THE TWO DIMENSIONAL REPRESENTATION OF THREE DIMENSIONAL INTERFEROMETER MEASUREMENTS

R.H. Frater
School of Electrical Engineering
University of Sydney

SUMMARY

A convolution technique for the reduction of three dimensional interferometer measurements to two dimensions is described. With the addition of relatively simple hardware to a general purpose computer the technique allows fast, efficient processing of three dimensional data.

1. INTRODUCTION

In the reduction of rotational synthesis data, measurements are often projected onto a plane normal to the direction of the source, then Fourier transformed for the final map. If the measurements are themselves on a plane having the same origin as the projected plane, a known distortion of the map co-ordinates results. When the measurements are not on such a plane, dispersion effects result. The production of a correct map in this situation requires the use of a transform from a three dimensional measurement set to a two dimensional map plane (Brouw, 1975, Clark,1975, Frater, 1978)

To compute the required transform, a set of two dimensional transforms may be carried out on the three dimensional data. A final stage involving multiplication and summation operations leads to the required map. Alternately, this final stage can be replaced by convolution and summation operations before the transforms. The field size limits in radio interferometry are such that the required convolution process can be greatly simplified resulting in an efficient approach to the processing of three dimensional data.

2. MEASUREMENTS IN (u,v,w) SPACE

If a set of measurements is made of a spatial autocorrelation distribution P(u,v,w) in a co-ordinate system (u,v,w) with the w co-ordinate directed to the centre of the field to be mapped (see Fig.1),

C. van Schooneveld (ed.), Image Formation from Coherence Functions in Astronomy, 19-26.

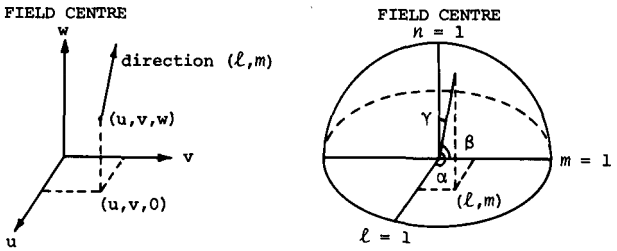

Figure 1. The co-ordinate system used for measurements made in (u,v,w) space

an estimate of the sky brightness distribution $B(\ell,m)$ can be obtained from the relationship

$$B(\ell,m) = \iiint P(u,v,w)\,g(u,v,w)\exp -2\pi i(u\ell+vm+w(1-\ell^2-m^2)^{\frac{1}{2}})\,dudvdw \quad (1)$$

where ℓ,m are direction cosines specifying a point in the map (ℓ=0, m=0 at the map centre) and $g(u,v,w)$ is a graded sampling function.

The term $-2\pi(u\ell+vm+w(1-\ell^2-m^2)^{\frac{1}{2}})$ may be identified as a phase. The w component may be expanded to give

$$w(1-\ell^2-m^2)^{\frac{1}{2}} = w - w(\ell^2+m^2)/2 - w(\ell^2+m^2)^2/8 - \ldots . \quad (2)$$

The error resulting if only the first two terms are retained is small in practical cases. For example if $w=10^5$ wavelengths an error of 1^o occurs at the corner of a 1.8^o field while for $w=10^4$ wavelengths the error is 1^o at the corner of a 3.1^o field.

The first w in equation 2 can be represented as a phase correction to the data included in $g(u,v,w)$ so that equation 1 can be rewritten

$$B(\ell,m) = \iiint P(u,v,w)\,g(u,v,w)\exp -2\pi i(u\ell+vm-w(\ell^2+m^2)/2)\,dudvdw \quad (3)$$

To clarify operations in u and v for a fixed w, write

$$P_w(u,v)\,g_w(u,v) = P(u,v,w)\,g(u,v,w) \quad (4)$$

and for a fixed w using F to denote Fourier transformation, write

$$\exp 2\pi iw(\ell^2+m^2)/2 = F^{-1}\{C_w(u)\,C_w(v)\} \quad (5)$$

Now using ** to denote two dimensional convolution, the convolution theorem yields

$$B(\ell,m) = \iiint\{P_w(u,v)\,g_w(u,v)\}**\{C_w(u)\,C_w(v)\}\exp -2\pi i(u\ell+vm)\,dudvdw \quad (6)$$

$$= \iint P_c(u,v)\exp -2\pi i(u\ell+vm)\,dudv \quad (7)$$

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The required convolution will be recognised as successive one dimensional convolutions in u and v with functions of the form

$$C_w(u) = F(\exp 2\pi i w \ell^2/2) \tag{8}$$

Equation 6 can perhaps be interpreted best by considering discrete values of w. For each w there are convolving functions $C_w(u)$ and $C_w(v)$ which are convolved with the correlation function for that w. The resultant values for each (u,v) are summed in w completing the outer integral and yielding a two dimensional correlation function $P_C(u,v)$ which represents the original three dimensional data set. $B(\ell,m)$ is now obtained by a two dimensional transform. The grading may, in fact, be conveniently applied at this stage.

At first sight the above procedure would seem to offer little when compared to a conventional three dimensional transform. Substantial advantage arises, however, by suitable choice of the convolving functions $C_w(u)$ and $C_w(v)$ and the implementation chosen for the process.

3. THE CONVOLVING FUNCTION

In equation 6, no bounds were set for ℓ and m and the convolving function is indeterminate. In the practical situation, data in the (u,v,w) domain must be convolved onto a three dimensional grid. The grid spacing determines the map dimensions. If co-ordinates u,v and w are expressed in wavelengths and ℓ and m in radians, the map width ($2L$) will be the inverse of the grid spacing ΔF. The resultant transform will be replicated at intervals ($2L$).

It is seen from equation 8 that for maps of equal dimensions in ℓ and m the convolving functions $C_w(u)$ and $C_w(v)$ are identical. Further consideration will therefore be restricted to $C_w(u)$. The function

$$\phi(\ell) = w\ell^2/2 \tag{9}$$

can be approximated in the range $-L$ to L by the Fourier series.

$$H(\ell) = \Phi/3 + \sum_{n=1}^{\infty} ((-1)^n 4\Phi/(\pi n)^2)\cos(n\pi\ell/L) \tag{10}$$

where $\Phi = wL^2/2$

The convolving function $C_w(u)$ can now be written

$$C_w(u) = F\{\exp(2\pi i H(\ell))\} \tag{11}$$

$$= F\{\exp(2\pi i\Phi/3) \times \prod_{n=o}^{\infty} \exp((2\pi i(-1)^n 4\Phi/(n\pi)^2)\cos(n\pi\ell/L)) \tag{12}$$

$$= F\{\exp(2\pi i\Phi/3)\} * F\{\exp((-2\pi i 4\Phi/\pi^2)\cos(\pi\ell/L))\}$$

$$* F\{\exp((2\pi i 4\Phi/4\pi^2)\cos(2\pi\ell/L))\} * \ldots \tag{13}$$

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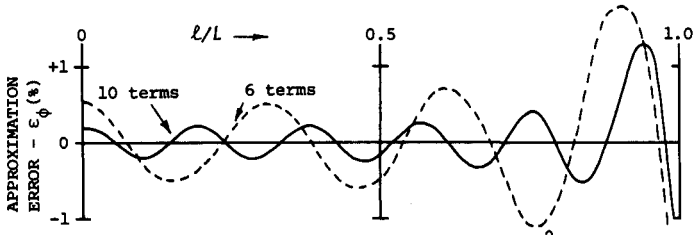

Figure 2. The error in the approximation of $w\ell^2/2$ by Fourier series with (a) 6 terms and (b) 10 terms.

Each of the transforms in equation 13 gives rise to a spectrum whose harmonic components have amplitude $J_n(4\Phi/\pi n^2)$ where J_n are Bessel functions of the first kind.

For small values of Φ (and hence small values for the arguments of the Bessel functions) the total spectrum from equation 13 approximates the sum of spectra resulting from the individual components since for small x

$$J_n(x) \approx (x/2)^n/n! \tag{14}$$

and only $J_1(x)$ need be considered. The spectrum thus falls of in amplitude as n^2.

The question now arises as to how many terms should be retained in order to have a manageable convolving function.

It is seen that the first order effect of restricting the number of components in the convolving function to 2n-1 is to restrict the number of terms in equation 10 to n. This gives rise to an error that can be calculated as

$$\varepsilon_\phi = 100\{H_n(\ell) - w\ell^2/2\}/(wL^2/2) \tag{15}$$

where subscript n denotes the number of terms retained in $H(\ell)$. This error is plotted in figure 2 for 10 terms and for 6 terms.

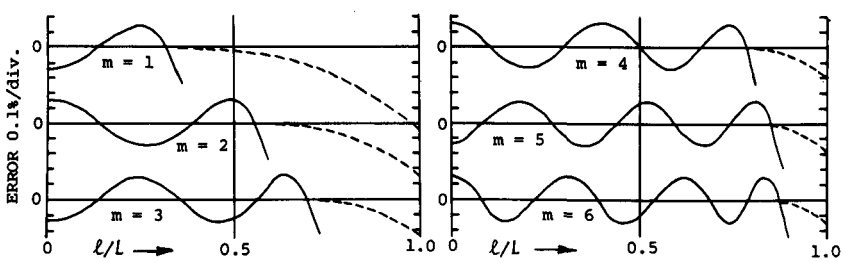

Figure 3. The error in the approximation for $w\ell^2/2$ as a percentage of the field edge phase Φ. The dotted curves are reduced in scale by 100.

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m	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	0.2221	0.2236					
2	0.2845	0.3160	0.0330				
3	0.3085	0.3580	0.0600	0.0120			
4	0.3194	0.3782	0.0761	0.0227	0.0067		
5	0.3248	0.3884	0.0852	0.0300	0.0117	0.0046	
6	0.3274	0.3936	0.0900	0.0342	0.0152	0.0071	0.0036

Table 1. Coefficients for the approximation of $w\ell^2/2$ for orders 1 to 6 with 0.15% ripple.

While the Fourier series gives a least squares fit to the phase in equation 11 it is by no means optimum for present needs. It is possible to optimise series having a limited number of terms to meet a prescribed error over the central part of the range in ℓ at the expense of the extremes in ℓ. The result of such an optimisation with an error limit of 0.15% is shown in Fig. 3 for phase approximation functions of order 1 to 6. These functions are described for order m by the equation

$$H_m(\ell) = \sum_{n=0}^{m} (-1)^n a_n \cos(n\pi\ell/L) \qquad (16)$$

normalised for unity phase at $\ell=L$. Coefficients for orders 1 to 6 are given in table 1.

The functions may be optimised at various levels of maximum error. However, it must be considered that these errors are not random and limits must be determined with this in mind.

The convolving function can now be calculated from the equation

$$C_w(u) = F\{\exp 2\pi i \Phi H_m(\ell)\} \qquad (17)$$

The spectrum of $C_w(u)$ will have (2m+1) terms for small Φ but other terms will become significant for larger Φ. If the convolving function is truncated to (2n+1) terms for all values of Φ then this will

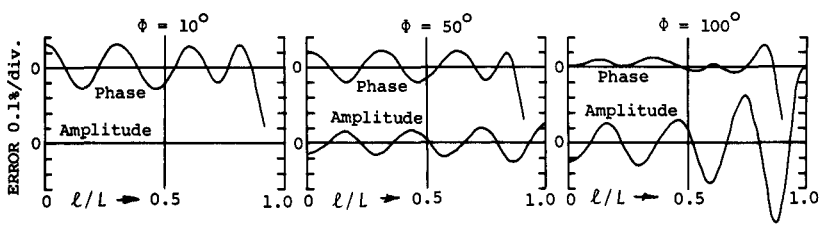

Figure 4. The effect of truncating the 6th order convolving function to 13 points. As Φ increases, the phase error reduces to a minimum at $\Phi=100^\circ$ while the amplitude error increases. The truncation error calculated in equation 18 corresponds to the region where ℓ is close to L.

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give rise to phase and amplitude errors in the final map. Fig. 4 shows the total error including the truncation error produced using the convolution approach when compared to the exact computation, for a sixth order approximation function and for three values of Φ. For small Φ the error is in fact the error of the approximation function itself. At $\Phi=50^{o}$ the percentage phase and amplitude errors are equal while at 100^{o}, the phase error has almost been nulled while the maximum amplitude error has risen to 0.5%. (0.3% within the range of interest). Similar behaviour occurs for other orders. The maximum truncation error ε_T is given approximately for values of n up to 6 by

$$\varepsilon_T(\%) = \{45.9/n + 0.85n - 7.05\}\Phi^2/10^5 \qquad (18)$$

In the range of interest, truncation principally affects the amplitude.

The final choice of convolving function will depend on the area of the map over which high accuracy is required and the value of Φ, which itself depends on the type of implementation chosen.

4. IMPLEMENTATION

In order to make use of either the convolution approach or a three dimensional transform, the measured data must be convolved onto a three dimensional "grid". The number of samples required in the w dimension depends on Φ and the extent of the convolving function used in the w dimension. For a two point function (corresponding to linear interpolation) the spacing in w giving 1^{o} maximum phase error would be that corresponding to a difference of 35^{o} in Φ for the two values of w.

There are two ways of utilizing the convolution approach. The first is to calculate a separate convolving function for each value of w, perform a two dimensional convolution with the appropriate function on each w "layer", then combine all data onto the w=0 plane. A two dimensional transform to obtain the map completes the process. This last transform requires only Real output. The second approach involves calculation of the appropriate convolving function for the grid spacing in w. Denoting the w=0 plane as w_o, the next as w_1 etc., this convolving function, which will be referred to as the Δw function, is that required for the w_1 plane. If the same convolution is performed twice, this corresponds to the required convolution for the w_2 plane.

To use this approach, the data from the w_n plane is first convolved with the Δw function. The data from the w_{n-1} plane is added and the convolution repeated. This process is repeated until the addition of the w_o data when again a two dimensional transform is used to produce the final map. The approach has the advantage that the truncation errors are very low. However, attention must be paid to rounding errors in the successive convolutions if a large number of w planes is involved. The best utilization of the Hermitian property of the measurements will be achieved by minimising the number of w layers.

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Otherwise, special attention will be needed to "wrap around" the convolution in an appropriate way.

To obtain maximum benefit from the convolution approach some special purpose hardware is required on a general purpose computer. The function of this hardware can be simply described. It accepts, under d.m.a. transfer, a row or column of data from the computer, convolves it with the appropriate function and returns it under d.m.a. to the computer. It can be shown that the effective real multiply time required for the convolver to handle the data as fast as it can be accessed is $t_c/(4(2m-1))$ where t_c is the "read plus write" time for the computer. If t_c is 1.5 usec the multiply time is 29nsec. This time can be effectively achieved in hardware using eight separate 16 bit 240nsec multipliers.

It can be seen that for the purpose of comparison with the three dimensional transform approach the convolution can be completed within the access times required to send data to FFT hardware, resulting in a time saving equal to the transform time.

5. CONCLUSION

A new approach to the processing of three dimensional data has been discussed in relation to rotational synthesis applications although the principles involved may be rather more widely applied where three dimensional transforms are used.

The approach will allow high speed processing with a saving both in time and fast memory requirements. The availability of a two dimensional data set from original three dimensional data may be of value in optical processing and indeed the processes leading to the two dimensional data set may themselves be implemented in optics.

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DISCUSSION.

Comment L.R. D'ADDARIO
As this is an alternative to the three-dimensional transform, I would like to comment on each method. 1) The three-dimensional integral over the visibility measurements written by several of today's speakers cannot represent correctly the brightness distribution, if only because the physical units are incorrect. This is a consequence of the fact that the sky is inherently two-dimensional. Nevertheless, the map obtained by summing over all measurements with the three-term exponential kernel has properties much better than those obtained with the two-term exponential (ignoring the w-term). 2) With respect to your method, it is clear that measurements made everywhere in a w = const. $\neq$ 0 planes can be "reduced" to the w = 0 plane by convolution. But what happens if the w = const. slice has measurements at only a few values of (u,v), as actually happens with rotation synthesis?
Reply R.H. FRATER
The method makes no assumption about the location of measurements in (u, v, w) space. Each measurement made for w $\neq$ 0 is represented in the w = 0 plane by a set of "measurements" which produce the same fringe as the original measurement over most of the map area.

Comment T.W. COLE
My background makes it easier to visualize the method as a Fresnel transform or as propagation of spherical waves from one plane to another But is it not true that the sampled measurements rather than continuous measurements still give you a map with beamshape varying over the field?
Reply R.H. FRATER
Since the measurement set in the w = 0 plane accurately represents the original set over most of the map the original property of a varying beamshape is retained. The new measurement set is not the set that would be obtained by measuring originally in the w = 0 plane.

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