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A GLOBAL ESTIMATE FOR THE DIEDERICH–FORNAESS INDEX OF WEAKLY PSEUDOCONVEX DOMAINS

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Abstract. A uniform upper bound for the Diederich–Fornaess index is given for weakly pseudoconvex domains whose Levi form of the boundary vanishes in ℓ-directions everywhere.

§1. Introduction

The aim of this article is to reveal a relation between the Diederich–Fornaess index of weakly pseudoconvex domains and the rank of the Levi form of their boundaries.

Let us first recall the definition of the Diederich–Fornaess index. Consider a complex manifold X and a relatively compact domain $\Omega \Subset X$ with $\mathcal{C}^2$-smooth boundary. A defining function of Ω is a $\mathcal{C}^2$-smooth function $\rho : \overline{\Omega} \to \mathbb{R}$ satisfying $\Omega = \{\rho < 0\}$ and whose gradient does not vanish on $\partial\Omega$. To avoid using too many minus signs, we will associate to a fixed defining function ρ the nonnegative function $\hat{\delta} = \hat{\delta}_\rho = -\rho$, which can be thought of as a boundary distance function of Ω with respect to a certain Hermitian metric on X (depending on ρ).

The *Diederich–Fornaess exponent* $\eta_{\hat{\delta}}$ of a defining function $-\hat{\delta}$ is the supremum of $\eta \in (0,1)$ such that $-\hat{\delta}^\eta$ is a bounded, strictly plurisubharmonic exhaustion function of Ω. If there is no such η, we let $\eta_{\hat{\delta}} := 0$. The *Diederich–Fornaess index* $\eta(\Omega)$ of Ω is the supremum of the Diederich–Fornaess exponents of defining functions of Ω.

The Diederich–Fornaess index is a numerical index on the strength of a certain pseudoconvexity, more precisely that of hyperconvexity. If $\partial\Omega$ is strictly pseudoconvex, we know that $\partial\Omega$ admits a strictly plurisubharmonic defining function; hence, $\eta(\Omega) = 1$. For Ω to have positive $\eta(\Omega)$, Ω must be

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Stein, and in fact we need more. A theorem of Ohsawa and Sibony [12, Theorem 1.1]; (see also [11]) tells us that $\eta_{\hat{\delta}} > 0$ if and only if $i\partial\overline{\partial}(-\log\hat{\delta}) \geq \omega_0$ in Ω for some Hermitian metric ω_0 of X. The domains Ω with positive $\eta(\Omega)$ should carry such a special exhaustion as if they are proper pseudoconvex domains in $X = \mathbb{CP}^n$, where Takeuchi's theorem guarantees this kind of exhaustion. Many techniques using such exhaustions have been developed for solving the $\overline{\partial}$-equation on weakly pseudoconvex domains (see, e.g., [2]–[5], [11]).

Let us give several examples to illustrate the situation we are considering. In a celebrated series of works, Diederich and Fornaess (see [7], [8]) showed that, if X is Stein, $\eta(\Omega) > 0$ for any domain $\Omega \Subset X$ with $\mathcal{C}^2$-smooth pseudoconvex boundary. Note that in this situation $\partial\Omega$ must have a strictly pseudoconvex point, for we can find a level set of a strictly plurisubharmonic exhaustion of X touching $\partial\Omega$ at some points and bounding Ω. They also showed that, for any $\varepsilon > 0$, there is $\Omega \Subset X = \mathbb{C}^2$ with $0 < \eta(\Omega) < \varepsilon$ by using the worm domains, where a Levi-flat portion sits on $\partial\Omega$. Adachi in [1] proved that certain holomorphic disk bundles Ω over compact Riemann surfaces in their associated flat ruled surfaces X satisfy $\eta(\Omega) > 0$ even though $\partial\Omega$ is totally Levi-flat.

A natural question, therefore, is to ask to what extent the Diederich–Fornaess exponent gets smaller when $\partial\Omega$ is nearly Levi-flat everywhere. Our answer is the following.

Main Theorem. *Let X be a complex manifold of dimension $n \geq 2$, and let $\Omega \Subset X$ be a relatively compact domain with $\mathcal{C}^3$-smooth boundary. Assume that the Levi form of the boundary $\partial\Omega$ has at least ℓ zero eigenvalues everywhere on $\partial\Omega$ where $0 \leq \ell \leq n-1$. Then $\eta(\Omega) \leq (n-\ell)/n$.*

In particular, we obtain the following.

Corollary 1.1. *If $\eta(\Omega) > 1/n$, then $\partial\Omega$ is not Levi-flat.*

Corollary 1.2. *If $\eta(\Omega) > (n-1)/n$, then $\partial\Omega$ has a strictly pseudoconvex point.*

Let us explain the idea of our proof of the Main Theorem. When X is Stein, we found a strictly pseudoconvex point on $\partial\Omega$ by approximating $\partial\Omega$ by strictly pseudoconvex real hypersurfaces from outside. Since no such approximation exists in general, we use the following method inside. We assume by contradiction that $\eta(\Omega) > (n-\ell)/n$. Then we show in Theorem 4.1, using weighted L^2-estimates, that any smooth, top-degree form with compact support in Ω is $\overline{\partial}$-exact in the sense of currents on X. This is impossible essentially because the top-degree cohomology with compact support does not vanish.

For the proof of Theorem 4.1, we use an estimate of Donnelly–Fefferman type (see [9]) to pass from an L^2 vanishing result in $L^2_{n,n}(\Omega, \hat{\delta}^{\eta})$ to an L^2 vanishing result in $L^2_{n,n}(\Omega, \hat{\delta}^{-\eta})$. We also modify this argument by using a special Kähler metric $\omega := i\partial\overline{\partial}(-\hat{\delta}^{\eta})$ in Ω for some $\eta \in (0, \eta_{\hat{\delta}})$. This metric respects the degeneracy of the Levi form of $\partial\Omega$ in a certain manner and permits the proof that the trivial extension of this solution is in fact a solution on all of X.

§2. Preliminaries on L^2-estimate

In this section we introduce some notation that we use throughout this article. Also, for the convenience of the reader, we recall some of the basic facts concerning a priori estimates and solvability results for the $\overline{\partial}$ operator.

Let X be a complex manifold equipped with a Hermitian metric ω_0, and let $\Omega \subset X$ be a domain with $\mathcal{C}^2$-smooth boundary. We let $-\hat{\delta} : \overline{\Omega} \to \mathbb{R}$ be a defining function.

We denote by $L^2_{p,q}(\Omega, \hat{\delta}^s)$ the Hilbert space of (p,q)-forms u which satisfy

$$\|u\|^2_{\hat{\delta}^s} := \int_{\Omega} |u|^2_{\omega_0} \hat{\delta}^s \, dV_{\omega_0} < +\infty.$$

Here dV_{ω_0} is the canonical volume element associated with the metric ω_0, and $|\cdot|_{\omega_0}$ is the norm of (p,q)-forms induced by ω_0. For $s = 0$ the L^2-spaces just defined coincide with the usual L^2-spaces on Ω; in this case, we will omit the index $\hat{\delta}^0$.

In our proofs it is sometimes necessary to replace the base metric ω_0 with a different metric ω. The corresponding Hilbert spaces (resp., norms) will then be denoted by $L^2_{p,q}(\Omega, \hat{\delta}^s, \omega)$ (resp., $\|\cdot\|_{\hat{\delta}^s,\omega}$).

For later use, we recall the well-known Bochner–Kodaira–Nakano inequality for Kähler metrics for the special case of the trivial line bundle $\mathbb{C}$ on Ω equipped with a weight function $\varphi \in \mathcal{C}^2(\Omega)$, which is the key point when establishing L^2 existence theorems for the $\overline{\partial}$ operator (see [6]), as follows.

Let ω be a Kähler metric on Ω. Then for every $u \in \mathcal{D}^{p,q}(\Omega)$ we have

$$\|\overline{\partial}u\|^2_{e^{-\varphi}} + \|\overline{\partial}^*_{e^{-\varphi}}u\|^2_{e^{-\varphi}} \geq \left\langle\!\left\langle [i\partial\overline{\partial}\varphi, \Lambda]u, u \right\rangle\!\right\rangle_{e^{-\varphi}}. \tag{2.1}$$

Here Λ is the adjoint of multiplication by ω.

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A standard computation for the curvature term yields that

$$\langle [i\partial\overline{\partial}\varphi, \Lambda]u, u\rangle \geq \Big(\lambda_1 + \cdots + \lambda_q - \sum_{j=1}^{n} \lambda_j\Big)|u|^2 \tag{2.2}$$

for any form $u \in \Lambda^{0,q}T^*\Omega$. Here $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of $i\partial\overline{\partial}\varphi$ with respect to ω.

§3. A special metric

When Ω has a defining function $-\hat{\delta}$ with positive Diederich–Fornaess exponent $\eta_{\hat{\delta}}$, taking $0 < \eta < \eta_{\hat{\delta}}$, we will equip the domain Ω with another Kähler metric $\omega := i\partial\overline{\partial}(-\hat{\delta}^\eta)$ different from ω_0.

Let us study the behavior of the metric ω near $\partial\Omega$ for later use.

LEMMA 3.1. *Suppose that $\partial\Omega$ is $\mathcal{C}^3$-smooth and that the Levi form of $\partial\Omega$ has at least ℓ zero eigenvalues everywhere. Then, we have*

$$dV_\omega \lesssim \hat{\delta}^{n\eta-2-(n-\ell-1)} dV_{\omega_0} \tag{3.1}$$

near $\partial\Omega$.

Proof. First fix a finite covering of $\partial\Omega$ by holomorphic charts $\{(U; z_U)\}$ equipped with the Euclidean metrics ω_U associated with their coordinates z_U. We can fix the covering so that

- $|d\hat{\delta}|_{\omega_U} > 1$ on each chart U;
- ω_U are uniformly comparable to ω_0; and
- a $\mathcal{C}^k$-norm for functions defined on a neighborhood of $\overline{\Omega}$, say, $\|\cdot\|_{\mathcal{C}^k(\overline{\Omega})}$, bounds the $\mathcal{C}^k$-norm associated with the coordinate z_U from above for functions compactly supported in U.

Let $p \in \partial\Omega$, and take one of the holomorphic charts that contains p, say, $(U; z_U = (z_1, z_2, \dots, z_n))$. For small $\varepsilon > 0$, consider a nontangential cone $\Gamma_{p,\varepsilon} := \{z \in U \cap \Omega \mid |z-p| < 2\hat{\delta}(z), |z-p| < \varepsilon\}$ with vertex at p. Note that $\Gamma_{p,\varepsilon}$ is nonempty as $\overline{\Gamma}_{p,\varepsilon}$ contains a segment starting from p normal to $\ker d\hat{\delta}_p$. It suffices to find a positive constant C independent of the choice of p so that

$$D_U := \frac{dV_\omega}{dV_{\omega_U}} \leq C\hat{\delta}^{n\eta-2-(n-\ell-1)}$$

holds on $\Gamma_{p,\varepsilon}$ for some $\varepsilon = \varepsilon(p) > 0$. That is because $\bigcup_{p\in\partial\Omega} \Gamma_{p,\varepsilon(p)} = W \cap \Omega$ for some neighborhood W of $\partial\Omega$ and ω_0 is comparable to every ω_U with a uniform constant; we can prove the desired inequality on $W \cap \Omega$.

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To compute dV_ω/dV_{ω_U}, we will select an orthonormal frame of $T^{1,0}U$. By a unitary transformation, we can suppose that $\ker d\hat{\delta}_p = \mathbb{C}^{n-1}\times\mathbb{R}$ and that $\mathbb{C}^\ell \times \{0'\}$ is contained in the kernel of the Levi form of $\partial\Omega$ at p. Define a $\mathcal{C}^2$-smooth frame $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_n)$ of $T^{1,0}U$ by

$$Y_j := \frac{\partial}{\partial z_j} - \frac{\partial\hat{\delta}/\partial z_j}{\partial\hat{\delta}/\partial z_n}\frac{\partial}{\partial z_n} \quad (j = 1, 2, \ldots, n-1), \qquad Y_n := \frac{\partial}{\partial z_n}.$$

Note that $\{Y_1, Y_2, \ldots, Y_{n-1}\}$ spans $\ker \partial\hat{\delta}$ on U. We apply the Gram–Schmidt procedure to $\mathcal{Y}$ and obtain an orthonormal frame $\mathcal{X} = (X_1, X_2, \ldots, X_n)$ with respect to ω_U. Denote by $A(z) = (a_{jk}(z))$ the change-of-base matrices at each point: $X_k = \sum_{j=1}^n Y_j a_{jk}$ on U.

We would like to estimate each $\lambda_{j\overline{k}} := \omega(X_j, \overline{X_k})$ on $\Gamma_{p,\varepsilon}$. To achieve it, we combine two estimates: one is about $\mu_{j\overline{k}} := \omega(Y_j, \overline{Y_k})$, and the other is about the change-of-base matrices $A(z)$.

First consider the behavior of $\mu_{j\overline{k}}$ on $\Gamma_{p,\varepsilon}$. The equality

$$\omega = i\eta\hat{\delta}^\eta\left\{\frac{\partial\overline{\partial}(-\hat{\delta})}{\hat{\delta}} + (1-\eta)\frac{\partial\hat{\delta}\wedge\overline{\partial}\hat{\delta}}{\hat{\delta}^2}\right\} \tag{3.2}$$

yields that, if $j = k = n$,

$$\lim_{z\to p, z\in U\cap\Omega}\frac{\mu_{n\overline{n}}(z)}{\hat{\delta}(z)^{\eta-2}} = \eta(1-\eta)\left|\partial\hat{\delta}\big(Y_n(p)\big)\right|^2 \le \|\hat{\delta}\|^2_{\mathcal{C}^1(\overline{\Omega})};$$

otherwise,

$$\lim_{z\to p, z\in U\cap\Omega}\frac{|\mu_{j\overline{k}}(z)|}{\hat{\delta}(z)^{\eta-1}} = \eta\left|\partial\overline{\partial}(-\hat{\delta})\big(Y_j(p), \overline{Y_k(p)}\big)\right| \le \|\hat{\delta}\|_{\mathcal{C}^2(\overline{\Omega})}.$$

We can say more for directions in which the Levi form vanishes. If $1 \le j \le \ell$, $1 \le k \le n-1$ or $1 \le j \le n-1$, $1 \le k \le \ell$,

$$\begin{aligned}
&\limsup_{z\to p, z\in\Gamma_{p,\varepsilon}}\frac{|\mu_{j\overline{k}}(z)|}{\hat{\delta}(z)^\eta}\\
&\quad= \limsup_{z\to p, z\in\Gamma_{p,\varepsilon}}\eta\left|\frac{\partial\overline{\partial}(-\hat{\delta})(Y_j(z), \overline{Y_k(z)})}{\hat{\delta}(z)}\right|\\
&\quad= \limsup_{z\to p, z\in\Gamma_{p,\varepsilon}}\eta\frac{|z-p|}{\hat{\delta}(z)}\left|\frac{\partial\overline{\partial}(-\hat{\delta})(Y_j(z), \overline{Y_k(z)}) - 0}{|z-p|}\right|
\end{aligned}$$

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$$\leq 2\big|d\big(\partial\overline{\partial}(-\hat{\delta})(Y_j,\overline{Y_k})\big)(p)\big|_{\omega_U}$$
$$\leq 2\big(\|\hat{\delta}\|_{\mathcal{C}^3(\overline{\Omega})} + 2\|\hat{\delta}\|^2_{\mathcal{C}^2(\overline{\Omega})}\big).$$

Next we proceed to estimate the change-of-base matrices $A(z)$. We identify an n-tuple of $(1,0)$-vectors with an $n\times n$ matrix by using our coordinate z_U. Then, we have $\mathcal{X}(p)=\mathcal{Y}(p)=I_n$ and $A(z)=\mathcal{Y}^{-1}(z)\cdot\mathcal{X}(z)$, where I_n denotes the identity matrix. As a matrix-valued 1-form, we have

$$dA(p)=\mathcal{Y}^{-1}(p)\cdot d\mathcal{X}(p)+d\mathcal{Y}^{-1}(p)\cdot X(p)=d\mathcal{X}(p)+d\mathcal{Y}^{-1}(p).$$

Since $I_n=\mathcal{Y}^{-1}(z)\cdot\mathcal{Y}(z)$, we also have

$$0=d(\mathcal{Y}^{-1}\cdot\mathcal{Y})(p)=d\mathcal{Y}^{-1}(p)+d\mathcal{Y}(p).$$

Now let $\mathrm{GS}:\mathrm{GL}(n,\mathbb{C})\to U(n)$ be the map determined by the Gram–Schmidt procedure. Its differential at I_n defines $d\,\mathrm{GS}_{I_n}:\mathfrak{gl}(n,\mathbb{C})\to\mathfrak{u}(n)$. We linearly extend this map on matrix-valued, that is, $\mathfrak{gl}(n,\mathbb{C})$-valued 1-forms, and we also write $d\,\mathrm{GS}_{I_n}$ for the extended linear map by abuse of notation. Then, $d\,\mathrm{GS}_{I_n}(d\mathcal{Y}(p))=d\mathcal{X}(p)$ follows from $\mathrm{GS}(\mathcal{Y}(z))=\mathcal{X}(z)$. Combining these equalities, we therefore have

$$dA(p)=d\,\mathrm{GS}_{I_n}\big(d\mathcal{Y}(p)\big)-d\mathcal{Y}(p).$$

We use the norm $|A|=\max_{j,k}|a_{jk}|$ for matrices, and we consider the induced norm for linear maps between spaces of matrices. Since a straightforward computation yields $|d\mathcal{Y}(p)|_{\omega_U}\leq\|\hat{\delta}\|_{\mathcal{C}^2(\overline{\Omega})}$, we have

$$\begin{aligned}\limsup_{z\to p,z\in\Gamma_{p,\varepsilon}}\frac{|A(z)-I_n|}{\hat{\delta}(z)} &= \limsup_{z\to p,z\in\Gamma_{p,\varepsilon}}\frac{|z-p|}{\hat{\delta}(z)}\frac{|A(z)-I_n|}{|z-p|}\\ &\leq 2\big|\,dA(p)\big|_{\omega_U}\\ &\leq 2\big(|d\,\mathrm{GS}_{I_n}|+1\big)\big|d\mathcal{Y}(p)\big|_{\omega_U}\\ &\leq 2\big(|d\,\mathrm{GS}_{I_n}|+1\big)\|\hat{\delta}\|_{\mathcal{C}^2(\overline{\Omega})}.\end{aligned}$$

Note that $|d\,\mathrm{GS}_{I_n}|$ is independent of p and depends only on n.

By combining the estimates on $\mu_{j\bar{k}}$ and $A(z)$ above, we can find a positive constant C depending only on $n = \dim X$ and $\|\hat{\delta}\|_{\mathcal{C}^3(\overline{\Omega})}$ so that

$$\begin{aligned} \left|\lambda_{j\bar{k}}(z)\right| &= \left|\sum_{l,m} \mu_{l\overline{m}}(z) a_{jl}(z) \overline{a_{km}(z)}\right| \\ &\leq \begin{cases} C\hat{\delta}^{\eta-2} & (\text{for } j=k=n) \\ C\hat{\delta}^{\eta} & (\text{for } 1 \leq j \leq \ell, 1 \leq k \leq n-1) \\ C\hat{\delta}^{\eta} & (\text{for } 1 \leq j \leq n-1, 1 \leq k \leq \ell) \\ C\hat{\delta}^{\eta-1} & (\text{otherwise}) \end{cases} \end{aligned} \tag{3.3}$$

holds on $\Gamma_{p,\varepsilon}$ for $0 < \varepsilon \ll 1$. It follows that

$$\begin{aligned} D_U &= \det(\lambda_{j\bar{k}})_{j,k=1}^n \\ &\leq n! C^n \hat{\delta}^{\ell\eta + (n-\ell-1)(\eta-1) + (\eta-2)} \\ &= n! C^n \hat{\delta}^{n\eta - 2 - (n-\ell-1)} \end{aligned}$$

on $\Gamma_{p,\varepsilon}$, which completes the proof. □

LEMMA 3.2. *Suppose that $\partial\Omega$ is $\mathcal{C}^3$-smooth and that the Levi form of $\partial\Omega$ has at least ℓ zero eigenvalues everywhere. Then, for any $(n, n-1)$-form u on Ω,*

$$|u|^2_{\omega_0} dV_{\omega_0} \lesssim |u|^2_{\omega} \hat{\delta}^{(n-1)\eta - 2 - (n-\ell-1)} dV_{\omega}$$

near $\partial\Omega$ with positive constant independent of u.

Proof. It suffices to prove the inequality on $\Gamma_{p,\varepsilon}$ with ω_U instead of ω_0, where we work in the same local situation as in the proof of Lemma 3.1. Consider the induced frame of $\wedge^n T^{1,0}U \otimes \wedge^{n-1} T^{0,1}U$ from $\{X_1, X_2, \ldots, X_n\}$ over U. It follows from (3.3) that

$$\begin{aligned} &|X_1 \wedge X_2 \wedge \cdots \wedge X_n \otimes \overline{X}_1 \wedge \overline{X}_2 \wedge \cdots \wedge \widehat{\overline{X}_k} \wedge \cdots \wedge \overline{X}_n|^2_{\omega} \\ &\quad = D_U |\overline{X}_1 \wedge \overline{X}_2 \wedge \cdots \wedge \widehat{\overline{X}_k} \wedge \cdots \wedge \overline{X}_n|^2_{\omega} \\ &\quad \leq D_U (n-1)! C^{n-1} \begin{cases} \hat{\delta}^{(\ell-1)\eta + (n-\ell-1)(\eta-1) + (\eta-2)} & (\text{for } 1 \leq k \leq \ell) \\ \hat{\delta}^{\ell\eta + (n-\ell-2)(\eta-1) + (\eta-2)} & (\text{for } \ell+1 \leq k \leq n-1) \\ \hat{\delta}^{\ell\eta + (n-\ell-1)(\eta-1)} & (\text{for } k = n) \end{cases} \\ &\quad \leq D_U (n-1)! C^{n-1} \hat{\delta}^{(n-1)\eta - 2 - (n-\ell-1)}. \end{aligned}$$

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Hence, we can estimate $|u|^2_\omega$ as

$$\begin{aligned}|u|^2_\omega &\geq \max_{1\leq k\leq n} \frac{|u(X_1, X_2, \ldots, X_n, \overline{X}_1, \overline{X}_2, \ldots, \widehat{\overline{X}}_k, \ldots, \overline{X}_n)|^2}{|X_1 \wedge X_2 \wedge \cdots \wedge X_n \otimes \overline{X}_1 \wedge \overline{X}_2 \wedge \cdots \wedge \widehat{\overline{X}}_k \wedge \cdots \wedge \overline{X}_n|^2_\omega} \\ &\geq \frac{\max_{1\leq k\leq n} |u(X_1, X_2, \ldots, X_n, \overline{X}_1, \overline{X}_2, \ldots, \widehat{\overline{X}}_k, \ldots, \overline{X}_n)|^2}{(n-1)!C^{n-1} D_U \hat{\delta}^{(n-1)\eta-2-(n-\ell-1)}} \\ &\geq C' \frac{|u|^2_{\omega_U}}{D_U \hat{\delta}^{(n-1)\eta-2-(n-\ell-1)}},\end{aligned}$$

with constant $C' > 0$ independent of u. We therefore have the desired inequality

$$\begin{aligned}|u|^2_\omega \, dV_\omega &\geq C' \frac{1}{D_U} |u|^2_{\omega_U} \hat{\delta}^{-(n-1)\eta+2+(n-\ell-1)} D_U dV_{\omega_U} \\ &= C' |u|^2_{\omega_U} \hat{\delta}^{-(n-1)\eta+2+(n-\ell-1)} dV_{\omega_U}.\end{aligned}$$ □

§4. The $\overline{\partial}$ equation in top degree

In this section we will study a version of an L^2 $\overline{\partial}$-Cauchy problem in top degree on a smoothly bounded domain with weakly pseudoconvex boundary, which, by duality, implies a restriction on the rank of the Levi form of $\partial\Omega$.

THEOREM 4.1. *Let X be a complex manifold of dimension $n \geq 2$, and let $\Omega \Subset X$ be a relatively compact domain with $\mathcal{C}^3$-smooth boundary. Suppose that the Levi form of $\partial\Omega$ has at least ℓ zero eigenvalues everywhere on $\partial\Omega$ for some $0 \leq \ell \leq n-1$. If $\eta(\Omega) > (n-\ell)/n$, then for any $f \in L^2_{n,n}(X)$ which is compactly supported in Ω there exists a current $T \in \mathcal{D}'_{0,1}(X)$ supported in $\overline{\Omega}$ such that $\overline{\partial} T = f$ in the distribution sense on X.*

Theorem 4.1 is based on the following estimate of Donnelly–Fefferman type.

THEOREM 4.2. *Let X be a complex manifold of dimension $n \geq 2$, and let $\Omega \Subset X$ be a relatively compact domain with $\mathcal{C}^2$-smooth boundary. Let $-\hat{\delta}$ be a defining function of Ω with Diederich–Fornaess exponent $\eta_{\hat{\delta}} > 0$. For an arbitrary but fixed $\eta \in (0, \eta_{\hat{\delta}})$ we define $\omega := i\partial\overline{\partial}(-\hat{\delta}^\eta)$. Then, for any $f \in L^2_{n,n}(\Omega, \hat{\delta}^{-\eta}, \omega)$, there exists $u \in L^2_{n,n-1}(\Omega, \hat{\delta}^{-\eta}, \omega)$ satisfying $\overline{\partial} u = f$ in the distribution sense in Ω.*

Proof. Let us first see that the conclusion follows in a standard manner from the following a priori estimate.

CLAIM. *There exists a constant* $C>0$ *such that*

$$\|v\|^2_{\hat{\delta}^{-\eta},\omega} \le C\|\overline{\partial}^* v\|^2_{\hat{\delta}^{-\eta},\omega} \tag{4.1}$$

for any $v\in\mathcal{D}^{n,n}(\Omega)$. *Here* $\overline{\partial}^*=\overline{\partial}^*_{\hat{\delta}^{-\eta},\omega}$ *is the adjoint of* $\overline{\partial}$ *with respect to the scalar product induced by* $\|\cdot\|_{\hat{\delta}^{-\eta},\omega}$.

Note that in the top degree we can work with noncomplete metrics, since there is no compatibility condition. Indeed, let us take $f\in L^2_{n,n}(\Omega,\hat{\delta}^{-\eta},\omega)$ and define a linear functional ϕ on $\overline{\partial}^*(\mathcal{D}^{n,n}(\Omega))\subset L^2_{n,n-1}(\Omega,\hat{\delta}^{-\eta},\omega)$ by $\phi(\overline{\partial}^* v)=\langle\!\langle v,f\rangle\!\rangle_{\hat{\delta}^{-\eta},\omega}$, which is well defined and bounded from (4.1). The Hahn–Banach theorem allows us to extend ϕ to a bounded linear functional on $L^2_{n,n-1}(\Omega,\hat{\delta}^{-\eta},\omega)$, and the Riesz representation theorem yields $u\in L^2_{n,n-1}(\Omega,\hat{\delta}^{-\eta},\omega)$ satisfying

$$\langle\!\langle\overline{\partial}^* v,u\rangle\!\rangle_{\hat{\delta}^{-\eta},\omega}=\langle\!\langle v,f\rangle\!\rangle_{\hat{\delta}^{-\eta},\omega}$$

for all $v\in\mathcal{D}^{n,n}(\Omega)$; that is, $\overline{\partial}u=f$ in the distribution sense in Ω.

Let us proceed to prove (4.1). For a direct proof of it, we would have to work with different adjoint operators. Therefore, it is somewhat more convenient to actually prove the dual a priori estimate

$$\|v\|_{\hat{\delta}^{\eta},\omega}\le C\|\overline{\partial}v\|_{\hat{\delta}^{\eta},\omega} \tag{4.2}$$

for any $v\in\mathcal{D}^{0,0}(\Omega)$. Equation (4.1) then follows from (4.2) using a weighted Hodge star operator.

So let us proceed to prove (4.2). Since $\eta<\eta_{\hat{\delta}}$, there exists some small $\varepsilon>0$ such that $\eta+\varepsilon<\eta_{\hat{\delta}}$, which means that

$$i\partial\overline{\partial}(-\hat{\delta}^{\eta+\varepsilon})\ge 0\quad\text{in }\Omega.$$

But then

$$i\partial\overline{\partial}\log\hat{\delta}^{\eta+\varepsilon}=\frac{i\partial\overline{\partial}\hat{\delta}^{\eta+\varepsilon}}{\hat{\delta}^{\eta+\varepsilon}}-i\partial\log\hat{\delta}^{\eta+\varepsilon}\wedge\overline{\partial}\log\hat{\delta}^{\eta+\varepsilon}\le -i\partial\log\hat{\delta}^{\eta+\varepsilon}\wedge\overline{\partial}\log\hat{\delta}^{\eta+\varepsilon}.$$

Hence, we get

$$\mathrm{Trace}_\omega(i\partial\overline{\partial}\log\hat{\delta}^{\eta+\varepsilon})\le -|\overline{\partial}\log\hat{\delta}^{\eta+\varepsilon}|^2_\omega\quad\text{in }\Omega.$$

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Putting $\psi = \hat{\delta}^\eta$, we have $i\partial\overline{\partial}\psi = -\omega$ by definition of ω; thus, $\mathrm{Trace}_\omega(i\partial\overline{\partial}\psi) = -n$. Hence, we get

$$\mathrm{Trace}_\omega(i\partial\overline{\partial}\psi + i\partial\overline{\partial}\log\hat{\delta}^{\eta+\varepsilon}) \leq -n - |\partial\log\hat{\delta}^{\eta+\varepsilon}|^2_\omega \quad \text{on } \Omega. \tag{4.3}$$

On Ω, we consider the weight function $e^{-\psi}$. Since $e^{-\psi}$ is bounded from below and from above by positive constants on Ω, we can replace the norm $\|\cdot\|$ by $\|\cdot\|_{e^{-\psi}}$ for forms on Ω.

Multiplying the metric of the trivial bundle $\mathbb{C}$ further by $\hat{\delta}^{-(\eta+\varepsilon)} = e^{-\log\hat{\delta}^{\eta+\varepsilon}}$ on Ω, it then follows from (2.1) and (2.2) that for $u \in \mathcal{D}^{0,0}(\Omega)$ one has

$$\langle\!\langle -\mathrm{Trace}_\omega(i\partial\overline{\partial}\psi + i\partial\overline{\partial}\log\hat{\delta}^{\eta+\varepsilon})u, u\rangle\!\rangle_{e^{-\psi}\hat{\delta}^{-(\eta+\varepsilon)},\omega} \leq \|\overline{\partial}u\|^2_{e^{-\psi}\hat{\delta}^{-(\eta+\varepsilon)},\omega}.$$

Using (4.3) we obtain

$$\langle\!\langle (n + |\overline{\partial}\log\hat{\delta}^{\eta+\varepsilon}|^2_\omega)u, u\rangle\!\rangle_{e^{-\psi}\hat{\delta}^{-(\eta+\varepsilon)},\omega} \leq \|\overline{\partial}u\|^2_{e^{-\psi}\hat{\delta}^{-(\eta+\varepsilon)},\omega}$$

for $u \in \mathcal{D}^{0,0}(\Omega)$. Observing that $\partial\log\hat{\delta}^{\eta+\varepsilon} = (\eta+\varepsilon)\partial\log\hat{\delta}$ and setting $u = v\hat{\delta}^{\eta+\varepsilon/2}$, we obtain

$$\begin{aligned} &\langle\!\langle (n + (\eta+\varepsilon)^2|\overline{\partial}\log\hat{\delta}|^2_\omega)v, v\rangle\!\rangle_{e^{-\psi}\hat{\delta}^\eta,\omega} \\ &\leq \left\|\overline{\partial}v + \left(\eta + \frac{\varepsilon}{2}\right)v\overline{\partial}\log\hat{\delta}\right\|^2_{e^{-\psi}\hat{\delta}^\eta,\omega} \\ &\leq \left(1 + \frac{1}{a}\right)\|\overline{\partial}v\|^2_{e^{-\psi}\hat{\delta}^\eta,\omega} + (1+a)\left(\eta + \frac{\varepsilon}{2}\right)^2\|v\overline{\partial}\log\hat{\delta}\|^2_{e^{-\psi}\hat{\delta}^\eta,\omega}. \end{aligned} \tag{4.4}$$

Choosing a so small that $(1+a)(\eta+\varepsilon/2)^2 \leq (\eta+\varepsilon)^2$, we can thus absorb the last term in (4.4) in the left-hand side, which immediately gives the a priori estimate (4.2). ∎

Now let us give the proof of Theorem 4.1.

Proof of Theorem 4.1. By the assumption on Ω, we can find a defining function $-\hat{\delta}$ with $\eta_{\hat{\delta}} > (n-\ell)/n$. We fix some real η such that $(n-\ell)/n < \eta < \eta_{\hat{\delta}}$, and we apply Theorem 4.2 with this choice of η.

Now let $f \in L^2_{n,n}(X)$ be compactly supported in Ω, which implies that $f \in L^2_{n,n}(\Omega, \hat{\delta}^{-\eta}, \omega)$. Hence, it follows from Theorem 4.2 that there exists $u \in L^2_{n,n-1}(\Omega, \hat{\delta}^{-\eta}, \omega)$ satisfying $\overline{\partial} u = f$ in Ω.

We first claim that, if we extend u by zero outside Ω, then it defines a current $T = T_u \in \mathcal{D}'_{0,1}(X)$. Indeed, we see from Lemma 3.2 that

$$\int_\Omega |u|^2_{\omega_0} \hat{\delta}^{1-\nu} \, dV_{\omega_0} \lesssim \int_\Omega |u|^2_\omega \hat{\delta}^{1-\nu} \hat{\delta}^{(n-1)\eta - 2 - (n-\ell-1)} \, dV_\omega.$$

Now a straightforward computation shows that the last integral can be estimated by $\int_\Omega |u|^2_\omega \hat{\delta}^{-\eta} \, dV_\omega < +\infty$ if $\nu \leq n\eta - n + \ell$. But by assumption on η we have $n\eta - n + \ell > 0$; hence, we may deduce that for some small $\nu > 0$ we have $u \in L^2_{n,n-1}(\Omega, \hat{\delta}^{1-\nu})$.

But then for any $v \in \mathcal{C}^\infty_{0,1}(X)$ we have

$$\begin{aligned} \left| \int_\Omega u \wedge v \right|^2 &\leq \left(\int_\Omega |u|^2_{\omega_0} \hat{\delta}^{1-\nu} \, dV_{\omega_0} \right) \cdot \left(\int_\Omega |v|^2_{\omega_0} \hat{\delta}^{-1+\nu} \, dV_{\omega_0} \right) \\ &\leq \|u\|^2_{\hat{\delta}^{1-\nu}} \cdot \left(\int_\Omega \hat{\delta}^{-1+\nu} \, dV_{\omega_0} \right) \sup_\Omega |v|^2_{\omega_0}. \end{aligned} \tag{4.5}$$

Since $\nu > 0$, we have $\int_\Omega \hat{\delta}^{-1+\nu} \, dV_{\omega_0} < +\infty$. Therefore, u defines a current $T \in \mathcal{D}'_{0,1}(X)$.

It remains to see that $T = T_u$ satisfies $\overline{\partial} T = f$ in the sense of distributions on X. Let $\alpha \in \mathcal{C}^\infty_{0,0}(X)$. We must show that

$$\int_\Omega u \wedge \overline{\partial} \alpha = \int_\Omega f \wedge \alpha. \tag{4.6}$$

Let $\chi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ be a function such that $\chi(t) = 0$ for $t \leq 1/2$ and $\chi(t) = 1$ for $t \geq 1$. Set $\chi_j = \chi(j\hat{\delta}) \in \mathcal{D}^{0,0}(\Omega)$. Then $\chi_j \alpha \in \mathcal{D}^{0,0}(\Omega)$, and since $\overline{\partial} u = f$ in Ω, we therefore have

$$\int_\Omega f \wedge \chi_j \alpha = \int_\Omega u \wedge \overline{\partial}(\chi_j \alpha) = \int_\Omega u \wedge (\alpha \overline{\partial} \chi_j + \chi_j \wedge \overline{\partial} \alpha).$$

As f has L^2 coefficients on Ω, the integral of $f \wedge \chi_j \alpha$ converges to the integral of $f \wedge \alpha$ as j tends to infinity. The convergence of the integral of $u \wedge \chi_j \overline{\partial} \alpha$ to the integral of $u \wedge \overline{\partial} \alpha$ follows from $u \in L^2_{n,n-1}(\Omega, \hat{\delta}^{1-\nu})$ (use the estimate (4.5)).

The remaining term can be estimated as follows. Using the Cauchy–Schwarz inequality we have

$$\begin{aligned}\Big|\int_\Omega u\wedge\alpha\overline{\partial}\chi_j\Big|^2 &= \Big|\int_{\{\frac{1}{2j}\le\hat\delta\le\frac{1}{j}\}}\langle u\hat\delta^{-\eta/2},\overline{\star_\omega\alpha\overline{\partial}\chi_j\hat\delta^{\eta/2}}\rangle_\omega\,dV_\omega\Big|^2\\ &\le\int_{\{\frac{1}{2j}\le\hat\delta\le\frac{1}{j}\}}|u\hat\delta^{-\eta/2}|^2_\omega\,dV_\omega\cdot\int_{\{\frac{1}{2j}\le\hat\delta\le\frac{1}{j}\}}|\star_\omega\alpha\overline{\partial}\chi_j\hat\delta^{\eta/2}|^2_\omega\,dV_\omega\\ &\le\sup_\Omega|\alpha|^2\int_{\{\hat\delta\le\frac{1}{j}\}}|u|^2_\omega\hat\delta^{-\eta}\,dV_\omega\cdot\int_\Omega|\overline{\partial}\chi_j|^2_\omega\hat\delta^\eta\,dV_\omega,\end{aligned}$$

where $\star_\omega$ denotes the Hodge star operator with respect to ω in Ω. Since $u\in L^2_{n,n-1}(\Omega,\hat\delta^{-\eta},\omega)$, the integral $\int_{\{\hat\delta\le 1/j\}}|u|^2_\omega\hat\delta^{-\eta}\,dV_\omega$ converges to 0 when j tends to infinity.

To estimate the second integral, we look at the behavior of its integrand $|\overline{\partial}\chi_j|^2_\omega$ near $\partial\Omega$. From $\overline{\partial}\chi_j=j\chi'\overline{\partial}\hat\delta$,

$$\begin{aligned}\hat\delta(z)^{\eta-2}|\overline{\partial}\chi_j|^2_\omega(z)&\le j^2\|\chi'\|^2_{\mathcal{C}^1(\mathbb{R})}|\overline{\partial}\hat\delta|^2_{\hat\delta^{2-\eta}\omega}(z)\\ &=j^2\|\chi'\|^2_{\mathcal{C}^1(\mathbb{R})}\max_{0\ne v\in T^{1,0}_zX}\frac{|\partial\hat\delta(v)|^2}{\eta(\hat\delta(z)i\partial\overline{\partial}(-\hat\delta)(v,v)+|\partial\hat\delta(v)|^2)}\\ &\to j^2\|\chi'\|^2_{\mathcal{C}^1(\mathbb{R})}\frac{1}{\eta}\quad\text{as } z\to\partial\Omega.\end{aligned}$$

Therefore, $|\overline{\partial}\chi_j|^2_\omega\lesssim j^2\hat\delta^{2-\eta}$ near $\partial\Omega$. Since the Levi form of $\partial\Omega$ has ℓ zero eigenvalues, we can estimate it with Lemma 3.1 as

$$\begin{aligned}\int_\Omega|\overline{\partial}\chi_j|^2_\omega\hat\delta^\eta\,dV_\omega&\lesssim\int_{\{\hat\delta\le\frac{1}{j}\}}j^2\hat\delta^{2-\eta}\hat\delta^\eta\hat\delta^{n\eta-2-(n-\ell-1)}\,dV_{\omega_0}\\ &=\int_{\{\hat\delta\le\frac{1}{j}\}}j^2\hat\delta^{1+n\eta-(n-\ell)}\,dV_{\omega_0}\\ &\lesssim j^{2-(2+n\eta-(n-\ell))}\\ &=j^{-n\eta+n-\ell}\to 0\end{aligned}$$

as $j\to\infty$ since $-n\eta+n-\ell<0$ by the assumption that $\eta>(n-\ell)/n$.

Therefore, $\int_\Omega u\wedge\alpha\overline{\partial}\chi_j$ converges to 0 when j tends to infinity. Equation (4.6) follows. ∎

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§5. Proof of the Main Theorem

The proof of the Main Theorem easily follows from Theorem 4.1 using a duality argument.

Proof of the Main Theorem. Assume by contradiction that the Levi form of the boundary $\partial\Omega$ has ℓ zero eigenvalues, and assume that $\eta(\Omega) > (n-\ell)/n$. Let $f \in \mathcal{D}^{n,n}(\Omega)$ be a smooth form of top degree with compact support in Ω satisfying $\int_\Omega f = 1$. Applying Theorem 4.1, we can find a current $T \in \mathcal{D}'_{0,1}(X)$ satisfying $\overline{\partial}T = f$ in the current sense. Let χ be a compactly supported smooth function on X which is equal to one on $\overline{\Omega}$. But then

$$1 = \int_\Omega f = \langle f, \chi\rangle = \langle T, \overline{\partial}\chi\rangle = 0.$$

This contradiction proves that $\eta(\Omega) \leq (n-\ell)/n$. ∎

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