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STRICT REGULARITY FOR 2-COCYCLES OF FINITE GROUPS

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Abstract

Let α be a complex-valued 2-cocycle of a finite group G. A new concept of strict α-regularity is introduced and its basic properties are investigated. To illustrate the potential use of this concept, a new proof is offered to show that the number of orbits of G under its action on the set of complex-valued irreducible α_N-characters of N equals the number of α-regular conjugacy classes of G contained in N, where N is a normal subgroup of G.

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1. Introduction

Throughout this paper, G will denote a finite group and it will be implicitly assumed that all projective representations affording projective characters are defined over the field of complex numbers $\mathbb{C}$.

DEFINITION 1.1. A 2-*cocycle* of G over $\mathbb{C}$ is a function $\alpha : G \times G \to \mathbb{C}^*$ such that $\alpha(1, 1) = 1$ and $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$.

The set of all such 2-cocycles of G form a group $Z^2(G, \mathbb{C}^*)$ under multiplication. Let $\delta : G \to \mathbb{C}^*$ be any function with $\delta(1) = 1$. Then $t(\delta)(x, y) = \delta(x)\delta(y)/\delta(xy)$ for all $x, y \in G$ is a 2-cocycle of G, which is called a *coboundary*. Two 2-cocycles α and β are *cohomologous* if there exists a coboundary $t(\delta)$ such that $\beta = t(\delta)\alpha$. This defines an equivalence relation on $Z^2(G, \mathbb{C}^*)$ and the *cohomology classes* $[\alpha]$ form a finite abelian group, called the *Schur multiplier* $M(G)$.

DEFINITION 1.2. Let α be a 2-cocycle of G.

(a) Define $f_\alpha : G \times G \to \mathbb{C}^*$ by

$$f_\alpha(g, x) = \frac{\alpha(g, x)\alpha(gx, g^{-1})}{\alpha(g, g^{-1})}.$$

(b) For each $x \in G$, define $\alpha_x : C_G(x) \to \mathbb{C}^*$ by $\alpha_x(g) = \alpha(g, x)/\alpha(x, g)$.

These two functions arise naturally in the twisted group algebra $(\mathbb{C}(G))_\alpha$ in which $\bar{x}\bar{y} = \alpha(x, y)\overline{xy}$ for all $x, y \in G$ (see [4, page 66]). Here, $\bar{g}\bar{x}\bar{g}^{-1} = f_\alpha(g, x)\overline{gxg^{-1}}$ for $g, x \in G$ and $\bar{g}\bar{x}\bar{g}^{-1} = \alpha_x(g)\bar{x}$ if $g \in C_G(x)$. Also, if $\beta = t(\delta)\alpha$, then $f_\beta(g, x) = (\delta(x)/\delta(gxg^{-1}))f_\alpha(g, x)$ for all $g, x \in G$ and consequently $\alpha_x = \beta_x$.

Now $\alpha_x \in \mathrm{Lin}(C_G(x))$ from [6, Lemma 4.2], where $\mathrm{Lin}(C_G(x))$ is the group of linear characters of $C_G(x)$. The kernel of α_x is the *absolute centraliser* $C_\alpha(x)$ of x with respect to α and $C_G(x)/C_\alpha(x) \cong \langle\alpha_x\rangle$.

DEFINITION 1.3. Let α be a 2-cocycle of G. Then $x \in G$ is *α-regular* if α_x is the trivial character of $C_G(x)$ (or equivalently $C_\alpha(x) = C_G(x)$).

First, every element of G is α-regular if $[\alpha]$ is trivial. Second, setting $y = 1$ and $z = 1$ in Definition 1.1 yields $\alpha(x, 1) = 1$ and similarly $\alpha(1, x) = 1$ for all $x \in G$, and hence 1 is always α-regular. Third, if $x \in G$ is α-regular, then it is α^k-regular for any integer k. Finally, if $x \in G$ is α-regular, then so too is any conjugate of x (see [4, Lemma 2.6.1]), so that one may refer to the α-regular conjugacy classes of G.

Now let $\mathrm{Proj}(G, \alpha)$ denote the set of all irreducible α-characters of G (see [4, page 184]). Then $x \in G$ is α-regular if and only if $\xi(x) \neq 0$ for some $\xi \in \mathrm{Proj}(G, \alpha)$ (see [5, Proposition 1.6.3]) and $|\mathrm{Proj}(G, \alpha)|$ is the number of α-regular conjugacy classes of G (see [5, Theorem 1.3.6]).

Let N be a normal subgroup of G. Then G acts on $\mathrm{Proj}(N, \alpha_N)$ by

$$\zeta^g(x) = f_\alpha(g, x)\zeta(gxg^{-1})$$

for $\zeta \in \mathrm{Proj}(N, \alpha_N)$, $g \in G$ and all $x \in N$. Clifford's theorem for projective characters applies to this action (see [5, Theorem 2.2.1]).

A new concept of strict α^d-regularity, which refines the notion of α^d-regularity, will be defined and investigated in Section 2 for d a divisor of the order of $[\alpha]$. This concept will be used in Section 3 to give an alternative proof that the number of orbits of G under its action on $\mathrm{Proj}(N, \alpha_N)$, for N a normal subgroup of G, is equal to the number of α-regular conjugacy classes of G contained in N from [2, Lemma 3.1]. It is also easy to show that this result is independent of the choice of 2-cocycle from $[\alpha]$. The result is well known when α is trivial (see [3, Corollary 6.33]); the method employed will be to apply this to the orbits of an α-covering group of G under its action on the irreducible characters and conjugacy classes of a normal subgroup, but to decompose these orbits into corresponding sets.

2. Strictly α^d-regular elements

Let $o(\,)$ denote the order of an element in a group. Then for $[\beta] \in M(G)$, there exists $\alpha \in [\beta]$ such that $o(\alpha) = o([\beta])$ and α is a *class-function* cocycle, that is, the elements of $\mathrm{Proj}(G, \alpha)$ are class functions (see [5, Corollary 4.1.6]). To avoid repetition throughout the rest of this paper, it will be assumed that α has these two properties with $n = o(\alpha)$. A consequence of the second property is that $x \in G$ is α-regular if and only if

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$f_\alpha(g, x) = 1$ for all $g \in G$ (see [5, page 33]). The first property allows us to make the following definition in terms of α^d rather than for the more clumsy $\beta \in [\alpha]^d$.

DEFINITION 2.1. Define $x \in G$ to be *strictly* α^d-regular if d is the smallest integer with $1 \leq d \leq n$ such that x is α^d-regular.

Next suppose $o(\alpha^d) = o(\alpha^k) = m$. If ω is a primitive mth root of unity, then there exists a field automorphism τ of $\mathbb{Q}(\omega)$ over $\mathbb{Q}$ such that $\tau(\alpha^d) = \alpha^k$. Consequently, $x \in G$ is α^d-regular if and only if it is α^k-regular. Thus, $d \mid n$ in Definition 2.1.

Let $\pi(d)$ denote the set of prime numbers that divide d and let d_p denote the pth part of d for any prime number p.

LEMMA 2.2. *We have $x \in G$ is strictly α^d-regular if and only if either:*

(a) *x is α^d-regular but not $\alpha^{d/p}$-regular for each $p \in \pi(d)$; or*
(b) *$o(\alpha_x) = d$ in* $\mathrm{Lin}(C_G(x))$.

PROOF. For condition (a), if x is not $\alpha^{d/p}$-regular, then it is not α^t-regular for all positive integers t with $t \mid d/p$. For condition (b), observe that x is α^d-regular if and only if α_x^d is trivial, that is, $o(\alpha_x) \mid d$. Now for $d > 1$, x is strictly α^d-regular if and only if $o(\alpha_x) \mid d$, but $\alpha_x^{d/p} \neq 1$ for each prime $p \in \pi(d)$ from condition (a). The latter is true if and only if $d_p \mid o(\alpha_x)$ for each prime $p \in \pi(d)$, that is, if and only if $d \mid o(\alpha_x)$. □

An equivalent way of stating Lemma 2.2(b) is that $x \in G$ is strictly α^d-regular if and only if $|C_G(x)/C_\alpha(x)| = d$.

Now by definition for each $x \in G$, there exists a unique $d \mid n$ such that x is strictly α^d-regular. Thus, the conjugacy classes of G are partitioned into strictly α^d-regular conjugacy classes. So for $d \mid n$ and N a normal subgroup of G, let t_d be the number of strictly α^d-regular conjugacy classes of G contained in N. Thus, the number of α^d-regular conjugacy classes of G contained in N is $\sum_{s \mid d} t_s$; in particular, $\sum_{d \mid n} t_d = t(N)$, where $t(N)$ is the number of conjugacy classes of G contained in N.

The choice of 2-cocycle α allows the construction of an α-covering group H of G with the following three properties (see [4, Section 4.1]):

(a) H has a cyclic subgroup $A \leq Z(H) \cap H'$ of order n;
(b) there exists a conjugacy-preserving transversal (see below) $\{r(g) : g \in G\}$ of A in H such that $\theta : H \to G$ defined by $\theta(r(g)a) = g$ for all $g \in G$ and all $a \in A$ is a homomorphism with kernel A;
(c) there exists a faithful character $\lambda \in \mathrm{Lin}(A)$ such that $\alpha(x, y) = \lambda(A(x, y))$ for all $x, y \in G$, where $r(x)r(y) = A(x, y)r(xy)$.

A *conjugacy-preserving transversal* means that $r(x)$ and $r(y)$ are conjugate in H if and only if x and y are conjugate in G (see [5, Lemma 4.1.1]).

It is easy to see that $\theta(C_H(r(x))) = C_\alpha(x)$ for $x \in G$ and $\theta(C_H(r(x)A)) = C_G(x)$. Thus, working in H, we see that x is strictly α^d-regular if and only if the cyclic group $C_H(r(x)A)/C_H(r(x))$ has order d.

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PROPOSITION 2.3. *Let H be an α-covering group of G. Then $x \in G$ is strictly α^d-regular if and only if either:*

(a) *$r(x)\langle z^m\rangle$ are the conjugates of $r(x)$ in $r(x)A$, where $\langle z\rangle = A$ and $dm = n$; or*
(b) *$\{r(x)z^i : i = 1, \ldots, m\}$ is a maximal set of conjugacy class representatives of H in $r(x)A$.*

PROOF. Define $k_{r(x)} : C_H(r(x)A) \to A$ by $k_{r(x)}(h) = hr(x)h^{-1}(r(x))^{-1}$. Then $k_{r(x)}$ is a homomorphism with kernel $C_H(r(x))$, since $\lambda(k_{r(x)}) = \alpha_x$. Now let z be a generator of A. Then $r(x)z^i$ and $r(x)z^j$ are conjugate if and only if $z^{j-i} \in \mathrm{Im}(k_{r(x)})$, that is, if and only if $z^i\,\mathrm{Im}(k_{r(x)}) = z^j\,\mathrm{Im}(k_{r(x)})$.

Now x is strictly α^d-regular if and only if $\mathrm{Im}(k_{r(x)}) = \langle z^m\rangle$, that is, if and only if the cosets of $\mathrm{Im}(k_{r(x)})$ in A are $z^i\langle z^m\rangle$ for $i = 1, \ldots, m$. □

3. Counting orbits of projective characters

Let N be a subgroup of G. Let H be an α-covering group of G and, using the notation of Section 2, let M be the subgroup of H containing A such that $\theta(M) = N$. Finally, for any integer k, let $\mathrm{Irr}(M|\lambda^k) = \{\chi \in \mathrm{Irr}(M) : \chi_A = \chi(1)\lambda^k\}$, where $\mathrm{Irr}(M)$ is the set of irreducible characters of M. Then the mapping from $\mathrm{Proj}(N, \alpha_N^k)$ to $\mathrm{Irr}(M|\lambda^k), \zeta \mapsto \chi$ is a bijection, where $\zeta(x) = \chi(r(x))$ for all $x \in N$ (see [4, pages 134–135] or [5, Corollary 4.1.3]). Now suppose N is normal in G, then it is easy to check that $\zeta^g = \chi^{r(g)}$ for all $g \in G$ and hence the orbit length of ζ under the action of G equals that of χ under the action of H. By definition, for each $x \in G$, there exists a unique $d \mid n$ such that x is strictly α^d-regular. Thus, the conjugacy classes of H are partitioned according to $|C_H(r(x)A)/C_H(r(x)|$ for $r(x)a$, where $x \in G$ and $a \in A$. However, if x is a strictly α^d-regular conjugacy class representative of G, then n/d corresponding conjugacy class representatives of H are obtained as detailed in Proposition 2.3. So the number of conjugacy classes of H in M corresponding to the number of α^d-regular conjugacy classes of G contained in N is $\sum_{s|d}(n/s)t_s$; in particular, $\sum_{d|n}(n/d)t_d = t(M)$, where $t(M)$ is the number of conjugacy classes of H contained in M.

LEMMA 3.1. *Let N be a normal subgroup of G and suppose that $o(\alpha^d) = o(\alpha^k)$. Let σ be a field automorphism of $\mathbb{C}$ that extends τ, as described in Section 2, so that $\sigma(\alpha^d) = \alpha^k$. Then $\zeta^g = \zeta'$ if and only if $\sigma(\zeta)^g = \sigma(\zeta')$ for $g \in G$ and $\zeta \in \mathrm{Proj}(N, \alpha_N^d)$.*

PROOF. If $\zeta \in \mathrm{Proj}(N, \alpha_N^d)$, then $\sigma(\zeta) \in \mathrm{Proj}(N, \sigma(\alpha_N^d))$. Now

$$\sigma(\zeta)^g(x) = f_{\sigma(\alpha)}(g, x)\sigma(\zeta(gxg^{-1})) = \sigma(f_\alpha(g, x)\zeta(gxg^{-1}))$$

for all $x \in N$. □

Lemma 3.1 sets up a one-to-one correspondence between the orbits of G under its action on $\mathrm{Proj}(N, \alpha_N^d)$ and those under its action on $\mathrm{Proj}(N, \alpha_N^k)$ in which orbit lengths are preserved. We next just restate Lemma 3.1 for an α-covering group H of G.

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COROLLARY 3.2. *Suppose that* $o(\lambda^d) = o(\lambda^k)$ *in* $\langle\lambda\rangle = \mathrm{Lin}(A)$. *Let* σ *be as in Lemma 3.1, so that* $\sigma(\lambda^d) = \lambda^k$. *Then* $\chi^h = \chi'$ *if and only if* $\sigma(\chi)^h = \sigma(\chi')$ *for* $h \in H$ *and* $\chi \in \mathrm{Irr}(M|\lambda^d)$.

Let ϕ denote Euler's totient function. We use the well-known result from number theory that $\sum_{d|n} \phi(d) = \sum_{d|n} \phi(n/d) = n$.

THEOREM 3.3. *Let* N *be a normal subgroup of* G. *Then the number of orbits of* G *under its action on* $\mathrm{Proj}(N, \alpha_N)$ *is equal to the number of* α*-regular conjugacy classes of* G *contained in* N.

PROOF. Proceeding by induction, we count the number of α^d-regular conjugacy classes of G contained in N. First, if $d = n$, then, as previously stated, the number of conjugacy classes of G contained in N is equal to the number of orbits of G under its action on $\mathrm{Irr}(N)$. So assume by induction that the number of orbits of G under its action on $\mathrm{Proj}(N, \alpha_N^d)$ is equal to the number of α^d-regular conjugacy classes of G contained in N for each $d \mid n$ with $d \neq 1$. Let H be an α-covering group of G and let M denote the subgroup of H containing A such that $\theta(M) = N$.

Now for $d \mid n$ and $d \neq 1$, G has $\sum_{s|d} t_s$ orbits under its action on $\mathrm{Proj}(N, \alpha_N^d)$. Thus, H has the same number of orbits under its action on $\mathrm{Irr}(M|\lambda^d)$. Now $o(\lambda^k) = o(\lambda^d)$ for $\phi(n/d)$ values of k with $1 \leq k \leq n$. Thus, using Corollary 3.2, the total number of orbits of H under its actions on $\mathrm{Irr}(M|\lambda^c)$, for the $n - \phi(n)$ values of c with $1 \leq c \leq n$ that are not relatively prime to n, is

$$\begin{aligned}\sum_{\substack{d|n\\ d\neq 1}} \phi\Big(\frac{n}{d}\Big)\Big(\sum_{s|d} t_s\Big) &= \sum_{s|d} t_s\Big(\sum_{\substack{d|n\\ d\neq 1}} \phi\Big(\frac{n}{d}\Big)\Big)\\ &= \sum_{s|n} t_s\Big(\sum_{\substack{r|(n/s)\\ (r,s)\neq(1,1)}} \phi\Big(\frac{n/s}{r}\Big)\Big)\\ &= t_1(n - \phi(n)) + \sum_{\substack{s|n\\ s\neq 1}} t_s\frac{n}{s}.\end{aligned}$$

The total number of orbits of H under its action on $\mathrm{Irr}(M)$ is $t(M)$, so the total number of orbits of H under its actions on $\mathrm{Irr}(M|\lambda^c)$, for the $\phi(n)$ values of c with $1 \leq c \leq n$ that are relatively prime to n, is

$$t(M) - t_1(n - \phi(n)) - \sum_{\substack{s|n\\ s\neq 1}} t_s\frac{n}{s} = t_1\phi(n).$$

Hence, the number of orbits of H under its action on $\mathrm{Irr}(M|\lambda)$ (and the number of orbits of G under its action on $\mathrm{Proj}(N, \alpha_N)$) is t_1, as required. □

Suppose that $\beta = t(\delta)\alpha$. Then from [1, Lemma 1.4], we see that $\mathrm{Proj}(N, \beta_N) = \{\delta_N\zeta : \zeta \in \mathrm{Proj}(N, \alpha_N)\}$ and, for $g \in G$, $\zeta^g = \zeta'$ if and only if $(\delta_N\zeta)^g = \delta_N\zeta'$ for $\zeta \in \mathrm{Proj}(N, \alpha_N)$. In particular, this establishes a one-to-one correspondence between

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the orbits of G under its action on $\mathrm{Proj}(N, \beta_N)$ and those under its action on $\mathrm{Proj}(N, \alpha_N)$ in which orbit lengths are preserved. So from this and Lemma 3.1, the result of Theorem 3.3 is independent of the choice of 2-cocycle from $[\alpha]^c$ for c relatively prime to n.

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