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A Theorem on Rational Integral Symmetric Functions.

An identity involving symmetric functions of n letters may in a certain class of cases be extended immediately to a greater number of letters.

For example, the theorem

$$(a+b)^2 = a^2 + b^2 + 2ab$$

may be written

$$(\Sigma\, a)^2 = \Sigma\, a^2 + 2\,\Sigma\, ab;$$

and in the latter form it is true for any number of letters.

Similarly with the theorem

$$(a+b+c)^3$$

$$= a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(c+a) + 3c^2(a+b) + 6abc; \quad \ldots . (1)$$

in the form

$$(\Sigma\, a)^3 = \Sigma\, a^3 + 3\,\Sigma\, a^2 b + 6\,\Sigma\, abc \qquad \ldots\ldots\ldots (2)$$

it is true for any number of letters.

The symmetric functions which occur in these results, such as $\Sigma\, a^3$, $\Sigma\, a^2 b$, $\Sigma\, abc$ are *monomial* symmetric functions, being of the form

$$\Sigma\, a^p\, b^q\, c^r \,\ldots .$$

Symmetric functions occur which are not monomial; with three letters, for instance, the function

$$(b+c)(c+a)(a+b).$$

Such a function, however, may always be expressed, usually in many ways, in terms of monomials; thus the function just written is

$$\Sigma\, a^2 b + 2\,\Sigma\, abc.$$

The following theorem is by no means new, but it is not made prominent in the text-books, and yet is quite useful even in elementary work.

Theorem: If a homogeneous rational integral equation, of degree n in the letters involved, and expressed in terms of monomial symmetric functions is true when the number of letters is n, it is true whatever the number of letters may be.

Formal proof is scarcely necessary. A simple example will show the idea. Suppose we know of the correctness of (1) above. Write it in the form (2), and then suppose that the number of letters is four, say a, b, c, d. The equation in four letters reduces to the equation originally given if a or b or c or d is 0. Hence if all the terms be brought to one side, the aggregate is either 0 or divisible by $abcd$. Considerations of degree rule out the latter alternative. Equation (4) is therefore true for four letters. The equation is next extended in exactly the same way from four letters to five, and thus finally to any number of letters.

Ex. 1. The familiar identity

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$$

gives for any number of letters

$$\Sigma\, a^3 - 3\,\Sigma\, abc = \Sigma\, a\,(\Sigma\, a^2 - \Sigma\, ab),$$

or
$$s_3 - s_2\,\Sigma\, a + s_1\,\Sigma\, ab - 3\,\Sigma\, abc = 0,$$

where $s_r \equiv$ sum of rth powers of $a, b, \ldots$.

Ex. 2. The last result is obviously a special case of Newton's well-known formula for the sum of the powers of the roots of an equation. The general case of this formula of Newton's is in fact one of the best examples of the principle before us. Thus, if α is a root of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$$

we have

$$\alpha^n + p_1 \alpha^{n-1} + p_2 \alpha^{n-2} + \ldots + p_n = 0.$$

Adding the n equations of this form, we get

$$s_n + p_1 s_{n-1} + p_2 s_{n-2} + \ldots + np_n = 0.$$

This formula, being of the nth degree and true for n letters, is true for any number of letters; which is Newton's theorem.[1]

[1] *Cf.* Dr R. F. Muirhead, "Some Proofs of Newton's Theorem on Sums of Powers of Roots," *Proc. Edin. Math. Soc.*, **23**, 1904–05.

Ex. 3. The principle may be of use in cases not coming under the above theorem. For example, the simple elementary identity

$$(x^2 + xy + y^2)(x^2 - xy + y^2) = x^4 + x^2 y^2 + y^4$$

may be written

$$(\Sigma\, x^2 + \Sigma\, xy)(\Sigma\, x^2 - \Sigma\, xy) \quad = \Sigma x^4 + \Sigma\, x^2 y^2,$$

for two letters.

For three letters, we deduce easily by the above method

$$(\Sigma\, x^2 + \Sigma\, xy)(\Sigma\, x^2 - \Sigma\, xy) = \Sigma\, x^4 + \Sigma\, x^2 y^2 - 2\, \Sigma\, x\, \Sigma\, xyz,$$

the coefficient of the last term being determined by putting each letter equal to 1.

This again gives for four letters

$$(\Sigma\, x^2 + \Sigma\, xy)(\Sigma\, x^2 - \Sigma\, xy)$$
$$= \Sigma\, x^4 + \Sigma\, x^2 y^2 - 2\, \Sigma\, x\, \Sigma\, xyz + 2\, \Sigma\, xyzu,$$

which is now true for any number of letters.

JOHN DOUGALL.

A physical solution of the Apollonian problem.

This celebrated problem—*to describe a circle to touch three given circles*—admits of a simple experimental solution which might be found interesting by a mathematical class.

The surface on which the circles lie may be either a plane or a sphere, but the latter case is the more convenient to deal with, and will be the one taken.

We suppose also that the points of intersection of the circles are all real—though this is not essential—so that eight triangles are formed, the sides of which are circular arcs. The circles are to be fitted together in the form of wires or thin strips of wood.

Now take a spherical ball of any (not too small) size, and drop it into the opening formed within one of the triangles ABC, so that it meets the arcs BC, CA, AB at X, Y, Z respectively. Then the circle through the points X, Y, Z is one of the circles touching the three given circles.

Proof. The points X, Y, Z lie on both spheres, and the circle XYZ is therefore the common section of the spheres. The circles