1 = \xi(f(x))$ and $f|X \setminus U \equiv 0$.*

Then $\xi \circ \delta_x \in \mathrm{ext}(A^*)$ for each $\xi \in \mathrm{ext}(E^*)$. In particular, if each point of $\mathrm{Ch}_E(A)$ satisfies the above conditions (a)–(c), then A satisfies Condition (ext).

PROOF. The following proof uses the ideas of [1, 8, 10]. Take an arbitrary $\xi \in \text{ext}(E^*)$ and suppose that $\xi \circ \delta_x = \frac{1}{2}(\eta + \zeta)$ for some $\eta, \zeta \in B(A^*)$. First, we show that, for $f \in A$,

$$f(x) = 0 \Longrightarrow \eta(f) = \zeta(f) = 0. \tag{2.2}$$

Case 1. Assume that $\|f\|_\infty \leq 1$ and, moreover, assume that $f|N \equiv 0$ for some neighborhood N of x.

By the assumption (c), there exists a $g \in A$ such that $\|g\|_\infty = 1 = \xi(g(x))$ and $g|X \setminus N \equiv 0$. Then $|\eta(g)| \leq \|\eta\| \|g\|_\infty = 1$ and, similarly, $|\zeta(g)| \leq 1$. From these, together with the equality $1 = \xi(g(x)) = \frac{1}{2}(\eta(g) + \zeta(g))$, we conclude that $\eta(g) = \zeta(g) = 1$. Let $h = f + g \in A$. We see easily that $\|h\|_\infty \leq 1$ and thus $|\eta(h)| \leq 1$ and $|\zeta(h)| \leq 1$. Also, from the above,

$$1 = \xi(h(x)) = \xi \circ \delta_x(h) = \frac{1}{2}(\eta(h) + \zeta(h)),$$

and this leads to the equality $\eta(h) = \zeta(h) = 1$. Combining this with $\eta(g) = \zeta(g) = 1$, we conclude that $\eta(f) = \zeta(f) = 0$.

Case 2. If the function $f \in A$ has the norm at most one, then, for each $\varepsilon > 0$, we apply the assumption (b) to find a function f_ε such that

$$\|f_\varepsilon - f\|_\infty < \varepsilon, \quad f_\varepsilon|U_\varepsilon \equiv 0$$

for some neighborhood U_ε of x. By what has been proved in Case 1, we see that $\eta(f_\varepsilon) = 0$ from which we derive $|\eta(f)| < \varepsilon$ (note $\|\eta\| \leq 1$). Since ε is an arbitrary positive number, we conclude that $\eta(f) = 0$.

Case 3. General case: for a function $f \in A$ with $f(x) = 0$, take a sufficiently large $N > 0$ such that $\|f/N\|_\infty \leq 1$ and apply the conclusion in Case 2. We see that $\eta(f/N) = 0$ and therefore $\eta(f) = 0$. This proves (2.2).

By (2.2), we may find $\bar\eta, \bar\zeta \in E^*$ such that

$$\eta = \bar\eta \circ \delta_x, \quad \zeta = \bar\zeta \circ \delta_x.$$

By assumption (1), $c_u \in A$ and thus, for $u \in E$,

$$|\bar\eta(u)| = |\eta \circ \delta_x(c_u)| \leq \|\eta\| \|c_u\|_\infty = \|u\|.$$

Hence $\|\bar\eta\| \leq 1$ and, similarly, $\|\bar\zeta\| \leq 1$. Now

$$\xi \circ \delta_x = \frac{1}{2}(\eta + \zeta) = \frac{1}{2}(\bar\eta \circ \delta_x + \bar\zeta \circ \delta_x),$$

and the assumption $\xi \in \text{ext}(E^*)$ forces $\xi = \bar\eta = \bar\zeta$ and hence $\xi \circ \delta_x = \eta = \zeta$, which is to be shown. □

It should be noticed here that Condition (ext) is satisfied by subspaces of scalar-valued continuous functions (see, for example, [16, Corollary 2.3.6] for a proof). This fact is naturally generalized to vector-valued function settings, as follows. For a Banach space E and a subgroup Γ of $\mathcal{L}(E, E)$, the group Γ naturally acts on the space $C(X, E)$ by the action

$$(\gamma \cdot f)(x) = \gamma(f(x)), \quad \gamma \in \Gamma, \quad f \in C(X, E), \quad x \in X.$$

We say that a subspace A of $C(X, E)$ is Γ-*invariant* if $\gamma \cdot f \in A$ whenever $\gamma \in \Gamma$ and $f \in A$. Let $U(E)$ be the group of linear isometries on E. A $U(E)$-invariant subspace is called a *unitarily invariant subspace*. Our generalization is stated in the Hilbert-space setting.

LEMMA 2.5. *Let E be a Hilbert space and let Γ be a subgroup of $U(E)$ which acts transitively on $S(E)$. Then every Γ-invariant subspace A of $C(X, E)$ satisfies Condition (ext).*

PROOF. Let $\langle \cdot, \cdot \rangle$ be the inner product on E. For an element $\gamma \in \Gamma$, define $\gamma_A^* : A^* \to A^*$ to be the linear isometry by the formula

$$(\gamma_A^* \xi)(f) = \xi(\gamma^{-1} f), \quad \xi \in A^*, \quad f \in A.$$

The above $\gamma \mapsto \gamma_A^*$ defines an action of Γ on A^* by linear isometries. We show that

$$\begin{aligned} &\text{for each } \xi, \eta \in S(E^*) \text{ and for each } x \in X \quad \text{there exists a } \gamma \in \Gamma \\ &\text{such that } \gamma_A^*(\xi \circ \delta) = \eta \circ \delta. \end{aligned} \tag{2.3}$$

PROOF OF (2.3). By the Riesz representation theorem, there exist $a_\xi, a_\eta \in E$ such that $\xi(u) = \langle u, a_\xi \rangle$ and $\eta(u) = \langle u, a_\eta \rangle$ for each $u \in E$. Since $\xi, \eta \in S(E^*)$, $a_\xi, a_\eta \in S(E)$. Take an element γ of Γ such that $\gamma(a_\xi) = a_\eta$. Then, for each $f \in A$,

$$\begin{aligned} \gamma_A^*(\xi \circ \delta_x)(f) &= (\xi \circ \delta_x)(\gamma^{-1} f) \\ &= \xi((\gamma^{-1} f)(x)) \\ &= \xi(\gamma^{-1}(f(x))) = \langle \gamma^{-1}(f(x)), a_\xi \rangle \\ &= \langle f(x), \gamma(a_\xi) \rangle \\ &= \langle f(x), a_\eta \rangle = \eta(f(x)) \\ &= \eta \circ \delta_x(f). \end{aligned}$$

This proves (2.3).

For each $x \in \mathrm{Ch}_E(A)$, there exists a $\xi \in \mathrm{ext}(E^*)$ such that $\xi \circ \delta_x \in \mathrm{ext}(A^*)$. For an arbitrary $\eta \in \mathrm{ext}(E^*)$, choose an element $\gamma \in \Gamma$ such that $\gamma_A^*(\xi \circ \delta_x) = \eta \circ \delta_x$, by (2.3). Since γ_A^* is an isometry, $\gamma_A^*(\mathrm{ext}(A^*)) = \mathrm{ext}(A^*)$ and thus we see that $\eta \circ \delta_x \in \mathrm{ext}(A^*)$, as desired. □

We conclude this section with a well-known lemma.

LEMMA 2.6. *Let E be a Banach space. For each $u \in E$ there exists $\xi \in \mathrm{ext}(E^*)$ such that $\xi(u) = \|u\|$.*

PROOF. Let $\mathcal{E}$ be the subset of E^* defined by

$$\mathcal{E} = \{\xi \in E^* \mid \|\xi\| \leq 1, \ \xi(u) = \|u\|\},$$

which is nonempty by the Hahn–Banach theorem. It is also a convex and weak*-compact subset and hence, by the Krein–Milman theorem, it has an extreme point $\xi \in \mathrm{ext}(\mathcal{E})$. We show that ξ is an extreme point of $B(E^*)$.

Suppose that $\xi = \frac{1}{2}(\alpha + \beta)$ with $\alpha, \beta \in E^*$ and $\|\alpha\|, \|\beta\| \le 1$. Then $\|u\| = \xi(u) = \frac{1}{2}(\alpha(u) + \beta(u))$ and $|\alpha(u)|, |\beta(u)| \le \|u\|$. The last equality implies that $\alpha(u) = \beta(u) = \|u\|$ and hence $\alpha, \beta \in \mathcal{E}$. Then we obtain the desired conclusion: that is, $\alpha = \beta = \xi$. □

3. Linear isometries of function spaces and induced homeomorphisms of extreme points

Let $T : A \to B$ be a surjective isometry between linear subspaces A and B of $C(X, E)$ and $C(Y, E)$, respectively, where E is a real or complex Banach space. The Mazur–Ulam theorem [37] states that T is a real-linear isometry whenever $T(0) = 0$. If E is a complex Banach space, A and B are complex subspaces and T is a *real*-linear isometry, then we regard E, $C(X, E)$ and $C(Y, E)$ as *real* Banach spaces and consider the *real* duals.

Fix a scalar field $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Let E be a Banach space over $\mathbb{F}$ and let A and B be $\mathbb{F}$-subspaces of $C(X, E)$ and $C(Y, E)$, respectively, that satisfy Condition (S1), Condition (S2) and Condition (ext). Assume that $T : A \to B$ is an $\mathbb{F}$-linear surjective isometry and we consider the $\mathbb{F}$-duals $A^*_{\mathbb{F}}$ and $B^*_{\mathbb{F}}$ and the $\mathbb{F}$-adjoint $T^*_{\mathbb{F}} : B^*_{\mathbb{F}} \to A^*_{\mathbb{F}}$, which is a surjective $\mathbb{F}$-linear isometry and thus satisfies $T^*_{\mathbb{F}}(\mathrm{ext}(B^*_{\mathbb{F}})) = \mathrm{ext}(A^*_{\mathbb{F}})$. In what follows until Example 3.7, we will omit the scalar field $\mathbb{F}$ from the notation. Proposition 2.3 implies that

$$\Phi_A : \mathrm{ext}(E^*) \times \mathrm{Ch}_E(A) \to \mathrm{ext}(A^*),$$
$$\Phi_B : \mathrm{ext}(E^*) \times \mathrm{Ch}_E(B) \to \mathrm{ext}(B^*)$$

are both homeomorphisms. They, together with the adjoint T^*, induce a homeomorphism

$$\tau = \Phi_A^{-1} \circ T^* \circ \Phi_B : \mathrm{ext}(E^*) \times \mathrm{Ch}_E(B) \to \mathrm{ext}(E^*) \times \mathrm{Ch}_E(A).$$

Let

$$\tau(\eta, y) = (\alpha(\eta, y), \varphi(\eta, y)), \quad (\eta, y) \in \mathrm{ext}(E^*) \times \mathrm{Ch}_E(B),$$

where $(\alpha(\eta, y), \varphi(\eta, y)) \in \mathrm{ext}(E^*) \times \mathrm{Ch}_E(B)$. This defines continuous maps

$$\alpha : \mathrm{ext}(E^*) \times \mathrm{Ch}_E(B) \to \mathrm{ext}(E^*)$$

and

$$\varphi : \mathrm{ext}(E^*) \times \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A).$$

It is convenient to use the following notation to rewrite the above as

$$T^*(\eta \circ \delta_y) = \alpha(\eta, y)\delta_{\varphi(\eta, y)}, \quad (y, \eta) \in \mathrm{ext}(E^*) \times \mathrm{Ch}_E(B). \tag{3.1}$$

For $y \in \mathrm{Ch}_E(B)$ and $\eta \in \mathrm{ext}(E^*)$, let $\alpha_y : \mathrm{ext}(E^*) \to \mathrm{ext}(E^*)$ and $\varphi_\eta : \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A)$ be continuous maps defined by

$$\begin{aligned}
\alpha_y(\eta) &= \alpha(\eta, y), \quad \eta \in \mathrm{ext}(E^*), \\
\varphi_\eta(y) &= \varphi(\eta, y), \quad y \in \mathrm{Ch}_E(B).
\end{aligned} \tag{3.2}$$

Similarly, the adjoint operator $(T^{-1})^* = (T^*)^{-1} : A^* \to B^*$ induces two continuous maps

$$\beta : \text{ext}(E^*) \times \text{Ch}_E(A) \to \text{ext}(E^*),$$
$$\psi : \text{ext}(E^*) \times \text{Ch}_E(A) \to \text{Ch}_E(B),$$

which are defined by the formula

$$(T^*)^{-1}(\xi \circ \delta_x) = \beta(\xi, x) \circ \delta_{\psi(\xi, x)}, \quad (\xi, x) \in \text{ext}(E^*) \times \text{Ch}_E(A). \tag{3.3}$$

As in (3.2), set

$$\begin{aligned}
\beta_x(\xi) &= \beta(\xi, x), \quad \xi \in \text{ext}(E^*), \\
\psi_\xi(x) &= \psi(\xi, x), \quad x \in \text{Ch}_E(A).
\end{aligned} \tag{3.4}$$

Writing down the equalities $\eta \circ \delta_y = (T^*)^{-1} T^* (\eta \circ \delta_y)$ and $\xi \circ \delta_x = T^* (T^*)^{-1} (\xi \circ \delta_x)$, according to (3.1) and (3.4), we obtain, for each $y \in \text{Ch}_E(B)$, $x \in \text{Ch}_E(A)$, and for each $\eta, \xi \in \text{ext}(E^*)$,

$$\eta = \beta(\alpha(\eta, y), \varphi(\eta, y)), \tag{3.5}$$
$$y = \psi(\alpha(\eta, y), \varphi(\eta, y)), \tag{3.6}$$
$$\xi = \alpha(\beta(\xi, x), \psi(\xi, x)), \tag{3.7}$$
$$x = \varphi(\beta(\xi, x), \psi(\xi, x)). \tag{3.8}$$

The following theorem reduces a derivation of a Banach–Stone type theorem to a problem of verifying a set of conditions on the above maps. This is a common framework in previous works, for example, [1, 3, 9, 10, 27, 34].

THEOREM 3.1. *Let X and Y be compact Hausdorff spaces and let E be a Banach space over $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Assume that A and B are subspaces of $C(X, E)$ and $C(Y, E)$, respectively, both satisfying Condition (S1), Condition (S2) and Condition (ext). For a surjective $\mathbb{F}$-linear isometry $T : A \to B$, we have the following.*

(a) *For an arbitrary $y \in \text{Ch}_E(B)$, let $V_y : E \to E$ be the linear operator defined by*

$$V_y(u) = T c_u(y), \quad u \in E, \tag{3.9}$$

and let $A_y = V_y^* : E^* \to E^*$, the adjoint operator of V_y. Then we have the following.

    (a.1) $\|V_y\| = \|A_y\| \le 1.$

    (a.2) $A_y \mid \text{ext}(E^*) = \alpha_y.$

    (a.3) *The map $y \mapsto V_y : \text{Ch}_E(B) \to \mathcal{L}(E, E)$ is continuous where the space $\mathcal{L}(E, E)$ is endowed with the strong operator topology.*

(b) *Suppose that $\varphi_{\eta_1} = \varphi_{\eta_2}$ for each $\eta_1, \eta_2 \in \text{ext}(E^*)$ and let $\varphi_* = \varphi_\eta : \text{Ch}_E(B) \to \text{Ch}_E(A)$. Then we have the following.*

    (b.1) *For each $y \in \text{Ch}_E(B)$, $\|V_y\| = 1$.*

    (b.2) *For each $f \in A$ and for each $y \in \text{Ch}_E(B)$,*

$$Tf(y) = V_y(f(\varphi_*(y)). \tag{3.10}$$

(b.3) *The map $\varphi_* : \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A)$ is a continuous surjection. If the maps α_y and β_x are injective for each $x \in \mathrm{Ch}_E(A)$ and $y \in \mathrm{Ch}_E(B)$, then φ_* is a homeomorphism.*

(c) *Assume that E^* is strictly convex so that $S(E^*) = \mathrm{ext}(E^*)$. Then we have the following.*

(c.1) *The maps $\alpha_y : S(E^*) \to S(E^*)$ and $\beta_x : S(E^*) \to S(E^*)$ are injective for each $x \in \mathrm{Ch}_E(A)$ and for each $y \in \mathrm{Ch}_E(B)$.*

(c.2) *Suppose that $\varphi_{\eta_1} = \varphi_{\eta_2}$ for each $\eta_1, \eta_2 \in S(E^*)$ and let $\varphi_* = \varphi_\eta$. Then α_y is a homeomorphism for each $y \in \mathrm{Ch}_E(B)$ and $A_y = V_y^*$ is an isometric isomorphism. If, moreover, E is reflexive, then V_y is also an isometric isomorphism.*

The injectivity/bijectivity of maps involved in the above theorem are summarized in the following Lemma.

LEMMA 3.2. (a) *We have the following implications.*

(a.1) *Suppose that $\varphi_{\eta_1} = \varphi_{\eta_2}$ for each $\eta_1, \eta_2 \in \mathrm{ext}(E^*)$ and let $\varphi_* = \varphi_\eta$. Then α_y is injective for each $y \in \mathrm{Ch}_E(B)$, β_x is surjective for each $x \in \mathrm{Ch}_E(A)$ and φ_* is surjective.*

(a.2) *Suppose that $\psi_{\xi_1} = \psi_{\xi_2}$ for each $\xi_1, \xi_2 \in \mathrm{ext}(E^*)$ and let $\psi_* = \psi_\xi$. Then β_x is injective for each $x \in \mathrm{Ch}_E(A)$, α_y is surjective for each $y \in \mathrm{Ch}_E(B)$ and ψ_* is surjective.*

(b) *Assuming that α_y and β_x are injective for each $x \in \mathrm{Ch}_E(A)$ and for each $y \in \mathrm{Ch}_E(B)$, consider the following eight conditions.*

(1)   *$\varphi_{\eta_1} = \varphi_{\eta_2}$ for each $\eta_1, \eta_2 \in \mathrm{ext}(E^*)$.*
(2)   *φ_η is bijective for each $\eta \in \mathrm{ext}(E^*)$.*
(3)   *$\psi_{\xi_1} = \psi_{\xi_2}$ for each $\xi_1, \xi_2 \in \mathrm{ext}(E^*)$.*
(4)   *ψ_ξ is bijective for each $\xi \in \mathrm{ext}(E^*)$.*
(5)   *$\alpha_{y_1} = \alpha_{y_2}$ for each $y_1, y_2 \in \mathrm{Ch}_E(B)$.*
(6)   *α_y is bijective for each $y \in \mathrm{Ch}_E(B)$.*
(7)   *$\beta_{x_1} = \beta_{x_2}$ for each $x_1, x_2 \in \mathrm{Ch}_E(A)$.*
(8)   *β_x is bijective for each $x \in \mathrm{Ch}_E(A)$.*

We have the following implications and equivalences.

$$
\begin{array}{ccccccc}
(1) & \Rightarrow & (2) & \Leftrightarrow & (8) & \Leftarrow & (7) \\
\Updownarrow & & & & & & \Updownarrow \\
(3) & \Rightarrow & (4) & \Leftrightarrow & (6) & \Leftarrow & (5)
\end{array}
$$

(c) *When the above condition (1)($\Longleftrightarrow$ (3)) holds, the map $\varphi_* := \varphi_\eta$ is a homeomorphism with the inverse $\varphi_*^{-1} = \psi_* := \psi_\xi$.*

Theorem 3.1 is proved after the proof of Lemma 3.2. In what follows up to Example 3.7, we will keep the notation above and, also, Condition (S1), Condition (S2)

and Condition (ext) on subspaces A and B are always assumed, unless stated otherwise. The assumptions on Banach spaces will be explicitly stated each time. We start with a proof of Lemma 3.2.

PROOF OF LEMMA 3.2. (a) Suppose that $\varphi_{\eta_1} = \varphi_{\eta_2}$ for each $\eta_1, \eta_2 \in \text{ext}(E^*)$ and let $\varphi_* = \varphi_\eta$.

First, we show that α_y is injective. Assume that $\alpha_y(\eta_1) = \alpha_y(\eta_2)$. Then, by (3.5) and the assumption,

$$\eta_1 = \beta(\alpha_y(\eta_1), \varphi_*(y)) = \beta(\alpha_y(\eta_2), \varphi_*(y)) = \eta_2,$$

and hence α_y is injective. Next, we show the surjectivity of β_x. For each $\eta \in \text{ext}(E^*)$, let $y = \psi(\eta, x)$. By the assumption and (3.8),

$$\varphi(\eta, y) = \varphi(\beta(\eta, x), y) = \varphi(\beta(\eta, x), \psi(\eta, x)) = x.$$

Hence, by (3.5),

$$\beta_x(\alpha(\eta, y)) = \beta(\alpha(\eta, y), x) = \beta(\alpha(\eta, y), \varphi(\eta, y)) = \eta,$$

as desired. In order to prove that φ_* is surjective, fix an arbitrary η. For each $x \in \text{Ch}_E(A)$, use the surjectivity of β_x to take a ξ such that $\beta_x(\xi) = \eta$. Then

$$\varphi_*(\psi(\xi, x)) = \varphi(\beta(\xi, x), \psi(\xi, x)) = x,$$

by (3.8). Hence φ_* is surjective.

This proves (a.1) and a symmetric argument proves (a.2).

(b) Assume that α_y and β_x are injective for each $x \in \text{Ch}_E(A)$ and for each $y \in \text{Ch}_E(B)$. We show that the implications and equivalences given by

$$(1) \Longrightarrow (8) \Longleftrightarrow (2), \quad (1) \Longrightarrow (3), \quad (5) \Longleftrightarrow (7) \quad \text{and} \quad (7) \Longrightarrow (8)$$

hold. The other implications/equivalences are proved by symmetric arguments.

$(1) \Longrightarrow (8)$. Assume that $\varphi_{\eta_1} = \varphi_{\eta_2}$ for each η_1, η_2. By (a.1), we see that β_x is surjective, and is bijective because β_x is injective, by our starting assumption. The inverse map is given by $\beta_x^{-1}(\eta) = \alpha(\eta, \psi(\eta, x))$.

$(8) \Longrightarrow (2)$. We suppose that β_x is bijective for each $x \in \text{Ch}_E(A)$ and prove that φ_η is a bijection. For an arbitrarily fixed $\eta \in \text{ext}(E^*)$, let $\bar{\psi} : \text{Ch}_E(B) \to \text{Ch}_E(A)$ be the map defined by

$$\bar{\psi}(x) = \psi(\beta_x^{-1}(\eta), x), \quad x \in \text{Ch}_E(A).$$

We verify the equalities $\bar{\psi} \circ \varphi_\eta = \text{id}$ and $\varphi_\eta \circ \bar{\psi} = \text{id}$. For an arbitrary $y \in \text{Ch}_E(B)$, let $\xi = (\beta_{\varphi_\eta(y)})^{-1}(\eta)$. Then, by (3.5),

$$\beta_{\varphi_\eta(y)}(\xi) = \eta = \beta(\alpha(\eta, y), \varphi(\eta, y)) = \beta_{\varphi_\eta(y)}(\alpha(\eta, y)),$$

and, by the injectivity of $\beta_{\varphi_\eta(y)}$, we obtain $\xi = \alpha(\eta, y)$. Thus we see that

$$\begin{aligned} \bar{\psi}(\varphi_\eta(y)) &= \psi((\beta_{\varphi_\eta(y)})^{-1}(\eta), \varphi_\eta(y))) \\ &= \psi(\xi, \varphi_\eta(y)) = \psi(\alpha(\eta, y), \varphi(\eta, y)) \\ &= y \quad \text{(by (3.6)),} \end{aligned}$$

which proves that $\bar{\psi} \circ \varphi_\eta = \mathrm{id}$. For the other equality, take x and let $\xi = (\beta_x)^{-1}(\eta)$ so that $\bar{\psi}(x) = \psi(\xi, x)$. Then we obtain

$$
\begin{aligned}
\varphi_\eta(\bar{\psi}(x)) &= \varphi(\eta, \bar{\psi}(x)) \\
&= \varphi(\beta(\xi, x), \psi(\xi, x)) \\
&= x \quad \text{(by (3.8))},
\end{aligned}
$$

as required. This proves that $\bar{\psi}$ is the inverse map of φ_η.

(2) $\Longrightarrow$ (8). Suppose that φ_η is bijective for each η. In order to prove that β_x is a bijection for each $x \in \mathrm{ext}(E^*)$, take an arbitrary $\eta \in \mathrm{ext}(E^*)$ and let $y = (\varphi_\eta)^{-1}(x)$. Then, by (3.5),

$$
\eta = \beta(\alpha(\eta, y), \varphi(\eta, y)) = \beta_x(\alpha(\eta, y)),
$$

and hence β_x is surjective and thus is bijective. The inverse map β_x^{-1} is written as

$$
\beta_x^{-1}(\eta) = \alpha(\eta, \varphi_\eta^{-1}(x)).
$$

(1) $\Longrightarrow$ (3). We assume that $\varphi_{\eta_1} = \varphi_{\eta_2}$ for each η_1, η_2 and let $\varphi_* = \varphi_\eta$. In order to show that $\psi_{\xi_1} = \psi_{\xi_2}$, take an arbitrary $x \in \mathrm{Ch}_E(A)$ and apply (3.8) and the assumption to see that

$$
\varphi(\beta(\xi_2, x), \psi(\xi_2, x)) = x = \varphi(\beta(\xi_1, x), \psi(\xi_1, x)) = \varphi(\beta(\xi_2, x), \psi(\xi_1, x)).
$$

By the implication (1) $\Longrightarrow$ (8) $\Longleftrightarrow$ (2) proved above, we know that $\varphi_{\beta(\xi_2, x)} = \varphi_*$ is a bijection, from which we obtain $\psi(\xi_1, x) = \psi(\xi_2, x)$. This proves that $\psi_{\xi_1} = \psi_{\xi_2}$.

(7) $\Longleftrightarrow$ (5). We suppose that $\beta_{x_1} = \beta_{x_2}$ for each $x_1, x_2 \in \mathrm{Ch}_E(A)$ and show that $\alpha_{y_1} = \alpha_{y_2}$ for each $y_1, y_2 \in \mathrm{Ch}_E(B)$. For each η, let $x_1 = \varphi(\eta, y_1)$ and $x_2 = \varphi(\eta, y_2)$. By the assumption and (3.5),

$$
\begin{aligned}
\beta_{x_2}(\alpha(\eta, y_1)) = \beta_{x_1}(\alpha(\eta, y_1)) &= \beta(\alpha(\eta, y_1), \varphi(\eta, y_1)) \\
&= \eta = \beta(\alpha(\eta, y_2), \varphi(\eta, y_2)) \\
&= \beta_{x_2}(\alpha(\eta, y_2)).
\end{aligned}
$$

The injectivity of β_{x_2} implies that $\alpha(\eta, y_1) = \alpha(\eta, y_2)$. Hence $\alpha_{y_1} = \alpha_{y_2}$, as required. The converse implication is proved by an argument that is symmetric to the above.

(7) $\Longrightarrow$ (8). Suppose that $\beta_{x_1} = \beta_{x_2}$ for each $x_1, x_2 \in \mathrm{Ch}_E(A)$ and we show that $\beta_* := \beta_x$ is a bijection. In view of the equivalence (7) $\Leftrightarrow$ (5), let $\alpha_* = \alpha_y$ for an arbitrary y. Take an $\eta \in \mathrm{ext}(E^*)$ and let $\xi = \alpha_*(\eta)$. Then, by (3.5),

$$
\begin{aligned}
\beta_*(\xi) = \beta_{\varphi(\eta, y)}(\xi) &= \beta(\xi, \varphi(\eta, y)) = \eta.
\end{aligned}
$$

Hence β_* is surjective and thus is bijective, and the inverse is given by $\beta_*^{-1} = \alpha$.

(c) Assume that the condition (1)($\iff$ (3)) holds in (b) and let $\varphi_* := \varphi_\eta$ and $\psi_* = \psi_\xi$. We see directly from (3.6) and (3.8) that $\varphi_*^{-1} = \psi_*$. The continuity of these maps φ_* and ψ_* are a direct consequence of that of φ and ψ ((3.1) and (3.3)) and hence each of those maps is a homeomorphism that has the other as its inverse.

This completes the proof of Lemma 3.2.   □

PROOF OF THEOREM 3.1.

(a) Let V_y be the operator defined by (3.9) and let A_y be the adjoint operator of V_y. By definition, for each $\eta \in E^*$ and for each $u \in E$,

$$A_y(\eta)(u) = \eta(V_y(u))$$
$$= \eta(Tc_u(y)) = \eta \circ \delta_y(c_u)$$
$$= T^*(\eta \circ \delta_y)(c_u).$$

If $\eta \in \mathrm{ext}(E^*)$, then the last term of the above is equal to

$$\alpha(\eta, y)\delta_{\varphi(\eta,y)}(c_u) = \alpha(\eta, y)(u) = \alpha_y(\eta)(u)$$

(see (3.1) and (3.2)) and hence $\alpha_y(\eta) = A_y(\eta)$. This proves statement (a.2). The continuity of the map $y \mapsto V_y : \mathrm{Ch}_E(B) \to \mathcal{L}(E, E)$ is a direct consequence of (3.9). In order to show the inequality $\|V_y\| \le 1$, observe that

$$\|V_y(u)\| = \|Tc_u(y)\|$$
$$\le \|Tc_u\|_\infty = \|c_u\|_\infty$$
$$= \|u\|.$$

Hence we have $\|V_y\| \le 1$. This completes the proof of (a).

(b) and (c). Assume that $\varphi_{\eta_1} = \varphi_{\eta_2}$ for each $\eta_1, \eta_2 \in S(E^*)$ and let $\varphi_* = \varphi_\eta : \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A)$. In order to prove (3.10), we first show that

$$\text{for each } f \in A \text{ with } f(\varphi_*(y)) = 0, \text{ we have } Tf(y) = 0. \tag{3.11}$$

In fact, for each $\eta \in \mathrm{ext}(E^*)$,

$$\eta(Tf(y)) = T^*(\eta \circ \delta_y)f$$
$$= \alpha(\eta, y)\delta_{\varphi_*(y)}(f) = \alpha_y(\eta)(f(\varphi_*(y)))$$
$$= 0.$$

Applying Lemma 2.6, we obtain the conclusion (3.11). Notice here that, in order to obtain the above conclusion from Lemma 2.6, we need to have the equality $\eta(Tf(y)) = 0$ for *each* $\eta \in \mathrm{ext}(E^*)$. This is where we rely on Condition (ext).

The above (3.11) implies that

$$Tf_1(y) = Tf_2(y) \quad \text{whenever } f_1(\varphi_*(y)) = f_2(\varphi_*(y)), f_1, f_2 \in A.$$

In particular, $Tf(y) = Tc_{f(\varphi_*(y))}$ for $f \in A$. Then, by definition (3.9), we obtain the desired formula (3.10).

For the proof of the equality of (b.1), it suffices to show that $\|V_y\| \geq 1$. For the proof, take a $u \in E$ such that $\|u\| = 1$ and let $f = T^{-1} c_u$. Then $\|f\|_\infty = \|c_u\|_\infty = 1$. Also, by (3.10),

$$
\begin{aligned}
1 = \|u\| = \|Tf(y)\| &= \|V_y(f(\varphi_*(y)))\| \\
&\leq \|V_y\| \, \|f(\varphi_*(y))\| \\
&\leq \|V_y\| \, \|f\|_\infty = \|V_y\|,
\end{aligned}
$$

and hence we obtain $\|V_y\| \geq 1$, as desired.

By Lemma 3.2(a), φ_* is a surjection. If β_x and α_y are injective for each $x \in \mathrm{Ch}_E(A)$ and $y \in \mathrm{Ch}_E(B)$, then Lemma 3.2(b) and (c) apply to conclude that φ_* is a homeomorphism. This proves the statements of (b).

Assume, further, that E^* is strictly convex so that $S(E^*) = \mathrm{ext}(E^*)$. First, we show that, for each $\eta \in E^*$,

$$\|A_y(\eta)\| = \|\eta\|. \tag{3.12}$$

For each $\eta \in E^* \setminus \{0\}$, $\eta/\|\eta\| \in S(E^*) = \mathrm{ext}(E^*)$. Hence its image $\alpha_y(\eta/\|\eta\|)$ is in $\mathrm{ext}(E^*)$ and, in particular, has norm one. Thus we see that

$$
\begin{aligned}
\|A_y(\eta)\| = \|\eta\| \left\| A_y\!\left(\frac{\eta}{\|\eta\|} \right) \right\| \\
= \|\eta\| \left\| \alpha_y\!\left(\frac{\eta}{\|\eta\|} \right) \right\| \\
= \|\eta\|.
\end{aligned}
$$

This proves (3.12). Since A_y is an extension of α_y, it follows from this that α_y is injective. By repeating the above argument for T^{-1}, we obtain the same conclusion for β_x. This proves (c.1). In order to prove (c.2), assume $\varphi_{\eta_1} = \varphi_{\eta_2}$ for each $\eta_1, \eta_2 \in S(E^*)$ and let $\varphi_* = \varphi_\eta : \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A)$. The implication $(1) \Longrightarrow (6)$ of Lemma 3.2(b) shows that $\alpha_y : S(E^*) \to S(E^*)$ is a bijection on the unit sphere of E^*. Then A_y, as a linear extension of α_y, is a linear isomorphism as well as an isometry (by (3.12)). If, moreover, E is reflexive, the operator $V_y : E \cong E^{**} \to E$ is also an isometric isomorphism. This completes the proof of (c) and completes the proof of Theorem 3.1. $\square$

In [6, Proposition 3.2], Botelho and Jamison proved the condition '$\alpha_{y_1} = \alpha_{y_2}$', that is, the condition (5) of Lemma 3.2(b), under the hypothesis that T preserves the constant functions. Then they appeal to the Ding–Liu extension theorem [30, Corollary 2, page 963] to obtain an linear extension of α_y. Assuming, further, that E is reflexive, they obtain a linear map $V_y : E \to E$ to prove that T is a generalized weighted composition operator. The idea of applying the reflexivity in the proof of Theorem 3.1(c) was adopted from their argument.

Some of the previous works study sufficient conditions that guarantee the condition '$\varphi_{\eta_1} = \varphi_{\eta_2}$' from which we conclude that T is a generalized weighted composition operator. These conditions have generally been stated in terms of some separation

properties of the subspaces A and B. Theorem 3.4 is a result in this direction and its proof follows arguments given in [1, 10, 18]. For a subspace A of $C(X, E)$, we consider the following condition.

(S3) For each $x \in \mathrm{Ch}_E(A)$, for each neighborhood U of x and for each $u \in E$, there exists an $f \in A$ such that $\|f\|_\infty = \|u\|$, $f(x) = u$ and $f|X \setminus U \equiv 0$.

Notice that Condition (S3) implies Condition (S2).

THEOREM 3.3 (Cf. [1, 10, 18]). *Assume that E is strictly convex and reflexive, and assume, further, that A satisfies Condition (S1), Condition (S3) and Condition (ext). Then, for each $\eta_1, \eta_2 \in \mathrm{ext}(E^*)$, $\varphi_{\eta_1} = \varphi_{\eta_2}$.*

PROOF. Let $y \in \mathrm{Ch}_E(B)$. We prove that there exists an $x \in \mathrm{Ch}_E(A)$ such that for each $\eta \in \mathrm{ext}(E^*)$,

$$T^*(\eta \circ \delta_y) = \xi \circ \delta_x \text{ for some } \xi \in \mathrm{ext}(E^*).$$

From this, and Lemma 2.1, we conclude that $\varphi_\eta(y) = x$ for each $\eta \in \mathrm{ext}(E^*)$. First, we show the following.

CLAIM 1. *Suppose that $T^*(\eta \circ \delta_y) = \xi \circ \delta_x$ for $\|\eta\| = \|\xi\| = 1$. If a function $f \in A$ satisfies $f|U \equiv 0$ for a neighborhood U of x, then $(Tf)(y) = 0$.*

PROOF OF CLAIM 1 (CF. PROOF OF PROPOSITION 2.4). We may assume at the outset that $\|f\|_\infty = 1$. By the reflexivity, there exists a vector $u \in E$ with $\|u\| = 1$ such that $\xi(u) = \|u\| = \|\xi\| = 1$. By Condition (S3), there exists a function $f_1 \in A$ such that $\|f_1\|_\infty = 1$, $f_1(x) = u$ and $f_1|X \setminus U \equiv 0$. Let $g = f + f_1$ and $h = \frac{1}{2}(g + f_1) = f_1 + \frac{1}{2}f$. Then we see that $g(x) = h(x) = u$ and also $\|g\|_\infty = \|h\|_\infty = 1$. Furthermore, we see that

$$\eta(Tf_1(y)) = T^*(\eta \circ \delta_y) = \xi(f_1(x)) = \xi(u) = \|u\| = 1,$$

and we obtain, in the same way, that $\eta(Th(y)) = \eta(Tg(y)) = 1$. It follows that $\|Tf_1(y)\| = \|Tg(y)\| = \|Th(y)\| = 1$ and in, particular, $Th(y)$ is an extreme point of $B(E)$, by the strict convexity of E. Now the equation

$$Th(y) = \frac{1}{2}Tf_1(y) + \frac{1}{2}Tg(y)$$

forces that $Th(y) = Tf_1(y) = Tg(y)$. This implies that $\frac{1}{2}(Tf)(y) = 0$, as desired.

The proof will be complete once we have shown the following.

CLAIM 2. *If $T^*(\eta_1 \circ \delta_y) = \xi_1 \circ \delta_{x_1}$ and $T^*(\eta_2 \circ \delta_y) = \xi_2 \circ \delta_{x_2}$, then $x_1 = x_2$.*

PROOF OF CLAIM 2. Suppose that $x_1 \neq x_2$ and take $u \in E$ such that $\xi_2(u) \neq 0$. By Condition (S3), we may find a neighborhood U of x_1 and a map $f \in A$ such that $\|f\|_\infty = \|u\|$, $f(x_2) = u$ and $f|U \equiv 0$. By Claim 1, we see that $Tf(y) = 0$, which leads to the contradiction

$$0 = \eta_2(Tf(y)) = T^*(\eta_2 \circ \delta_y)(f) = \xi_2(f(x_2)) = \xi_2(u) \neq 0.$$

Thus $x_1 = x_2$ and the proof is complete.   □

In summary, we obtain the following theorem.

THEOREM 3.4. *Let X and Y be compact Hausdorff spaces and let E be a strictly convex, reflexive Banach space. Assume that A and B are subspaces of $C(X, E)$ and $C(Y, E)$, respectively, and that both satisfy Condition (S1), Condition (S3) and also:*

(M) *for each $f \in A$ with $f(x) = 0$ and for each $\varepsilon > 0$, there exist a neighborhood U of x and $f_\varepsilon \in A$ such that $\|f - f_\varepsilon\| < \varepsilon$ and $f_\varepsilon|U \equiv 0$.*

Let $T : A \to B$ be a surjective linear isometry.

(a) *There exists a continuous surjection $\varphi_* : \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A)$ and a collection $(V_y : E \to E)$ of linear operators with $\|V_y\| = 1$ and*

$$(Tf)(y) = V_y(f(\varphi_*(y))) \quad \text{for each } f \in A \text{ and for each } y \in \mathrm{Ch}_E(B).$$

The map $y \mapsto V_y$ is continuous with respect to the strong operator topology on $\mathcal{L}(E, E)$.

(b) *If, furthermore, the dual E^* is strictly convex, then the above map φ_* is a homeomorphism and V_y is an isometric isomorphism for each $y \in \mathrm{Ch}_E(B)$.*

In particular, if A and B are function-algebra modules in the sense of Section 2, then the above conclusions hold.

The last assertion has been known in the literature under weaker assumptions on the Banach space E. See, for example, [18] and [17, Theorem 7.5.9], while the subset on which the homeomorphism φ_* is defined does not necessarily coincide with the Choquet boundary. See also [1, Theorem 4.3].

PROOF. Condition (M) is exactly the hypothesis (b) of Proposition 2.4. We show that A and B satisfy the hypothesis (c) of Proposition 2.4;

• for each $\xi \in S(E^*)$ and for each neighborhood U of x, there exists an $f \in A$ such that $\|f\|_\infty = 1 = \xi(f(x))$ and $f|X \setminus U \equiv 0$.

Indeed. there exists a $u \in E$ such that $\xi(u) = 1 = \|u\|$, by the reflexivity of E. Take an $f \in A$ such that $\|f\|_\infty = 1$, $f(x) = u$ and $f|X \setminus U \equiv 0$, by (S3). Then $\xi(f(x)) = 1$ and f is the required function. The above, with Condition (S1) via Proposition 2.4, implies that A satisfies Condition (ext). The same is true for B.

Applying Theorem 3.3, we see that $\varphi_{\eta_1} = \varphi_{\eta_2}$ for each $\eta_1, \eta_2 \in \mathrm{ext}(E^*)$. Let $\varphi_* = \varphi_\eta$. By Theorem 3.1(b), (c) and the reflexivity of E, there exists a continuous surjection $\varphi_* : \mathrm{Ch}_E(E) \to \mathrm{Ch}_E(A)$ and a collection $(V_y)_{y \in \mathrm{Ch}_E(B)}$ of linear operators with the desired properties. This completes the proof.   □

By the use of Lemma 2.5, we may replace the assumptions on E, A and B in the above theorem by

E is a Hilbert space and A and B are unitarily invariant subspaces satisfying Condition (S1) and Condition (S3),

to obtain the same conclusion.

As mentioned in Section 1, the above theorem follows the same lines as that of previous results, for example [1, 6, 18, 25, 28]. In these previous works, deriving the existence of well-defined maps φ_* and V_y from (3.1), and verifying their continuity, requires involved arguments. In our set-up, these verifications are built into the statements of Propositions 2.3 and 2.4 and Theorem 3.1.

Homotopical rigidity is another condition on a compact Hausdorff space X which forces every linear isometry between function spaces over X to be a generalized weighted composition opeaator. We say that a topological space M is *homotopically rigid* if each continuous map $h : [0, 1] \to M$ is a constant map. If M is homotopically rigid, then every homotopy $H : Z \times [0, 1] \to M$ of a space Z to M must be trivial in the sense that $H(z, t) = H(z, 0)$ for each $(z, t) \in Z \times [0, 1]$. Besides totally disconnected spaces, connected examples include hereditarily indecomposable continua (see, for example, [11, 13]). There even exists a planar hereritarily indecomposable, and hence homotopically rigid, compact connected space [4]. Let X and Y be compact Hausdorff spaces and let A and B be subspaces of $C(X, E)$ and $C(Y, E)$, respectively. Assume that X is homotopically rigid, E is reflexive and that E^* is strictly convex so that $S(E^*) = \text{ext}(E^*)$. A surjective isometry $T : A \to B$ induces a map $\varphi : S(E^*) \times \text{Ch}_E(B) = \text{ext}(E^*) \times \text{Ch}_E(B) \to \text{Ch}_E(A) \subset X$, as in (3.1). For each $\eta_0, \eta_1 \in S(E^*)$, there exists a path $(\eta_t)_{0 \le t \le 1}$ from η_0 to η_1 in $S(E^*)$. Then $(\varphi_{\eta_t})_{0 \le t \le 1}$ is a homotopy and, by the homotopical rigidity, $\varphi_{\eta_0} = \varphi_{\eta_1}$. Therefore Theorem 3.1 yields the following theorem.

THEOREM 3.5. *Let X and Y be compact Hausdorff spaces and let A and B be subspaces of $C(X, E)$ and $C(Y, E)$, respectively. Assume that A and B satisfy Condition (S1), Condition (S2) and Condition (ext) and assume, further, that E is reflexive and E^* is strictly convex. If X is homotopically rigid, then every surjective linear isometry $T : A \to B$ is a generalized weighted composition operator*

$$T f(y) = V_y(f(\varphi_*(y))), \quad f \in A, \ y \in Y,$$

where $\varphi_* : Y \to X$ is a homeomorphism and $(V_y : E \to E)_{y \in \text{Ch}_E(B)}$ is a continuous collection of isometric linear isomorphisms.

REMARK 3.6. In view of the comment after Theorem 3.4, we see that assumptions on E, A and B are replaced by 'E is a Hilbert space, A and B are unitarily invariant subspaces satisfying (S1) and (S2)' to obtain the same conclusion.

EXAMPLE 3.7. As an illustration of previous results, we review a Banach–Stone type theorem for $\mathbb{C}$-valued continuous function spaces.

Let A and B be $\mathbb{C}$-subspaces of $C(X, \mathbb{C})$ and $C(Y, \mathbb{C})$ where X and Y are compact Hausdorff spaces. We assume that A and B satisfy Condition (S1), and also that they separate the points of X and Y, respectively, in the sense of (s2). Let $T : A \to B$ be a surjective isometry such that $T(0) = 0$. The Mazur–Ulam theorem [37] implies that T is a real-linear isometry. We show that T is a generalized weighted composition operator under some additional hypotheses on A and B.

First, let us make some preliminary considerations. Let $R : \mathbb{C} \to \mathbb{R}^2$ be the map defined by

$$R(a + ib) = {}^t(a, b), \quad a, b \in \mathbb{R}.$$

The map R is an $\mathbb{R}$-linear isometry and induces an $\mathbb{R}$-linear isometry $R_X : C(X, \mathbb{C}) \to C(X, \mathbb{R}^2)$, defined by $R_X(f) = R \circ f$ for $f \in C(X, \mathbb{C})$. Let $A_{\mathbb{R}} = R_X(A)$. Since the space A satisfies Condition (S1) and Condition (S2), by Lemma 1.1, so does $A_{\mathbb{R}}$. By Lemma 2.5, A, and hence also $A_{\mathbb{R}}$, satisfy Condition (ext). Via the standard isometry between $\mathbb{R}^2$ and $(\mathbb{R}^2)^*$, the unit sphere $S((\mathbb{R}^2)^*)$ is isometric to the unit circle $S = R(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $A_{\mathbb{C}}^*$ be the $\mathbb{C}$-dual of A and let $A_{\mathbb{R}}^* = \mathcal{L}_{\mathbb{R}}(A_{\mathbb{R}}, \mathbb{R})$, the $\mathbb{R}$-dual of $A_{\mathbb{R}}$.

It is known that there exists an $\mathbb{R}$-isometry $r_A : A_{\mathbb{C}}^* \to \mathcal{L}_{\mathbb{R}}(A, \mathbb{R})$ defined by

$$r_A(\xi)(f) = \mathfrak{R}\xi(f), \quad \xi \in A_{\mathbb{C}}^*, \ f \in A,$$

whose inverse $s_A = r_A^{-1}$ is given by

$$s_A(\rho)(g) = \mathfrak{R}(\rho(g)) - i\mathfrak{R}(\rho(ig)), \quad \rho \in \mathcal{L}_{\mathbb{R}}(A, \mathbb{R}), \ g \in A$$

(see, for example, [27]). The maps R_X and r_A naturally induce an $\mathbb{R}$-linear isometry $\rho_A : A_{\mathbb{C}}^* \to A_{\mathbb{R}}^* = \mathcal{L}_{\mathbb{R}}(A_{\mathbb{R}}, \mathbb{R})$ defined by

$$\rho_A(\xi)({}^t(u, v)) = \mathfrak{R}\xi(u + iv)$$

for $\xi \in A_{\mathbb{C}}^*$ and ${}^t(u, v) \in A_{\mathbb{R}}$. In particular,

$$\rho_A(\mathrm{ext}(A_{\mathbb{C}}^*)) = \mathrm{ext}(A_{\mathbb{R}}^*).$$

In view of the above isometry, we can prove the equality $\mathrm{Ch}_{\mathbb{R}^2}(A) = \mathrm{Ch}(A)$, the standard Choquet boundary of A. As we have already seen above, $A_{\mathbb{R}}^*$ satisfies Condition (S1), Condition (S2) and Condition (ext). Combining these with Proposition 2.3, we have a homeomorphism

$$\Phi_{A_{\mathbb{R}}} : S \times \mathrm{Ch}(A) = \mathrm{ext}((\mathbb{R}^2)^*) \times \mathrm{Ch}(A) \to \mathrm{ext}(A_{\mathbb{R}}^*). \tag{3.13}$$

The same holds for the subspace B. The $\mathbb{R}$-adjoint of the $\mathbb{R}$-linear map $T_{\mathbb{R}} = R_Y \circ T \circ R_X^{-1} : A_{\mathbb{R}} \to B_{\mathbb{R}}$, denoted by $T_{\mathbb{R}}^* : B_{\mathbb{R}}^* \to A_{\mathbb{R}}^*$, is an $\mathbb{R}$-linear isometry and preserves the extreme points: that is,

$$T_{\mathbb{R}}^*(\mathrm{ext}(B_{\mathbb{R}}^*)) = \mathrm{ext}(A_{\mathbb{R}}^*).$$

These preliminary considerations enable us to apply Theorem 3.1. The homeomorphisms $T_{\mathbb{R}}^*$, $\Phi_{A_{\mathbb{R}}}$ and $\Phi_{B_{\mathbb{R}}}$ of (3.13) induce a homeomorphism $S \times \mathrm{Ch}(B) \to S \times \mathrm{Ch}(A)$ which comprises two maps $\varphi : S \times \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ and $\alpha : S \times \mathrm{Ch}(B) \to S$. By Theorem 3.1(a), for each $y \in \mathrm{Ch}(B)$, there exists an isometric $\mathbb{R}$-linear isomorphism $A_y : (\mathbb{R}^2)^* \to (\mathbb{R}^2)^*$, which is an extension of $\alpha_y : S \to S$.

Assume that $\varphi_{y_1} = \varphi_{y_2}$ for each pair y_1, y_2 of points of $\mathrm{Ch}(B)$: for example, assume that A and B satisfy Condition (S3), or A and B are strongly separating and strongly zero-separating in the sense of [27]. Then we see that $A_{\mathbb{R}}$ and $B_{\mathbb{R}}$ have the same

property. The map $\varphi_* := \varphi_\eta$ and the adjoint $V_y : \mathbb{R}^2 \to \mathbb{R}^2$ of A_y provide us with an expression of $T_\mathbb{R}$ as a generalized weighted composition operator, by Theorem 3.1(c): that is,

$$T_\mathbb{R} f_\mathbb{R}(y) = V_y(f_\mathbb{R}(\varphi_*(y))), \quad f_\mathbb{R} \in A_\mathbb{R}, \quad y \in \mathrm{Ch}_{\mathbb{R}^2}(B) = \mathrm{Ch}(B).$$

The linear map V_y is an isometry and, with respect to the standard basis of $\mathbb{R}^2$, it is represented as a two-by-two matrix of the form

$$V_y = \rho(\theta_y) \text{ or } J\rho(\theta_y),$$

depending on whether $\det V_y = 1$ or $\det V_y = -1$, where

$$\rho(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.14}$$

and

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $K_+ = \{y \in \mathrm{Ch}(B) \mid \det V_y = 1\}$ and $K_- = \{y \in \mathrm{Ch}(B) \mid \det V_y = -1\}$. They are closed and open in $\mathrm{Ch}(B)$, mutually disjoint, and cover $\mathrm{Ch}(B)$.

Let $W_y = R^{-1} \circ V_y \circ R : \mathbb{C} \to \mathbb{C}$. Noticing that the action of the rotation matrix $\rho(\theta)$ of (3.14) on $\mathbb{R}^2$ transfers, via the map R, to the multiplication of $e^{i\theta} \in \mathbb{T}$ on $\mathbb{C}$, we see that there exists a map $\omega : \mathrm{Ch}(B) \to \mathbb{T}$ such that

$$W_y(\lambda) = \begin{cases} \omega(y) \cdot \lambda & \text{if } y \in K_+, \\ \overline{\omega(y) \cdot \lambda} & \text{if } y \in K_-. \end{cases} \tag{3.15}$$

Therefore, for each $f \in A$, and with $f_\mathbb{R} = R_X(f) = R \circ f$,

$$\begin{aligned}
Tf(y) &= ((R_Y^{-1} \circ T_\mathbb{R} \circ R_X)f)(y) \\
&= R^{-1}(T_\mathbb{R} f_\mathbb{R}(y)) \\
&= R^{-1}(V_y(f_\mathbb{R}(\varphi_*(y)))) \\
&= (R^{-1} \circ V_y \circ R)(R^{-1}(f_\mathbb{R}(\varphi_*(y)))) \\
&= W_y(f(\varphi_*(y))).
\end{aligned}$$

Applying (3.15) to the last term of the above, gives

$$Tf(y) = \begin{cases} \omega(y) \cdot f(\varphi_*(y)) & \text{if } y \in K_+, \\ \overline{\omega(y) \cdot f(\varphi_*(y))} & \text{if } y \in K_-. \end{cases}$$

If T is complex-linear, it is easy to see that $K_- = \emptyset$. This is the Banach–Stone type theorem for complex-valued continuous functions that was proved in [27].

4. Topological dimensions of underlying spaces and generalized weighted composition operators

For an $\mathbb{F}$-linear surjective isometry $T : A \to B$ between function spaces $A \subset C(X, E)$ and $B \subset C(Y, E)$ satisfying Condition (S1), Condition (S2) and Condition (ext), we continue to examine the induced map $\varphi : \mathrm{ext}(E^*) \times \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A)$ (see (3.1)) in connection with the topological dimension $\dim \mathrm{Ch}_E(A)$ of $\mathrm{Ch}_E(A)$. For each $y \in \mathrm{Ch}_E(B)$, let $\phi_y : \mathrm{Ch}_E(B) \to \mathrm{Ch}_E(A)$ be the map defined by

$$\phi_y(\eta) = \varphi(\eta, y), \quad \eta \in \mathrm{ext}(E^*).$$

Assume that the dual E^* is strictly convex so that $\mathrm{ext}(E^*) = S(E^*)$. The next lemma states that each fiber of ϕ_y forms a spherically convex subset of $S(E^*)$. For $\eta_1, \ldots, \eta_r \in \mathrm{ext}(E^*) \subset S(E^*)$, let $H(\eta_1, \ldots, \eta_r) = \mathrm{span}_{\mathbb{F}}\langle \eta_1, \ldots, \eta_r \rangle$, which is the linear subspace of E^* spanned by $\{\eta_1, \ldots, \eta_r\}$. Moreover, let $S(\eta_1, \ldots, \eta_r) = H(\eta_1, \ldots, \eta_r) \cap S(E^*)$, which is the spherical convex hull spanned by $\eta_1, \ldots, \eta_r$.

LEMMA 4.1. *Assume that E^* is strictly convex. For an arbitrary point $x \in \phi_y(S(E^*))$ and for each finite subset $\{\eta_1, \ldots, \eta_r\} \subset \phi_y^{-1}(x)$,*

$$S(\eta_1, \ldots, \eta_r) \subset \phi_y^{-1}(x).$$

In particular, $\{\eta, -\eta\} \subset \phi_y^{-1}(x)$ for each $\eta \in \phi_y^{-1}(x)$.

PROOF. For each $\mathbb{F}$-linear combination $\eta = \sum_{i=1}^r t_i \eta_i$, we see that

$$\begin{aligned}
\alpha(\eta, y)\delta_{\phi_y(\eta)} = \alpha(\eta, y)\delta_{\varphi(\eta, y)} &= T^*(\eta \circ \delta_y) \\
&= \sum_{i=1}^r t_i T^*(\eta_i \circ \delta_y) \\
&= \sum_{i=1}^r t_i \alpha(\eta_i, y)\delta_{\phi_y(\eta_i)} \\
&= \left(\sum_{i=1}^r t_i \alpha(\eta_i, y) \right)\delta_x.
\end{aligned}$$

By Condition (S2) and Lemma 2.1, we obtain that $x = \phi_y(\eta)$, which completes the proof. □

Let L be a subspace of E^* and notice that $S(L) = L \cap S(E^*)$. Also, let $\phi_{y,L} = \phi_y | S(L) : S(L) \to \mathrm{Ch}_E(A)$. For $x \in \mathrm{Ch}_E(A)$, let $H_{x,L}$ be the subspace of E^* spanned by the set $\phi_{y,L}^{-1}(x)$. When $L = E^*$, $H_{x,L}$ is simply denoted by H_x. The above lemma implies the following proposition.

PROPOSITION 4.2. *For each $x \in \phi_{y,L}(S(L))$,*

$$\phi_{y,L}^{-1}(x) = H_x \cap S(L).$$

Take an $\mathbb{F}$-linear isometry $T : A \to B$ which is *not* a generalized weighted composition operator. By Theorem 3.1(b), the map $\eta \mapsto \varphi_\eta$ is not a constant map and thus $\phi_y : S(E^*) \to \mathrm{Ch}_E(A)$ is not a constant map for some $y \in \mathrm{Ch}_E(B)$. Fix such y arbitrarily and take a finite dimensional subspace L such that $\phi_{y,L}$ is not constant. The map $\phi_{y,L}$ is examined in the next lemma.

LEMMA 4.3. *Assume that E^* is strictly convex and let L be a finite dimensional subspace of E^* such that $\phi_{y,L}$ is not a constant map. Let $n_L = \dim_\mathbb{F} L$. Then we have the following.*

(a) *The inequality*

$$1 \le \dim_\mathbb{F} H_{x,L} \le n_L - 1$$

  holds for each $x \in \phi_{y,L}(S(L))$.
(b) *Assume that $n_L \ge 2$. Then, either:*

  (b.1) *$\dim H_{x,L} < n_L - 1$ for each $x \in \mathrm{Ch}_E(A)$; or*
  (b.2) *there exists a unique $x_0 \in \mathrm{Ch}_E(A)$ such that $\dim_\mathrm{F} H_{x_0,L} = n_L - 1$ and $\dim_\mathbb{F} H_{x,L} = 1$ for each $x \in \mathrm{Ch}_E(A) \setminus \{x_0\}$.*

PROOF. (a) Note that $\dim H_{x,L}$ cannot be zero for each $x \in \phi_{y,L}(S(L))$ because $\{\eta, -\eta\} \subset \phi_{y,L}^{-1}(x)$. Also, $H_{x,L}$ cannot be the whole L because $\phi_{y,L}$ is not a constant map. The conclusion (a) follows from these.

(b) The set $\{\phi_{y,L}^{-1}(x) \mid x \in \phi_{y,L}(S_L)\}$ is a mutually disjoint collection. This implies that $H_{x_1,L} \cap H_{x_2,L} = \{0\}$ for each pair of distinct points x_1 and x_2. Thus

$$\dim_\mathbb{F} H_{x_1,L} + \dim_\mathbb{F} H_{x_2,L} = \dim_\mathbb{F}(H_{x_1,L} + H_{x_2,L}) \le n_L \tag{4.1}$$

for each pair of points $x_1 \ne x_2$ of $\phi_{y,L}(S(L))$. This, together with (a), yields that $\dim_\mathbb{F} H_{x,L} \le n_L - 1$.

Suppose that $\dim_\mathbb{F} H_{x_0,L} = n_L - 1$ for some $x_0 \in \mathrm{Ch}_E(A)$. Then, for each x other than x_0, we see from (4.1) and (a) that $\dim_\mathbb{F} H_{x,L} = 1$. This proves (b). $\square$

In order to apply the following theorem from topological dimension theory, we assume that the underlying space X is metrizable and the Banach space E is separable in Theorems 4.5 and 4.6 below. For a separable metrizable space M, $\dim M$ denotes the topological dimension of M. See [15] for a thorough treatment of topological dimension theory.

THEOREM 4.4 [15, Theorem 1.12.2]. *Let $f : M \to N$ be a closed map between separable metrizable spaces M and N. Then*

$$\dim M \le \dim N + \sup_{q \in N} \dim f^{-1}(q).$$

THEOREM 4.5. *Let X and Y be compact metrizable spaces and let E be a separable Banach space over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ with the strictly convex dual. Let A and B be subspaces of $C(X, E)$ and $C(Y, E)$, respectively, satisfying Condition (S1), Condition (S2) and Condition (ext). Let $T : A \to B$ be an $\mathbb{F}$-linear isometry such that $\phi_y : S(E^*) \to \mathrm{Ch}_E(A)$ is not a constant map for some $y \in \mathrm{Ch}_E(B)$.*

(a) *Assume that* $\dim_{\mathbb{F}} E \geq 3$. *Then* $\dim \mathrm{Ch}_E(A) \geq 2$.
(b) *Assume that* $\dim_{\mathbb{F}} E \leq 2$.

    (b.1) *If* $\mathbb{F} = \mathbb{R}$, *then* $\dim_{\mathbb{R}} E = 2$ *and* $\dim \mathrm{Ch}_E(A) \geq 1$.
    (b.2) *If* $\mathbb{F} = \mathbb{C}$, *then*

$$\dim \mathrm{Ch}_E(A) \geq \begin{cases} 1 & \text{if } \dim_{\mathbb{C}} E = 1, \\ 2 & \text{if } \dim_{\mathbb{C}} E = 2. \end{cases}$$

PROOF. (a) First we assume that $\mathbb{F} = \mathbb{R}$. Take a subspace L of E^* such that $3 \leq \dim_{\mathbb{R}} L < \infty$ and $\phi_{y,L} : S(L) \to \mathrm{Ch}_E(A)$ is not a constant map. Let $n = \dim_{\mathbb{R}} L$ and observe, from Lemma 4.3, that either:

(i) $\dim_{\mathbb{R}} H_{x,L} < n - 1$ for each $x \in \phi_{y,L}(S(L))$; or
(ii) there exists a unique $x_0 \in \mathrm{Ch}_E(A)$ such that $\dim_{\mathbb{F}} H_{x_0,L} = n - 1$ and $\dim_{\mathbb{R}} H_{x,L} = 1$ for each $x \in \phi_{y,L}(S(L))$, other than x_0.

In Case (i), the dimension of $\phi_y^{-1}(x)$ satisfies

$$\dim \phi_y^{-1}(x) = \dim_{\mathbb{R}}(H_{x,L} \cap S(L)) \leq n - 2 - 1 = \dim S(L) - 2.$$

Since L is finite dimensional, the sphere $S(L)$ is compact and the map $\phi_y : S(L) \to \mathrm{Ch}_E(A)$ is closed. Applying Theorem 4.4 to the map ϕ_y, we obtain

$$\dim S(L) \leq \dim \mathrm{Ch}_E(A) + \sup_{x \in \mathrm{Ch}_E(A)} \dim \phi_y^{-1}(x)$$
$$\leq \dim \mathrm{Ch}_E(A) + \dim S(L) - 2,$$

from which we conclude that $\dim \mathrm{Ch}_E(A) \geq 2$.

In Case (ii), we see that the restriction $\phi_{y,L}|S(L) \setminus (\phi_{y,L})^{-1}(x_0)$ of $\phi_{y,L}$ to the subset $S(L) \setminus (\phi_{y,L})^{-1}(x_0)$ satisfies the condition

$$\phi_y(\eta_1) = \phi_y(\eta_2) \implies \eta_2 = \pm\eta_1. \tag{4.2}$$

Let $PS(L)$ be the projective space $PS(L) = S(L)/(\eta \sim -\eta)$ with the standard quotient map

$$[\cdot] : S(L) \to PS(L) = S(L)/(\eta \sim -\eta).$$

The above (4.2) yields that $\phi_{y,L} : S(L) \to \mathrm{Ch}_E(A)$ induces a map $[\phi_{y,L}] : PS(L) \to \mathrm{Ch}_E(A)$ such that the restriction $[\phi_{y,L}]|PS(L) \setminus [\phi_y^{-1}(x)] : PS(L) \setminus [\phi_y^{-1}(x)] \to \mathrm{Ch}_E(A)$ is a topological embedding. Hence

$$\dim \mathrm{Ch}_E(A) \geq \dim_{\mathbb{R}} S(L) = n - 1 \geq 2.$$

This proves the desired conclusion for the case $\mathbb{F} = \mathbb{R}$.

When $\mathbb{F} = \mathbb{C}$, the inequality $\dim_{\mathbb{C}} H_{x,L} \leq n - 1$ of Lemma 4.3 implies that $\dim_{\mathbb{R}} H_{x,L} \leq 2n - 2$ and hence $\dim \phi_{y,L}^{-1}(x) \leq 2n - 3 = \dim S(L) - 2$ for each $x \in \phi_{y,L}(S(L))$. Therefore, the same proof as that of Case(i) above can be used to prove that $\dim \mathrm{Ch}_E(A) \geq 2$.

(b.1) If $\dim_{\mathbb{R}} E = 1$, then $S(E^*)$ consists of two antipodal points and hence $\phi_y(S(E^*))$ consists of a singleton, by Lemma 4.1, which contradicts the assumption that ϕ_y is not a constant map. Assume that $\dim_{\mathbb{R}} E = 2$. We show that

$$\text{the induced map } [\phi_y] : PS(E^*) \to \text{Ch}_E(A) \text{ is an embedding.}$$

Suppose that there exist $\eta_1, \eta_2 \in S(E^*)$ with $\eta_2 \neq \pm \eta_1$ such that $\phi_y(\eta_1) = \phi_y(\eta_2) := x$. By Proposition 4.2 and the two-dimensionality of E^*, we see that $\phi_y^{-1}(x) = H_x \cap S(E^*) = E^* \cap S(E^*) = S(E^*)$. Hence ϕ_y is a constant map, which is a contradiction. Then we see that $\text{Ch}_E(A)$ contains a homeomorphic copy of $PS(E^*)$ and thus has dimension at least one.

(b.2) The *real* dimension of E, $\dim_{\mathbb{R}}(E^*)$, is equal to two or four. When $\dim_{\mathbb{R}} E^* = 2$, we may repeat the proof of (b.1) to show that $[\phi_y]$ is an embedding of the projective space $PS(L)$ into $\text{Ch}_E(A)$ and $\dim \text{Ch}_E(A) \geq 1$. When $\dim S(E^*) = 3$, the proof of (a) can be repeated to conclude that $\dim \text{Ch}_E(A) \geq 2$. □

In summary, we obtain the following theorem.

THEOREM 4.6. *Let X and Y be compact metrizable spaces and let E be a separable, reflexive Banach space with E^* being strictly convex. Let A and B be subspaces of $C(X, E)$ and $C(Y, E)$, respectively, satisfying Condition (S1), Condition (S2) and Condition (ext). If one of the conditions:*

(a) $\dim_{\mathbb{R}} E = 1$*; or*
(b) $\dim_{\mathbb{R}} E \geq 2$ *and* $\dim X = 0$*; or*
(c) $\dim_{\mathbb{R}} E \geq 3$ *and* $\dim X \leq 1$

holds, then every surjective linear isometry $T : A \to B$ between A and B is a generalized weighted composition operator

$$T f(y) = V_y(f \varphi_*(y)), \quad f \in A, \ y \in \text{Ch}_E(B),$$

where $\varphi_* : \text{Ch}_E(B) \to \text{Ch}_E(A)$ is a homeomorphism and $(V_y)_{y \in \text{Ch}_E(B)}$ is a continuous collection of isometric linear isomorphisms $E \to E$ such that $\text{Ch}_E(B) \ni y \mapsto V_y \in \mathcal{L}(E, E)$ is continuous with respect to the strong operator topology on $\mathcal{L}(E, E)$.

Suppose, further, that E is a Hilbert space, and A and B are unitarily invariant subspaces satisfying Condition (S1) and Condition (S2). Then, under each one of the assumptions (a)–(c), the same conclusion holds.

PROOF. Let $T : A \to B$ be a surjective linear isometry. Theorem 4.5 implies that, for each $y \in \text{Ch}_E(B)$, the map $\phi_y : S(E^*) \to S(E^*)$ must be a constant map. Let $\{\varphi_*(y)\} = \phi_y(S(E^*))$. It readily follows from this that $\varphi_\eta = \varphi_*$ and Theorem 3.1 can be applied to obtain the conclusion.

The statement on a Hilbert space and unitarily invariant subspaces follows from Lemma 2.5, in the same way as in Theorem 3.5. □

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References

[1] H. Al-Halees and R. J. Fleming, 'Extreme point methods and Banach–Stone theorem', *J. Aust. Math. Soc.* **75** (2003), 125–143.

[2] J. Araujo and J. J. Font, 'Linear isometries between subspaces of continuous functions', *Trans. Amer. Math. Soc.* **349** (1997), 413–428.

[3] E. Behrends, *M-Structures and the Banach–Stone Theorem*, Lecture Notes in Mathematics, 736 (Springer, Berlin, 1979).

[4] R. H. Bing, 'A homogeneous indecomposable plane continuum', *Duke Math. J.* **15** (1948), 729–742.

[5] F. Botelho, R. J. Fleming and J. E. Jamison, 'Extreme points and isometries on vector-valued Lipschitz spaces', *J. Math. Anal. Appl.* **381** (2011), 821–832.

[6] F. Botelho and J. E. Jamison, 'Surjective isometries on spaces of vector valued continuous and Lipschitz functions', *Positivity* **17** (2013), 395–497.

[7] F. Botelho, J. E. Jamison and V. Zheng, 'Isometries of spaces of vector valued Lipschitz functions', *Positivity* **17** (2013), 47–65.

[8] B. Brosowski and F. Deutsch, 'On some geometric properties of suns', *J. Approx. Theory* **10** (1974), 245–267.

[9] M. Cambern, 'Isometries of certain Banach algebras', *Studia Math.* **25** (1964–1965), 217–225.

[10] M. Cambern, 'A Holsztynski theorem for spaces of continuous vector-valued functions', *Studia Math.* **63** (1978), 213–217.

[11] M. G. Caralambous and J. Krzempek, 'Rigid continua and transfinite inductive dimension', *Topol. Appl.* **157** (2010), 1690–1702.

[12] J. B. Conway, *A Course in Functional Analysis*, 2nd edn, Graduate Texts in Mathematics, 96 (Springer, New York, 1990).

[13] H. Cook, 'Continua which admit only the identity mapping onto non-degenerate subcontinua', *Fund. Math.* **60** (1967), 241–249.

[14] N. Dunford and J. Schwarz, *Linear Operators: General Theory*, Vol. 1 (Interscience, New York, 1958).

[15] R. Engelking, *Theory of Dimensions, Finite and Infinite*, Sigma Series in Pure Mathematics, 10 (Helderman Verlag, Lemgo, 1995).

[16] R. J. Fleming and J. E. Jamison, *Isometries on Banach Spaces: Function Spaces*, Monographs and Surveys in Pure and Applied Mathematics, 129, Vol. 1 (Chapman & Hall/CRC, Boca Raton, 2003).

[17] R. J. Fleming and J. E. Jamison, *Isometries on Banach Spaces: Vector-Valued Function Spaces*, Monographs and Surveys in Pure and Applied Mathematics 138, Vol. 2 (Chapman & Hall/CRC, Boca Raton, 2008).

[18] J. J. Font, 'Linear isometries between certain subspaces of continuous vector-valued functions', *Illinois J. Math.* **42** (1998), 389–397.

[19] O. Hatori and T. Miura, 'Real linear isometries between function algebras. II', *Cent. Eur. J. Math.* **11** (2013), 1838–1842.

[20] W. Holsztynski, 'Continuous mappings induced by isometries of spaces of continuous functions', *Studia Math.* **26** (1966), 133–136.

[21] A. Jamshidi and F. Sady, 'Real-linear isometries between certain subspaces of continuous functions', *Cent. Eur. J. Math.* **11** (2013), 2034–2043.

[22] K. Jarosz and V. D. Pathak, 'Isometries between function spaces', *Trans. Amer. Math. Soc.* **305** (1988), 193–206.

[23] K. Jarosz and V. D. Pathak, 'Isometries and small bound isomorphisms of function spaces', in: *Function Spaces, Edwardsville Il 1990*, Lecture Notes in Pure and Applied Mathematics, 136 (Marcel Dekker, New York, 1992), 241–271.

[24] J.-S. Jeang and N.-C. Wong, 'On the Banach–Stone problem', *Studia Math.* **155** (2003), 95–105.

[25] M. Jerison, 'The space of bounded maps into Banach space', *Ann. of Math.* (2) **52** (1950), 309–327.

[26] A. Jiménez-Vargas and M. Villegas-Vallecillos, 'Linear isometries between spaces of vector-valued Lipschitz functions,', *Proc. Amer. Math. Soc.* **137** (2009), 1381–1388.

[27] H. Koshimizu, T. Miura, H. Takagi and S.-E. Takahasi, 'Real-linear isometries between subspaces of continuous functions', *J. Math. Anal. Appl.* **413** (2014), 229–241.

[28] K.-S. Lau, 'A representation theorem for isometries of $C(X, E)$', *Pacific J. Math.* **60** (1975), 229–233.

[29] A. Lima and G. Olsen, 'Extreme points in duals of complex operator spaces', *Proc. Amer. Math. Soc.* **94** (1985), 437–440.

[30] R. Liu, 'On extension of isometries between unit spheres of $\mathcal{L}^\infty(\Gamma)$ type spaces', *J. Math. Anal. Appl.* **333** (2007), 959–970.

[31] R. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics, 183 (Springer, New York, 1998).

[32] T. Miura, 'Real-linear isometries between function algebras', *Cent. Eur. J. Math.* **9** (2011), 778–788.

[33] T. Miura, 'Surjective isometries between function spaces', *Contemp. Math.* **645** (2015), 231–239.

[34] W. Novinger, 'Linear isometries of subspaces of spaces of continuous functions', *Studia Math.* **53** (1975), 273–276.

[35] W. M. Ruess and C. P. Stegall, 'Extreme points in duals of operator spaces', *Math. Ann.* **261** (1982), 535–546.

[36] K. Sundaresan, 'Spaces of continuous functions into Banach spaces', *Studia Math.* **48** (1973), 15–22.

[37] J. Väisälä, 'A proof of the Mazur–Ulam theorem', *Amer. Math. Monthly* **110** (2003), 633–635.

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