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Theoretical study of a φ-Hilfer fractional differential system in Banach spaces

Oualid Zentar, Mohamed Ziane, and Mohammed Al Horani

Abstract. In this work, we study the existence of solutions of nonlinear fractional coupled system of φ-Hilfer type in the frame of Banach spaces. We improve a property of a measure of noncompactness in a suitably selected Banach space. Darbo's fixed point theorem is applied to obtain a new existence result. Finally, the validity of our result is illustrated through an example.

1 Introduction

In recent decades, fractional differential equations are receiving great attention as a significant tool in pure and applied mathematics, finding applications in various fields such as propagation in complex mediums, epidemiology, biological tissues, computer vision (a survey), and the theory of viscoelasticity (see, for example, [9, 29]). Some basic results can be found in [1, 20, 37].

The concept of the fractional derivative (FD) with regard to another function in the sense of Riemann–Liouville was presented by Kilbas et al. in [20]. The authors in [30] proposed a φ-Hilfer FD and extended the work dealing with the Hilfer's FD in [17]. The φ-Hilfer's FD significance stems from the fact that it has as its special instances a number of widely used FD operators. As a matter of fact, the weakly singular kernel function in the fractional operator definition can be freely selected. In other words, it covers a wide range of cases for a specific function φ. For some recent developments, see [7–8, 27, 31, 33–35]. This kind of FD has been widely used in practical applications, such as, several anomalous diffusions, including ultra-slow processes [21], financial crisis [27], and random walks [16]. On the other side, the modeling of various natural phenomena in chemistry, biology, computer networks, and physics often involves different types of coupled fractional differential systems, as evidenced by references [28, 36]. Therefore, investigating of coupled systems within the context of the φ-Hilfer FD framework became recently crucial, for more background, see [2, 4, 24].

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This study investigates the following system:

$$\begin{cases} {}^{H}\mathcal{D}_{a^+}^{\alpha_1,\beta_1;\varphi} y_1(t) = g_1(t, y_1(t), y_2(t)), & t \in I' := (a, b], \\ {}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi} y_2(t) = g_2(t, y_1(t), y_2(t)), & t \in I' := (a, b], \\ \lim_{t\to a^+} \varphi(t, a)^{1-\gamma_1} y_1(t) = \xi_1, \\ \lim_{t\to a^+} \varphi(t, a)^{1-\gamma_2} y_2(t) = \xi_2, \end{cases} \tag{1.1}$$

where ${}^{H}\mathcal{D}_{a^+}^{\alpha_i,\beta_i;\varphi}$ (for $i = 1, 2$) denotes the φ-Hilfer FD of order $0 < \alpha_i < 1$ and type $0 \le \beta_i \le 1$, $0 < \gamma_i = \alpha_i + \beta_i(1 - \alpha_i) < 1$, $1 - \max_{1\le i\le 2}\{\gamma_i\} < \alpha_i - \mu_i$, $0 < \mu_i < \alpha_i$, $(\mathbb{E}, \|\cdot\|)$ is a Banach space and $g_i : [a, b] \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ $(i = 1, 2)$ satisfies some certain conditions, specified later, $\xi_1, \xi_2 \in \mathbb{E}$, and $\varphi(t, a) = \varphi(t) - \varphi(a)$, where φ be increasing and differentiable with $\varphi'(t) \neq 0$, for all $t \in [a, b]$.

In [6, 19, 37], the authors investigated some classes of coupled systems in the frame of φ-Hilfer FD. They obtained some quantitative and qualitative results by means of some classical fixed point theorems where the Lipschitz condition on the considered system is required. The proof of our existence theorem combines results from measure of noncompactness (MNC) and Darbo's fixed point theorem under fairly reasonable assumptions on the forcing terms taking values on infinite-dimensional Banach space. Some interesting features of this work are as follows:

- The MNC expression is rigorously characterized in the functional space on which we work, allowing us to provide a unified approach for treating various differential systems regardless of the kind of the singularity generated by the initial condition.
- Under rather general assumptions, namely, when the nonlinearities fulfill an L^p-Carathéodory type condition, a new existence criterion is proved.
- The results obtained in this work extend, refine, and generalize various related results appearing in the literature (see [13, 14, 32]).
- An illustrative example is discussed to show the applicability of our abstract results in treating differential systems in infinite-dimensional spaces driven by fractional derivatives [25].

This work is divided into three sections. Section 2 recalls some theoretical concepts which are used throughout this work. In particular, a reconstruction of the MNC in a suitably selected Banach space is established. A new existence result is stated and demonstrated in Section 3. The last section provides an example illustrating the validity of our results.

2 Preliminary results

Let $I := [a, b]$. Throughout this work, $C(I, \mathbb{E})$ denotes all $\mathbb{E}$-valued continuous functions on I with the sup norm

$$\|z\|_\infty = \sup_{t\in I} \|z(t)\|.$$

We endow the space $L^p_\varphi(I,\mathbb{E})$, $1 \le p < \infty$, of Bochner integrable functions z on I for which $\|z\|_{L^p_\varphi} < \infty$, with the norm $\|z\|_{L^p_\varphi} = \left(\int_a^b \varphi'(s)\|z(s)\|^p ds\right)^{\frac{1}{p}}$. If $\varphi(t) = t$, the space $L^p_\varphi(I,\mathbb{E})$ coincides with the usual $L^p(I,\mathbb{E})$ space.

If $p = \infty$, $L^\infty(I,\mathbb{E})$ is the Banach space of all equivalence classes of essentially bounded measurable functions on I equipped with the norm

$$\|z\|_{L^\infty} = \operatorname*{ess\,sup}_{t\in I}\|z(t)\| = \inf\{M > 0;\ \|z(t)\| \le M \text{ for almost every } t \in I\}.$$

We also define

$$\mathbb{S}^{1,+}_{a^+}(I,\mathbb{R}) = \{\varphi\ :\ \varphi \in C^1(I,\mathbb{R}) \text{ and } \varphi'(t) > 0 \text{ for all } t \in I\}.$$

For $\varphi \in \mathbb{S}^{1,+}_{a^+}(I,\mathbb{R})$ and $t, s \in I$, $(t > s)$, we pose

$$(\varphi(t) - \varphi(s))^\alpha = \varphi(t,s)^\alpha, \quad \text{for } \alpha \in \mathbb{R}.$$

Definition 2.1 [15] For $\alpha, \beta \in (0, +\infty)$, the gamma and beta functions are given by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \qquad B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt.$$

Lemma 2.2 [26] *Let $\psi \in \mathbb{S}^{1,+}_{a^+}(I,\mathbb{R})$, $0 < \alpha < 1$, $0 \leqslant \beta < \alpha$, $a \le \tau \le \zeta \le t \le b$, and let*

$$\Theta_{\alpha,\beta}(\tau,\zeta,t,\varphi) = \int_\tau^\zeta \varphi(t,s)^{\alpha-1}\varphi(s,a)^{-\beta}\varphi'(s)ds.$$

Then, for all $t \in I$, we have

$$\Theta_{\alpha,\beta}(a,t,t,\varphi) = \varphi(t,a)^{\alpha-\beta}B(\alpha, 1-\beta)$$

and

$$0 \le \Theta_{\alpha,\beta}(\tau,\zeta,t,\varphi) \le \left(\frac{2^\beta}{\alpha} + \frac{2^{1-\alpha}}{1-\beta}\right)\max\{1, \varphi(b,a)^{\alpha-\beta}\}\varphi(\zeta,\tau)^{\min\{\alpha,1-\beta,\alpha-\beta\}}.$$

Remark 2.3 [26] From Lemma 2.2, one has:

(i) $\Theta_{\alpha,\beta}(a,t,t,\varphi) \le \Theta_{\alpha,\beta}(a,b,b,\varphi)$ for $t \in I$,
(ii) $\Theta_{\alpha,\beta}(\tau,\zeta,t,\varphi) \to 0$ as $|\zeta - \tau| \to 0$.

Remark 2.4 By Lemma 2.2, we get

$$\psi(t,\cdot)^{\alpha_i-1}\psi(\cdot,a)^{\gamma_j-1}\psi'(\cdot) \in L^1(I,\mathbb{R}), \quad i,j = 1,2.$$

So it is possible to choose $\aleph$ such that

(2.1)
$$\mathcal{L}_{i,j}(\aleph) := \sup_{t\in I}\frac{2\|\eta_{i,j}\|_{L^\infty}\varphi(b,a)^{1-\gamma_i}}{\Gamma(\alpha_i)}\int_a^t \varphi'(s)\varphi(t,s)^{\alpha_i-1}\varphi(s,a)^{\gamma_j-1}e^{-\aleph(t-s)}ds$$
$$< 1/4.$$

Definition 2.5 The left-sided φ-fractional integral of a function f of order $\alpha > 0$ is defined as

$$\left(\mathcal{J}_{a^+}^{\alpha,\varphi} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \varphi(t,s)^{\alpha-1}\varphi'(s)f(s)ds, \quad t > a,$$

with $\varphi \in \mathbb{S}_{a^+}^{1,+}(I,\mathbb{R})$.

Definition 2.6 Let $n-1 < \alpha \le n$ with $n \in \mathbb{N}$, $\varphi \in \mathbb{S}_{a^+}^{1,+}(I,\mathbb{R})$. The left-sided φ-Hilfer FD of a function f of order α and type $0 \le \beta \le 1$ is defined as

$$\left({}^H\mathcal{D}_{a^+}^{\alpha,\beta;\varphi} f\right)(t) = \left(\mathcal{J}_{a^+}^{\beta(n-\alpha),\varphi}\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n \left(\mathcal{J}_{a^+}^{(1-\beta)(n-\alpha),\varphi} f\right)\right)(t).$$

To define a solution of the system (1.1), for each $i = 1,2$, we consider the Banach space

$$\mathcal{C}_{1-\gamma_i}^{\varphi}(I,\mathbb{E}) = \left\{z \in C(I',\mathbb{E}) : \varphi(\cdot,a)^{1-\gamma_i}z(\cdot) \in C(I,\mathbb{E})\right\},$$

normed by

$$\|z\|_{\mathcal{C}_{1-\gamma_i}^{\varphi}} = \sup_{t\in I} \varphi(t,a)^{1-\gamma_i}\|z(t)\|. \tag{2.2}$$

Also, by $\mathcal{C}_{1-\gamma_1}^{\varphi}(I,\mathbb{E}) \times \mathcal{C}_{1-\gamma_2}^{\varphi}(I,\mathbb{E})$ we denote the product weighted space with the norm

$$\|(z_1,z_2)\|_{\mathcal{C}_{1-\gamma_1}^{\varphi}\times\mathcal{C}_{1-\gamma_2}^{\varphi}} = \|z_1\|_{\mathcal{C}_{1-\gamma_1}^{\varphi}} + \|z_2\|_{\mathcal{C}_{1-\gamma_2}^{\varphi}}.$$

Henceforth, for a subset $\mathbb{U}$ of the space $\mathcal{C}_{1-\gamma_i}^{\varphi}(I,\mathbb{E})$, define $\mathbb{U}_{\gamma_i}$ by

$$\mathbb{U}_{\gamma_i} = \{z_{\gamma_i} : z \in \mathbb{U}\},$$

where

$$z_{\gamma_i}(t) = \begin{cases} \varphi(t,a)^{1-\gamma_i}z(t), & t \in I', \\ \lim_{t\to a^+} \varphi(t,a)^{1-\gamma_i}z(t), & t = a. \end{cases} \tag{2.3}$$

It is clear that $z_{\gamma_i} \in C(I,\mathbb{E})$.

Definition 2.7 [10] The Hausdorff MNC is the map $\Lambda : \mathcal{P}(\mathbb{E}) \to [0,\infty)$ defined by

$$\Lambda(\mathbb{U}) = \inf\left\{\varepsilon > 0 : \mathbb{U} \text{ has a finite } \varepsilon - \text{net in } \mathbb{E}\right\},$$

where $\mathcal{P}(\mathbb{E})$ denotes the family of all bounded subsets of $\mathbb{E}$.

Lemma 2.8 [10] *Let $\mathbb{E}$ be a real Banach space and $\mathbb{U}_0, \mathbb{U}_1, \mathbb{U}_2 \in \mathcal{P}(\mathbb{E})$. Then the following properties are satisfied:*

(1) $\Lambda(\mathbb{U}_0) \le \Lambda(\mathbb{U}_1)$ *if* $\mathbb{U}_0 \subset \mathbb{U}_1$,
(2) $\Lambda(\{a\} \cup \mathbb{U}) = \Lambda(\mathbb{U})$ *for every* $a \in \mathbb{E}$,
(4) $\Lambda(\mathbb{U}) = \Lambda(\overline{\text{conv}}\,\mathbb{U})$, *where* $\overline{\text{conv}}\,\mathbb{U}$ *is the closed convex hull of* $\mathbb{U}$,
(4) $\Lambda(\mu\mathbb{U}) = |\mu|\Lambda(\mathbb{U})$, *where* $\mu \in \mathbb{R}$,

(5) $\Lambda(\mathbb{U}) = 0$ *if and only if* $\mathbb{U}$ *is relatively compact,*
(6) $\Lambda(\mathbb{U}_2 \cup \mathbb{U}_1) = \max(\Lambda(\mathbb{U}_2), \Lambda(\mathbb{U}_1))$,
(7) $\Lambda(\mathbb{U}_2 + \mathbb{U}_1) \leq \Lambda(\mathbb{U}_2) + \Lambda(\mathbb{U}_1)$.

Lemma 2.9 [11] *Let $\mathbb{B} \subseteq C(I, \mathbb{E})$ be a bounded set. Then, for all $t \in I$,*

$$\Lambda(\mathbb{B}(t)) \leq \Lambda_c(\mathbb{B}),$$

where $\mathbb{B}(t) = \{u(t) : u \in \mathbb{B}\}$. Furthermore, if $\mathbb{B}$ is equicontinuous on I, then $\Lambda(\mathbb{B}(\cdot))$ is continuous on I and

$$\Lambda_c(\mathbb{B}) = \sup_{t \in I} \Lambda(\mathbb{B}(t)), \tag{2.4}$$

where Λ_c is the Hausdorff MNC in $C(I, \mathbb{E})$.

Next, we extend the result of Lemma 2.9 to the space $\mathcal{C}^{\varphi}_{1-\gamma}(I, \mathbb{E})$. Let us confirm that, in general, the expression (2.4) may not be well-defined, since bounded sets in $\mathcal{C}^{\varphi}_{1-\gamma}(I, \mathbb{E})$ are not necessarily bounded in $C(I, \mathbb{E})$. Consider, for instance, the set

$$\widetilde{\mathcal{Q}}(t) = \left\{ \varphi(t, a)^{\gamma - 1} z(t),\ z \in \mathcal{Q} \right\},$$

where $\mathcal{Q}$ is bounded in $C(I, \mathbb{E})$. Obviously, $\widetilde{\mathcal{Q}}$ is unbounded in $C(I, \mathbb{E})$, this indicates that the map $t \mapsto \Lambda(\widetilde{\mathcal{Q}}(t))$ is not well-defined, therefore it is wrong to consider the expression (2.4). However, clearly the set $\widetilde{\mathcal{Q}}$ is bounded with respect to the norm (2.2) (i.e., $\widetilde{\mathcal{Q}} \subset \mathcal{C}^{\varphi}_{1-\gamma}(I, \mathbb{E})$).

Lemma 2.10 *Let $\mathbb{B} \subseteq \mathcal{C}^{\varphi}_{1-\gamma}(I, \mathbb{E})$ be a bounded set. Then, for all $t \in I$, we have*

$$\Lambda(\mathbb{B}_\gamma(t)) \leq \Lambda_{\mathcal{C}^{\varphi}_{1-\gamma}}(\mathbb{B}).$$

Additionally, assume that $\mathbb{B}$ is equicontinuous on I, then $\Lambda(\mathbb{B}_\gamma(\cdot))$ is continuous on I and

$$\Lambda_{\mathcal{C}^{\varphi}_{1-\gamma}}(\mathbb{B}) = \sup_{t \in I} \Lambda(\mathbb{B}_\gamma(t)).$$

Proof For every $\varepsilon > 0$, there exists $\mathbb{B}_i \subseteq \mathcal{C}^{\varphi}_{1-\gamma}(I, \mathbb{E})$, $(i = 1, \ldots, n)$ such that $\mathbb{B} = \bigcup_{i=1}^{n} \mathbb{B}_i$ and

$$\delta(\mathbb{B}_i) \leq 2\Lambda(\mathbb{B}(t)) + 2\varepsilon, \quad i = 1, \ldots, n, \tag{2.5}$$

where $\delta(\cdot)$ denotes the diameter of a bounded set in $\mathcal{C}^{\varphi}_{1-\gamma}(I, \mathbb{E})$. So, we have

$$\mathbb{B}(t) = \bigcup_{i=1}^{n} \mathbb{B}_i(t) \ \text{ for each } \ t \in I$$

and

$$\|u_\gamma(t) - v_\gamma(t)\| \leq \|u - v\|_\gamma \leq \delta(\mathbb{B}_i), \quad \text{for } u, v \in \mathbb{B}_i,\ i = 1, \ldots, n. \tag{2.6}$$

From (2.5) and (2.6), it follows that

$$2\Lambda(\mathbb{B}_\gamma(t)) \le \delta(\mathbb{B}_i(t)) \le \delta(\mathbb{B}_i) \le 2\Lambda(\mathbb{B}(t)) + 2\varepsilon.$$

Since ε is arbitrary, one has

$$\Lambda(\mathbb{B}_\gamma(t)) \le \Lambda(\mathbb{B}(t)), \quad \text{for every } t \in I.$$

Consequently, we have

$$\sup_{t\in I} \Lambda(\mathbb{B}_\gamma(t)) \le \Lambda_{\mathcal{C}^{\varphi}_{1-\gamma}}(\mathbb{B}).$$

Now, let us prove the converse inequality. Assume that $\mathbb{B}$ is a bounded subset in $\mathcal{C}^{\varphi}_{1-\gamma}(I,\mathbb{E})$ and equicontinuous on I. Obviously, $\mathbb{B}_\gamma$ is a bounded subset in $C(I,\mathbb{E})$ and equicontinuous on I. From Lemma 2.9, we obtain that

$$\Lambda(\mathbb{B}_\gamma) \le \sup_{t\in I} \Lambda(\mathbb{B}_\gamma(t)).$$

Consider the isometric map $\Upsilon : \mathcal{C}^{\varphi}_{1-\gamma}(I,\mathbb{E}) \to C(I,\mathbb{E})$ defined by $z \mapsto z_\gamma$. Then, we get

$$\Lambda_{\mathcal{C}^{\varphi}_{1-\gamma}}(\mathbb{B}) = \sup_{t\in I} \Lambda(\mathbb{B}_\gamma(t)),$$

and the result is reached. ∎

Lemma 2.11 [18] *Let $\{x_n\}_{n=1}^{+\infty}$ belongs to $L^1(I,\mathbb{E})$ such that $\|x_n(t)\| \le \varsigma(t)$ almost everywhere on $I(n=1,2,\dots)$ for some $\varsigma \in L^1(I,\mathbb{R}_+)$. Then, the map $t \mapsto \Lambda(\{x_n(t)\}_{n=1}^{+\infty})$ is integrable on $\mathbb{R}_+$ and*

$$\Lambda\left(\left\{\int_0^t x_n(s)ds\right\}_{n=1}^{+\infty}\right) \le 2\int_0^t \Lambda(\{x_n(s)\}_{n=1}^{+\infty})ds. \tag{2.7}$$

Lemma 2.12 [3] *Let $\mathbb{B} \in \mathcal{P}(\mathbb{E})$. Then for each $\varepsilon > 0$, there exists a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq \mathbb{B}$, satisfies*

$$\Lambda_c(\mathbb{B}) \le 2\Lambda_c(\{x_n\}_{n=1}^{+\infty}) + \varepsilon. \tag{2.8}$$

Theorem 2.13 (Darbo[12]) *Let $\mathbb{E}$ be a Banach space, let $\mathbb{V} \subset \mathbb{E}$ be a nonempty, bounded, closed, convex, and let $N : \mathbb{V} \to \mathbb{V}$ be a continuous mapping. Assume that there exists $k \in [0,1)$ such that*

$$\Lambda(N(\mathbb{V})) \le k\Lambda(\mathbb{V}). \tag{2.9}$$

Then N admits a fixed point in $\mathbb{V}$.

Theorem 2.14 [10] *Suppose $\varpi_1, \varpi_2, \dots, \varpi_n$ are MNCs in the Banach spaces $\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_n$, respectively. Let $G : [0,\infty)^n \to [0,\infty)$ be a convex function such that $G(x_1, x_2, \dots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1,2,\dots,n$. Then*

$$\varpi(X) = G(\varpi_1(X_1), \varpi_2(X_2), \dots, \varpi_n(X_n)) \tag{2.10}$$

defines an MNC in $\mathbb{E}_1 \times \mathbb{E}_2 \times \cdots \times \mathbb{E}_n$, where X_i denotes the natural projection of X into $\mathbb{E}_i$ for $i = 1,2,\dots,n$.

Example 2.15 Let ϖ_1, ϖ_2 be MNCs in $\mathbb{E}_1, \mathbb{E}_2$, respectively and $G(x_1, x_2) = x_1 + x_2$ for $(x_1, x_2) \in [0, \infty)^2$. Then, G satisfies all properties of Theorem 2.14. Hence, $\varpi(X) = \varpi_1(X_1) + \varpi_2(X_2)$ is an MNC in the space $\mathbb{E}_1 \times \mathbb{E}_2$.

3 Main results

We initiate this section by introducing the following hypotheses which are needed in the sequel:

(H1) The functions $t \mapsto g_i(t, u, v); i = 1, 2$ are measurable on I for each $(u, v) \in C(I, \mathbb{E}) \times C(I, \mathbb{E})$, and the functions $(u, v) \mapsto g_i(t, u, v)$ are continuous for a.e. $t \in I$.

(H2) There exist functions $h_i \in L_\varphi^{\frac{1}{\mu_i}}(I, \mathbb{R}_+)$, $0 \le \mu_i < \alpha_i$ such that

$$\|g_i(t, v_1, v_2)\| \le h_i(t)\,(1 + \|v_1\| + \|v_2\|)\,, \quad i = 1, 2,$$

for $v_1, v_2 \in \mathbb{E}$, and a.e. $t \in I$.

(H3) There exist functions $\eta_i, \widehat{\eta}_i \in L^\infty(I, \mathbb{R}_+)$, $i = 1, 2$, such that for any bounded subsets $\mathbb{A}^1 \times \mathbb{A}^2 \subset \mathcal{C}_{1-\gamma_1}^{\varphi}(I, \mathbb{E}) \times \mathcal{C}_{1-\gamma_2}^{\varphi}(I, \mathbb{E})$, we have

$$\Lambda(g_i(t, \mathbb{A}^1, \mathbb{A}^2)) \le \sum_{j=1}^{2} \eta_{i,j}(t)\Lambda(\mathbb{A}^j), \quad \text{for all } t \in I. \tag{3.1}$$

(H4) The following inequality holds:

$$2\widehat{K}_i \le K, \quad i = 1, 2, \tag{3.2}$$

where

$$K_i := \|\xi_i\| + \frac{\varphi(b, a)^{1+\alpha_i-\gamma_i-\mu_i}\|h_i\|_{L_\varphi^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)\theta_i^{1-\mu_i}} + K\sum_{j=1}^{2}\frac{\varphi(b, a)^{1-\gamma_i}\|h_i\|_{L_\varphi^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)}\left(\Theta_{\theta_i, \vartheta_{i,j}}(a, b, b, \varphi)\right)^{1-\mu_i},$$

$$\theta_i = \frac{\alpha_i - \mu_i}{1 - \mu_i}, \quad \text{and} \quad \vartheta_{i,j} = \frac{1 - \gamma_j}{1 - \mu_i}, \quad i, j = 1, 2.$$

Now, we prove our main result for the system (1.1), which is based on Theorem 2.13.

Theorem 3.1 *Assume that (H1)–(H4) hold. Then, system (1.1) has at least one solution.*

Proof First, let us introduce an operator $\mathcal{H} : \mathcal{C}_{1-\gamma_1}^{\varphi}(I, \mathbb{E}) \times \mathcal{C}_{1-\gamma_2}^{\varphi}(I, \mathbb{E}) \to \mathcal{C}_{1-\gamma_1}^{\varphi}(I, \mathbb{E}) \times \mathcal{C}_{1-\gamma_2}^{\varphi}(I, \mathbb{E})$ associated with the system (1.1) as

$$(\mathcal{H}(y_1, y_2))(t) = ((\mathcal{H}_1(y_1, y_2))(t), (\mathcal{H}_2(y_1, y_2))(t))\,, \tag{3.3}$$

where the operators $\mathcal{H}_i : \mathcal{C}^{\varphi}_{1-\gamma_1}(I,\mathbb{E}) \times \mathcal{C}^{\varphi}_{1-\gamma_2}(I,\mathbb{E}) \to \mathcal{C}^{\varphi}_{1-\gamma_i}(I,\mathbb{E})$, $i = 1,2$ are defined by

$$(\mathcal{H}_i(y_1,y_2))(t) = \xi_i\varphi(t,a)^{\gamma_i-1} + \frac{1}{\Gamma(\alpha_i)}\int_a^t \varphi'(s)\varphi(t,s)^{\alpha_i-1}g_i(s,y_1(s),y_2(s))ds,\ t \in I'.$$

According to [22, Lemma 3.1], the solutions of the system (1.1) are fixed points of the operator $\mathcal{H}$. Consider the following bounded closed convex set:

$$\Omega_K = \left\{(y_1,y_2) \in \mathcal{C}^{\varphi}_{1-\gamma_1}(I,\mathbb{E}) \times \mathcal{C}^{\varphi}_{1-\gamma_2}(I,\mathbb{E})\ :\ \|y_1\|_{\mathcal{C}^{\varphi}_{1-\gamma_1}} \le K,\ \ \|y_2\|_{\mathcal{C}^{\varphi}_{1-\gamma_2}} \le K\right\}.$$

Then we divide the proof into four steps.

Step 1. $\mathcal{H}$ transforms Ω_K into itself. Indeed, for each $(y_1,y_2) \in \Omega_K$ and every $t \in I'$, one has

$$\begin{aligned}
&\left\|\varphi(t,a)^{1-\gamma_i}\left(\mathcal{H}_i(y_1,y_2)\right)(t)\right\| \\
&\le \|\xi_i\| + \frac{\varphi(t,a)^{1-\gamma_i}}{\Gamma(\alpha_i)}\int_a^t \varphi'(s)\varphi(t,s)^{\alpha_i-1}\|g_i(s,y_1(s),y_2(s))\|\,ds \\
&\le \|\xi_i\| + \frac{\varphi(t,a)^{1-\gamma_i}}{\Gamma(\alpha_i)}\int_a^t \varphi'(s)\varphi(t,s)^{\alpha_i-1}h_i(s)\left(1+\|y_1(s)\|+\|y_2(s)\|\right)ds \\
&\le \|\xi_i\| + \frac{\varphi(t,a)^{1-\gamma_i}}{\Gamma(\alpha_i)}\int_a^t \varphi'(s)\varphi(t,s)^{\alpha_i-1}h_i(s)(1+\varphi(s,a)^{\gamma_1-1}\|y_1\|_{\mathcal{C}^{\varphi}_{1-\gamma_1}} \\
&\qquad + \varphi(s,a)^{\gamma_2-1}\|y_2\|_{\mathcal{C}^{\varphi}_{1-\gamma_2}})ds \\
&\le \|\xi_i\| + \frac{\varphi(t,a)^{1-\gamma_i}}{\Gamma(\alpha_i)}\int_a^t \varphi(t,s)^{\alpha_i-1}h_i(s)\left(1+K\sum_{j=1}^{2}\varphi(s,a)^{\gamma_j-1}\right)\varphi'(s)ds.
\end{aligned}$$

Since $\mu_i + (1-\mu_i) = 1$, we can write $\varphi'(s)$ as the product

$$\varphi'(s) = \varphi'(s)^{\mu_i}\varphi'(s)^{1-\mu_i}.$$

Then, applying the Hölder inequality, we get

$$\begin{aligned}
&\left\|\varphi(t,a)^{1-\gamma_i}\left(\mathcal{H}_i(y_1,y_2)\right)(t)\right\| \\
&\le \|\xi_i\| + \frac{\varphi(t,a)^{1-\gamma_i}\|h_i\|_{L_{\varphi}^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)}\left(\int_a^t \varphi(t,s)^{\frac{\alpha_i-1}{1-\mu_i}}d\varphi(s)\right)^{1-\mu_i} \\
&\qquad + K\sum_{j=1}^{2}\frac{\varphi(t,a)^{1-\gamma_i}\|h_i\|_{L_{\varphi}^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)}\left(\int_a^t \varphi(t,s)^{\frac{\alpha_i-1}{1-\mu_i}}\varphi(s,a)^{\frac{\gamma_j-1}{1-\mu_i}}d\varphi(s)\right)^{1-\mu_i}.
\end{aligned}$$

Hence, by Lemma 2.2 and Remark 2.3, we obtain

$$\begin{aligned}\left\|\varphi(t,a)^{1-\gamma_i}\left(\mathcal{H}_i(y_1,y_2)\right)(t)\right\| &\le \|\xi_i\| + \frac{\|h_i\|_{L_\varphi^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)}\left(\frac{1-\mu_i}{\alpha_i-\mu_i}\right)^{1-\mu_i}\varphi(t,a)^{1+\alpha_i-\gamma_i-\mu_i}\\ &\quad + K\sum_{j=1}^{2}\frac{\varphi(t,a)^{1-\gamma_i}\|h_i\|_{L_\varphi^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)}\left(\Theta_{\frac{\alpha_i-\mu_i}{1-\mu_i},\frac{1-\gamma_j}{1-\mu_i}}(a,t,t,\varphi)\right)^{1-\mu_i}\\ &\le \widehat{K}_i, \quad i=1,2.\end{aligned}$$

Thus,

$$\|\mathcal{H}_i(y_1,y_2)\|_{\mathcal{C}^{\varphi}_{1-\gamma_i}} \le \widehat{K}_i, \quad i=1,2.$$

Consequently,

$$\|\mathcal{H}(y_1,y_2)\|_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}} \le \widehat{K}_1+\widehat{K}_2.$$

This shows that $\mathcal{H}$ transforms Ω_K into itself.

Step 2. The continuity of $\mathcal{H}(\cdot,\cdot)$.

Let $\{y_{1,n},y_{2,n}\}\in\Omega_q$ such that $(y_{1,n},y_{2,n})\to(y_1,y_2)$ as $n\longrightarrow\infty$. Making use of the Carathéodory condition of g_i, $i=1,2$, we easily have

$$\|g_i(s,y_{1,n}(s),y_{2,n}(s)) - g_i(s,y_1(s),y_2(s))\| \to 0, \quad \text{as } n\longrightarrow\infty.$$

Next, by (H2), one gets

$$\begin{aligned}&\|g_i(s,y_{1,n}(s),y_{2,n}(s)) - g_i(s,y_1(s),y_2(s))\|\\ &\quad\le \|g_i(s,y_{1,n}(s),y_{2,n}(s))\| + \|g_i(s,y_1(s),y_2(s))\|\\ &\quad\le 2h_i(s)(1+\|y_1(t)\|+\|y_2(t)\|)\\ &\quad\le 2h_i(s)\left(1+K\sum_{j=1}^{2}\varphi(s,a)^{\gamma_j-1}\right).\end{aligned}$$

Since, the function $s\mapsto h_i(s)\left(1+K\sum_{j=1}^{2}\varphi(s,a)^{\gamma_j-1}\right)$ is Lebesgue integrable over $[a,t]$, so is the function $s\mapsto h_i(s)\varphi(t,s)^{\alpha_i-1}\left(1+K\sum_{j=1}^{2}\varphi(s,a)^{\gamma_j-1}\right)$. Then it follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned}&\left\|\varphi(t,a)^{1-\gamma_i}\left[\mathcal{H}_i(y_{1,n},y_{2,n})(t) - \mathcal{H}_i(y_1,y_2)(t)\right]\right\|\\ &\le \frac{\varphi(t,a)^{1-\gamma_i}}{\Gamma(\alpha_i)}\int_a^t \varphi(t,s)^{\alpha_i-1}\varphi'(s)\,\|g_i(s,y_{1,n}(s),y_{2,n}(s)) - g_i(s,y_1(s),y_2(s))\|\,ds \to 0\end{aligned}$$

as $n\longrightarrow\infty$ for all $t\in I$, which leads to

$$\|(\mathcal{H}_i(y_{1,n},y_{2,n})) - (\mathcal{H}_i(y_1,y_2))\|_{\mathcal{C}^{\varphi}_{1-\gamma_i}} \xrightarrow[n\to\infty]{} 0$$

for any $t \in I$. Therefore,

$$\|\mathcal{H}(y_{1,n}, y_{2,n}) - \mathcal{H}(y_1, y_2)\|_{\mathcal{C}^{\varphi}_{1-\gamma_1} \times \mathcal{C}^{\varphi}_{1-\gamma_2}} \xrightarrow[n\to\infty]{} 0.$$

Accordingly, the operator $\mathcal{H}(\cdot,\cdot)$ is continuous.

Step 3. $\mathcal{H}(\Omega_q)$ is equicontinuous.

Let $(y_1, y_2) \in \Omega_q$ and $a < t_1 < t_2 \leq b$. Then for $i = 1, 2$, we have

$$\begin{aligned}
&\left\|\varphi(t_2,a)^{1-\gamma_i}(\mathcal{H}_i(y_1,y_2))(t_2) - \varphi(t_1,a)^{1-\gamma_i}(\mathcal{H}_i(y_1,y_2))(t_1)\right\| \\
&\leq \Bigg\| \frac{\varphi(t_2,a)^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_a^{t_2} \varphi'(s)\varphi(t_2,s)^{\alpha_i-1} g_i(s, y_1(s), y_2(s))ds \\
&\qquad - \frac{\varphi(t_1,a)^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_a^{t_1} \varphi'(s)\varphi(t_1,s)^{\alpha_i-1} g_i(s, y_1(s), y_2(s))ds \Bigg\| \\
&\leq J_0 + J_1,
\end{aligned}$$

where

$$J_0 = \frac{\varphi(t_2,a)^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_{t_1}^{t_2} \varphi'(s)\varphi(t_2,s)^{\alpha_i-1} \|g_i(s, y_1(s), y_2(s))\| ds,$$

and

$$\begin{aligned}
J_1 = &\Bigg\| \frac{\varphi(t_2,a)^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_a^{t_1} \varphi'(s)\varphi(t_2,s)^{\alpha_i-1} g_i(s, y_1(s), y_2(s))ds \\
&\qquad - \frac{\varphi(t_1,a)^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_a^{t_1} \varphi'(s)\varphi(t_1,s)^{\alpha_i-1} g_i(s, y_1(s), y_2(s))ds \Bigg\|.
\end{aligned}$$

Next, applying the Hölder inequality, we conclude that

$$\begin{aligned}
J_0 &\leq \frac{\varphi(t_2,a)^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_{t_1}^{t_2} \varphi'(s)\varphi(t_2,s)^{\alpha_i-1} h_i(s)(1 + \|y_1(s)\| + \|y_2(s)\|)ds \\
&\leq \frac{\varphi(t_2,a)^{1-\gamma_i}\|h_i\|_{L_{\varphi}^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)} \left(\int_{t_1}^{t_2} \varphi(t_2,s)^{\frac{\alpha_i-1}{1-\mu_i}} d\varphi(s)\right)^{1-\mu_i} \\
&\quad + K\sum_{j=1}^{2} \frac{\varphi(t_2,a)^{1-\gamma_i}\|h_i\|_{L_{\varphi}^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)} \left(\int_{t_1}^{t_2} \varphi(t_2,s)^{\frac{\alpha_i-1}{1-\mu_i}} \varphi(s,a)^{\frac{\gamma_j-1}{1-\mu_i}} d\varphi(s)\right)^{1-\mu_i},
\end{aligned}$$

and hence,

$$J_0 \leq \frac{\varphi(t_2,a)^{1-\gamma_i}\|h_i\|_{L_{\varphi}^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)} \left[\frac{\varphi(t_2,t_1)^{\alpha_i-1}}{\theta_i^{1-\mu_i}} + K\sum_{j=1}^{2}\left(\Theta_{\theta_i,\vartheta_{i,j}}(t_1,t_2,t_2,\varphi)\right)^{1-\mu_i}\right].$$

Now, using Remark 2.3, we obtain

$$J_0 \longrightarrow 0 \quad \text{as} \quad t_1 \longrightarrow t_2,$$

and

$$J_1 \leq J_2 + J_3,$$

where

$$J_2 = \frac{|\varphi(t_2,a)^{1-\gamma_i} - \varphi(t_1,a)^{1-\gamma_i}|}{\Gamma(\alpha_i)} \int_a^{t_1} \varphi'(s)\varphi(t_2,s)^{\alpha_i-1}\|g_i(s,y_1(s),y_2(s))\|ds,$$

$$J_3 = \frac{\varphi(t_1,a)^{1-\gamma_i}}{\Gamma(\alpha_i)} \int_a^{t_1} \varphi'(s)|\varphi(t_2,s)^{\alpha_i-1} - \varphi(t_1,s)^{\alpha_i-1}|\|g_i(s,y_1(s),y_2(s))\|ds.$$

Then,

$$J_2 \xrightarrow[n\to\infty]{} 0,$$

on the other hand, $\varphi(t_2,s)^{\alpha_i-1} \leq \varphi(t_1,s)^{\alpha_i-1}$. Therefore

$$J_3 \leq \frac{\varphi(t_1,a)^{1-\gamma_i}}{\Gamma(\alpha_i)}\left(\int_a^{t_1} h_i(s)\varphi'(s)\varphi(t_1,s)^{\alpha_i-1}ds - \int_a^{t_1} h_i(s)\varphi'(s)\varphi(t_2,s)^{\alpha_i-1}ds\right)$$
$$+ \frac{\varphi(t_1,a)^{1-\gamma_i}}{\Gamma(\alpha_i)} K\sum_{j=1}^{2}\int_a^{t_1} h_i(s)\varphi'(s)\left(\varphi(t_1,s)^{\alpha_i-1} - \varphi(t_2,s)^{\alpha_i-1}\right)\varphi(s,a)^{\gamma_j-1}ds.$$

Then,

$$J_3 \leq \frac{\varphi(t_1,a)^{1-\gamma_i}\|h_i\|_{L_\varphi^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)}\left[(\Theta_{\theta_i,0}(a,t_1,t_1,\varphi))^{1-\mu_i} - (\Theta_{\theta_i,0}(a,t_1,t_2,\varphi))^{1-\mu_i}\right]$$
$$+ \frac{\varphi(t_1,a)^{1-\gamma_i}\|h_i\|_{L_\varphi^{\frac{1}{\mu_i}}}}{\Gamma(\alpha_i)} K\sum_{j=1}^{2}\Bigg[\left(\Theta_{\theta_i,\vartheta_{i,j}}(a,t_1,t_1,\varphi)\right)^{1-\mu_i}$$
$$- \left(\Theta_{\theta_i,\vartheta_{i,j}}(a,t_1,t_2,\varphi)\right)^{1-\mu_i}\Bigg].$$

So,

$$J_3 \longrightarrow 0 \quad \text{as} \quad t_1 \longrightarrow t_2.$$

Hence,

$$\left\|\varphi(t_2,a)^{1-\gamma_i}(\mathcal{H}_i(y_1,y_2))(t_2) - \varphi(t_1,a)^{1-\gamma_i}(\mathcal{H}_i(y_1,y_2))(t_1)\right\| \longrightarrow 0 \quad \text{as} \quad t_1 \longrightarrow t_2.$$

Thus, we conclude that $\mathcal{H}(\Omega_q)$ is equicontinuous.

Step 4. Condition (2.9) holds. For every bounded subset $\mathbb{J}^1 \times \mathbb{J}^2 \subset \mathcal{C}^{\varphi}_{1-\gamma_1}(I,\mathbb{E}) \times \mathcal{C}^{\varphi}_{1-\gamma_2}(I,\mathbb{E})$, we define the MNC as

$$\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}(\mathbb{J}^1\times\mathbb{J}^2) = \widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}}(\mathbb{J}^1) + \widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_2}}(\mathbb{J}^2), \tag{3.4}$$

where

$$\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}(\mathbb{J}^i) = \sup_{t\in I} e^{-\aleph t}\Lambda(\mathbb{J}^i_{\gamma_i}(t)); \quad i=1,2, \ \aleph>0. \tag{3.5}$$

By using Lemma 2.10 and Example 2.15, $\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}$ satisfies all properties of the Hausdorff MNC mentioned in Lemma 2.8.

Now, let $\mathscr{A} = (\mathbb{A}^1, \mathbb{A}^2)$ be a bounded set belongs to $\overline{conv}\mathcal{H}(\Omega_K)$, using Lemma 2.12, it follows that for a given $\varepsilon_i > 0$ $(i=1,2)$, there exists $\{(y^{1,n}, y^{2,n})\}_{n=1}^{+\infty} \subseteq \mathscr{A}$ such that, for all $t \in I$,

$$\begin{aligned}\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}\Big(\mathcal{H}_i(\mathscr{A})(t)\Big) &= \widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}\Big(\big\{(\mathcal{H}_i(y^1,y^2))(t) \ : \ (y^1,y^2)\in\mathscr{A}\big\}\Big)\\ &\le 2\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}\Big(\big\{(\mathcal{H}_i(y^{1,n},y^{2,n}))(t)\big\}_{n=1}^{+\infty}\Big) + \varepsilon_i, \quad i=1,2.\end{aligned} \tag{3.6}$$

From

$$(\mathcal{H}_i(y^{1,n},y^{2,n}))(t) = \varphi(t,a)^{\gamma_i-1}\xi_i + \mathcal{J}^{\alpha_i,\varphi}_{a^+} g_i(t, y^{1,n}(t), y^{2,n}(t)), \quad i=1,2, \tag{3.7}$$

wet get

(3.8)

$$\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}\Big(\big\{(\mathcal{H}_i(y^{1,n},y^{2,n}))(\cdot)\big\}_{n=1}^{+\infty}\Big) = \widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}\Big(\big\{\mathcal{J}^{\alpha_i,\varphi}_{a^+} g_i(\cdot, y^{1,n}(\cdot), y^{2,n}(\cdot))\big\}_{n=1}^{+\infty}\Big), \ i=1,2.$$

Now, we estimate the quantity $\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}\Big(\big\{(\mathcal{H}_i(y^{1,n},y^{2,n}))(\cdot)\big\}_{n=1}^{+\infty}\Big)$. Using (H3), for all $s\in[a,t]$, one has for $i=1,2$

$$\begin{aligned}&\Lambda\Big(\Big\{\varphi'(s)\varphi(t,s)^{\alpha_i-1}g_i(s,y^{1,n}(s),y^{2,n}(s))\Big\}_{n=1}^{+\infty}\Big)\\ &\quad\le \sum_{j=1}^{2}\varphi'(s)\varphi(t,s)^{\alpha_i-1}\eta_{i,j}(s)\Lambda(\{y^{j,n}(s)\}_{n=1}^{+\infty})\\ &\quad\le \sum_{j=1}^{2}\eta_{i,j}(s)\varphi'(s)\varphi(t,s)^{\alpha_i-1}\varphi(s,a)^{\gamma_j-1}\Lambda(\{y^{j,n}_{\gamma_j}(s)\}_{n=1}^{+\infty})\\ &\quad\le \sum_{j=1}^{2}\eta_{i,j}(s)\varphi'(s)\varphi(t,s)^{\alpha_i-1}\varphi(s,a)^{\gamma_j-1}e^{\aleph s}\sup_{a\le s\le t} e^{-\aleph s}\Lambda(\{y^{j,n}_{\gamma_j}(s)\}_{n=1}^{+\infty})\\ &\quad\le \sum_{j=1}^{2}\eta_{i,j}(s)\varphi'(s)\varphi(t,s)^{\alpha_i-1}\varphi(s,a)^{\gamma_j-1}e^{\aleph s}\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_j}}(\{y^{j,n}\}_{n=1}^{+\infty}).\end{aligned}$$

Thus, Lemma 2.11 entails that, for all $t \in I$ and $s \le t$,

$$\Lambda\Big(\{\mathcal{J}_{a^+}^{\alpha_i,\varphi} g_i(t, y^{1,n}(t), y^{2,n}(t))\}_{n=1}^{+\infty}\Big) \\ \le \sum_{j=1}^{2} \frac{2\|\eta_{i,j}\|_{L^\infty}}{\Gamma(\alpha_i)} \widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_j}}(\{y^{j,n}\}_{n=1}^{+\infty}) \int_a^t \varphi'(s)\varphi(t,s)^{\alpha_i-1}\varphi(s,a)^{\gamma_j-1}e^{\aleph s}ds.$$

Hence,

$$\Lambda\left(\{\varphi(t,a)^{1-\gamma_i}\mathcal{H}_i(y^{1,n}(t), y^{2,n}(t))\}_{n=1}^{+\infty}\right) \\ \le \sum_{j=1}^{2} \frac{2\|\eta_{i,j}\|_{L^\infty}}{\Gamma(\alpha_i)} \widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_j}}(\{y^{j,n}\}_{n=1}^{+\infty})\varphi(b,a)^{1-\gamma_i} \int_a^t \varphi'(s)\varphi(t,s)^{\alpha_i-1}\varphi(s,a)^{\gamma_j-1}e^{\aleph s}ds.$$

Multiplying both sides by $e^{-\aleph t}$, we obtain

$$\sup_{t\in J} e^{-\aleph t}\Lambda\left(\{\varphi(t,a)^{1-\gamma_i}\mathcal{H}_i(y^{1,n}(t), y^{2,n}(t))\}_{n=1}^{+\infty}\right) \\ \le \sum_{j=1}^{2} \frac{2\|\eta_{i,j}\|_{L^\infty}}{\Gamma(\alpha_i)} \widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_j}}(\{y^{j,n}\}_{n=1}^{+\infty})\varphi(b,a)^{1-\gamma_i} \times \\ \sup_{t\in I} \int_a^t \varphi'(s)\varphi(t,s)^{\alpha_i-1}\varphi(s,a)^{\gamma_j-1}e^{-\aleph(t-s)}ds.$$

So,

$$\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}\left(\{\mathcal{H}_i(y^{1,n}, y^{2,n})\}_{n=1}^{+\infty}\right) \le \sum_{j=1}^{2} \mathcal{L}_{i,j}(\aleph)\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_j}}(\{y^{j,n}\}_{n=1}^{+\infty}),$$

where $\mathcal{L}_{i,j}(\aleph)$, $i = 1, 2$, $j = 1, 2$ are defined in (2.1). Hence,

$$\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}\left(\{\mathcal{H}_i(y^{1,n}, y^{2,n})\}_{n=1}^{+\infty}\right) \\ \le \max_{1\le j\le 2}\{\mathcal{L}_{i,j}(\aleph)\}\left(\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}}(\{y^{1,n}\}_{n=1}^{+\infty}) + \widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_2}}(\{y^{2,n}\}_{n=1}^{+\infty})\right) \\ = \max_{1\le j\le 2}\{\mathcal{L}_{i,j}(\aleph)\}\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}(\{(y^{1,n}, y^{2,n})\}_{n=1}^{+\infty}).$$

The last inequality together with the fact that

$$\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}(\{(y^{1,n}, y^{2,n})\}_{n=1}^{+\infty}) \le \widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}(\mathscr{A})$$

yields

$$\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}\left(\{\mathcal{H}_i(y^{1,n}, y^{2,n})\}_{n=1}^{+\infty}\right) \le \max_{1\le j\le 2}\{\mathcal{L}_{i,j}(\aleph)\}\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}(\mathscr{A}). \tag{3.9}$$

From (3.6) and (3.9), one gets

$$\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_i}}\left(\mathcal{H}_i(\mathscr{A})(t)\right) \le 2\max_{1\le j\le 2}\{\mathcal{L}_{i,j}(\aleph)\}\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}(\mathscr{A}) + \varepsilon_i, \quad i = 1, 2. \tag{3.10}$$

Then,

$$\begin{aligned}
&\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}\Big(\mathcal{H}(\mathscr{A})\Big)\\
&\quad=\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}}\Big(\mathcal{H}_i(\mathscr{A})\Big)+\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_2}}\Big(\mathcal{H}_i(\mathscr{A})\Big)\\
&\quad\le 2\Big(\max_{1\le j\le 2}\{\mathcal{L}_{1,j}(\aleph)\}+\max_{1\le j\le 2}\{\mathcal{L}_{2,j}(\aleph)\}\Big)\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}(\mathscr{A})+\varepsilon_3\\
&\quad\le 4\max_{1\le i\le 2}\{\max_{1\le j\le 2}\{\mathcal{L}_{i,j}(\aleph)\}\}\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}(\mathscr{A})+\varepsilon_3,
\end{aligned}$$

where $\varepsilon_3=\varepsilon_1+\varepsilon_2$. Since $\varepsilon_3>0$ is arbitrary, we have

$$\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}\Big(\mathcal{H}(\mathscr{A})\Big)\le 4\max_{1\le i\le 2}\{\max_{1\le j\le 2}\{\mathcal{L}_{i,j}(\aleph)\}\}\widehat{\Lambda}_{\mathcal{C}^{\varphi}_{1-\gamma_1}\times\mathcal{C}^{\varphi}_{1-\gamma_2}}(\mathscr{A}).$$

From Remark 2.4, we deduce $4\max_{1\le i\le 2}\{\max_{1\le j\le 2}\{\mathcal{L}_{i,j}(\aleph)\}\}<1$.

In view of Steps 1 to 4, we can apply Theorem 2.13 and deduce that $\mathcal{H}$ admits in Ω_K, at least one fixed point which is a solution of system (1.1). ■

4 An example

Consider the Banach space

$$\mathbb{E}=\{z=(z^1,z^2,\ldots,z^n,\ldots)\;:\;z^n\to 0\;\text{ as }\;n\to\infty\},$$

equipped with the norm $\|z\|_{\mathbb{E}}=\sup\limits_{n\ge 1}|z^n|$.

We recall that the Hausdorff MNC Λ in $(\mathbb{E},\|\cdot\|_{\mathbb{E}})$ is defined as follows:

$$\Lambda(\mathbb{A})=\lim_{n\to\infty}\sup_{z\in\mathbb{A}}\|(\mathrm{I}-\mathrm{P}_n)z\|_{\infty},$$

where $\mathbb{A}$ is a bounded subset in $\mathbb{E}$ and P_n is the projection onto the linear span of the first n vectors in the standard basis (see [5, 25]).

Consider the following coupled system:

$$(4.1)\qquad\begin{cases}
{}^{H}\mathcal{D}^{\alpha_1,\beta_1;t}_{a^+}y_1(t)=g_1(t,y_1(t),y_2(t)), & t\in I':=(0,b],\;\;0<b<\frac{1}{4e},\\
{}^{H}\mathcal{D}^{\alpha_2,\beta_2;t}_{a^+}y_2(t)=g_2(t,y_1(t),y_2(t)), & t\in I':=(0,b],\;\;0<b<\frac{1}{4e},\\
\left(t^{1-\gamma_1}y_1\right)(0^+)=(0,0,\ldots,0,\ldots),\\
\left(t^{1-\gamma_2}y_2\right)(0^+)=(0,0,\ldots,0,\ldots).
\end{cases}$$

Note that (4.1) is a particular case of (1.1), where:

$$a=0,\;\;\varphi(t)=L^{-1/\phi_{\min}}b^{\zeta}t,\;\;L=\max_{\substack{1\le i\le 2\\1\le j\le 2}}L_{i,j},\;\;\phi_{i,j}=\theta_i+\vartheta_{i,i}-\vartheta_{i,j},\;\;\mu_{\min}=\min_{1\le i\le 2}\mu_i$$

$$\zeta=\frac{1}{\phi_{\min}(1-\mu_{\min})}-1,\;\;L_{i,j}=\left\{\frac{2^{\vartheta_{i,j}}}{\theta_i}+\frac{2^{1-\theta_i}}{1-\vartheta_{i,j}}\right\},\;\;\phi_{\min}=\min_{\substack{1\le i\le 2\\1\le j\le 2}}\phi_{i,j},$$

and for $t \in I$, $y_i = \{y^{i,n}\}_n \in \mathbb{E}$,

$$g_1(t, y_1, y_2) = e^t \left\{ \arctan(|y^{1,n}| + |y^{2,n}|) + \frac{1}{n^2} \right\}_{n=1}^{\infty},$$

$$g_2(t, y_1, y_2) = e^t \left\{ \ln(|y^{1,n}| + 1) + |y^{2,n}| + \frac{1}{n^2} \right\}_{n=1}^{\infty}.$$

One can easily deduce that the functions g_i, $(i = 1, 2)$ satisfy the Carathéodory type hypotheses, so (H1) holds.

To justify hypothesis (H2), let $t \in I$ and $y_i = \{y^{i,n}\}_n \in \mathbb{E}$, $(i = 1, 2)$. We have

$$\begin{aligned}\|g_1(t, y_1, y_2)\|_{\mathbb{E}} &= e^t \left\| \left\{ \arctan(|y^{1,n}| + |y^{2,n}|) + \frac{1}{n^2} \right\}_{n=1}^{\infty} \right\|_{\mathbb{E}} \\ &\leq e^t \left(\sup_{n\geq 1} |y^{1,n}| + \sup_{n\geq 1} |y^{2,n}| + 1 \right) \\ &\leq e^t \left(\|y_1\|_{\mathbb{E}} + \|y_2\|_{\mathbb{E}} + 1 \right),\end{aligned}$$

and

$$\begin{aligned}\|g_2(t, y_1, y_2)\|_{\mathbb{E}} &= e^t \left\| \left\{ \ln(|1 + y^{1,n}|) + |y^{2,n}| + \frac{1}{n^2} \right\}_{n=1}^{\infty} \right\|_{\mathbb{E}} \\ &\leq e^t \left(\sup_{n\geq 1} |y^{1,n}| + \sup_{n\geq 1} |y^{2,n}| + 1 \right) \\ &\leq e^t \left(\|y_1\|_{\mathbb{E}} + \|y_2\|_{\mathbb{E}} + 1 \right).\end{aligned}$$

This shows that hypothesis (H2) holds, with

$$h(t) = h_1(t) = h_2(t) = e^t, \quad \text{for all } t \in I.$$

To prove (3.1), let $t \in I$ and $z_i = \{z^{i,n}\}_n \in \mathbb{A}^i \subseteq \mathbb{E}$, $i = 1, 2$. Fix $n \in \mathbb{N}$; then we have

$$\arctan(|z^{1,k}| + |z^{2,k}|) \leq |z^{1,k}| + |z^{2,k}| \leq \|(\mathrm{I} - \mathrm{P}_n)(z^{1,k})_k\|_\infty + \|(\mathrm{I} - \mathrm{P}_n)(z^{2,k})_k\|_\infty,$$

for all $k > n$, which implies, by taking the supremum, that

$$\begin{aligned}&\sup_{(z_1,z_2)\in\mathbb{A}^1\times\mathbb{A}^2} \|(\mathrm{I} - \mathrm{P}_n)(\arctan(|y^{1,k}| + |y^{2,k}|))_k\|_\infty \\ &\qquad \leq \sup_{z_1\in\mathbb{A}^1} \|(\mathrm{I} - \mathrm{P}_n)(z^{1,k})_k\|_\infty + \sup_{z_2\in\mathbb{A}^2} \|(\mathrm{I} - \mathrm{P}_n)(z^{2,k})_k\|_\infty.\end{aligned}$$

Then

$$\begin{aligned}&\lim_{n\to\infty} \sup_{(z_1,z_2)\in\mathbb{A}^1\times\mathbb{A}^2} \|(\mathrm{I} - \mathrm{P}_n)(\arctan(|y^{1,k}| + |y^{2,k}|))_k\|_\infty \\ &\quad \leq \lim_{n\to\infty} \sup_{z_1\in\mathbb{A}^1} \|(\mathrm{I} - \mathrm{P}_n)(z^{1,k})_k\|_\infty + \lim_{n\to\infty} \sup_{z_2\in\mathbb{A}^2} \|(\mathrm{I} - \mathrm{P}_n)(z^{2,k})_k\|_\infty,\end{aligned}$$

which yields

$$\begin{aligned}&\Lambda(g_1(t, \mathbb{A}^1, \mathbb{A}^2)) \\ &\quad = e^t \lim_{n\to\infty} \sup_{(z_1,z_2)\in\mathbb{A}^1\times\mathbb{A}^2} \left\| (\mathrm{I} - \mathrm{P}_n) \left(\arctan(|z^{1,k}| + |z^{2,k}|) + \frac{1}{k^2} \right)_k \right\|_\infty\end{aligned}$$

$$\leq e^t\Big[\lim_{n\to\infty}\sup_{z_1\in\mathbb{A}^1}\|(\mathrm{I}-\mathrm{P}_n)(z^{1,k})_k\|_\infty$$
$$+\lim_{n\to\infty}\sup_{z_2\in\mathbb{A}^2}\|(\mathrm{I}-\mathrm{P}_n)(z^{2,k})_k\|_\infty\Big]$$
$$\leq \eta_{1,1}(t)\Lambda(\mathbb{A}^1)+\eta_{1,2}(t)\Lambda(\mathbb{A}^2),$$

where

$$\eta_{1,1}=\eta_{1,2}=h.$$

Similarly, one can obtain

$$\Lambda(g_2(t,\mathbb{A}^1,\mathbb{A}^2))\leq \eta_{2,1}(t)\Lambda(\mathbb{A}^1)+\eta_{2,2}(t)\Lambda(\mathbb{A}^2),$$

where

$$\eta_{2,1}=\eta_{2,2}=h.$$

Hence condition (H3) is verified. Now, it remains to show that (3.2) holds. To do this, from $\Gamma(\alpha_i)>1$ for $0<\alpha_i<1$, we have

$$\widehat{K}_i\leq \|h\|_{L^{\frac{1}{\mu_i}}}\frac{\varphi(b,0)^{1+\alpha_i-\gamma_i-\mu_i}}{\theta_i^{1-\mu_i}}+K\sum_{j=1}^{2}\|h\|_{L^{\frac{1}{\mu_i}}}\left(\Theta_{\theta_i,\vartheta_{i,j}}(0,b,b,\varphi)\varphi(b,0)^{\vartheta_{i,i}}\right)^{1-\mu_i}.$$

Then, by Lemma 2.2 and $0<\varphi(b,0)<1$, we get

$$\widehat{K}_i\leq \|h\|_{L^{\frac{1}{\mu_i}}}\frac{\varphi(b,0)^{1+\alpha_i-\gamma_i-\mu_i}}{\theta_i^{1-\mu_i}}$$
$$+K\sum_{j=1}^{2}\|h\|_{L^{\frac{1}{\mu_i}}}\left(\left(\frac{2^{\vartheta_{i,j}}}{\theta_i}+\frac{2^{1-\theta_i}}{1-\vartheta_{i,j}}\right)\varphi(b,0)^{\theta_i-\vartheta_{i,j}}\varphi(b,0)^{\vartheta_{i,i}}\right)^{1-\mu_i}$$
$$\leq \|h\|_{L^{\frac{1}{\mu_i}}}\frac{\varphi(b,0)^{1+\alpha_i-\gamma_i-\mu_i}}{\theta_i^{1-\mu_i}}+2K\|h\|_{L^{\frac{1}{\mu_i}}}\left(L\left(L^{-1/\phi_{\min}}b^{\zeta+1}\right)^{\phi_{\min}}\right)^{1-\mu_i}$$
$$\leq \|h\|_{L^{\frac{1}{\mu_i}}}\frac{\varphi(b,0)^{1+\alpha_i-\gamma_i-\mu_i}}{\theta_i^{1-\mu_i}}+2K\|h\|_{L^{\frac{1}{\mu_i}}}\left(b^{\frac{1}{1-\mu_{\min}}}\right)^{1-\mu_{\min}}$$
$$\leq \frac{\varphi(b,0)^{1+\alpha_i-\gamma_i-\mu_i}}{\theta_i^{1-\mu_i}}\|h\|_{L^{\frac{1}{\mu_i}}}+2Kb\|h\|_{L^{\frac{1}{\mu_i}}}$$

and

$$\|h\|_{L^{\frac{1}{\mu_i}}}=\left(\mu_ie^{b/\mu_i}-\mu_i\right)^{\mu_i}\leq\left(\mu_ie^{b/\mu_i}\right)^{\mu_i}=\mu_i^{\mu_i}e^b\leq\mu_i^{\mu_i}e^1=e^{1+\mu_i\ln(\mu_i)}\leq e.$$

Now, choose

$$K\geq\frac{2e\varphi(b,0)^{1+\alpha_i-\gamma_i-\mu_i}}{\theta_i^{1-\mu_i}(1-4be)},\quad i=1,2.$$

Hence,

$$2\widehat{K}_i\leq K,\quad i=1,2.$$

Since, all hypotheses in Theorem 3.1 are verified, system (4.1) has at least one solution.

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Department of Computer Science, University of Tiaret, Tiaret, Algeria

and

Laboratoire de Recherche en Intelligence Artificielle et Systèmes (LRIAS), University of Tiaret, Tiaret, Algeria
e-mail: oualid.zentar@univ-tiaret.dz

Department of Mathematics, University of Tiaret, Tiaret, Algeria

and

Laboratoire de Recherche en Intelligence Artificielle et Systèmes (LRIAS), University of Tiaret, Tiaret, Algeria
e-mail: mohamed.ziane@univ-tiaret.dz

Department of Mathematics, The University of Jordan, Amman 11942, Jordan
e-mail: horani@ju.edu.jo