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PRESENTATIONS OF THE FREE METABELIAN GROUP OF RANK 2

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ABSTRACT. Let F_3 denote the free group of rank 3 and M_2 denote the free metabelian group of rank 2. We say that $x \in F_3$ is a primitive element of F_3 if it can be included in some basis of F_3. We establish the existence of presentations $N \hookrightarrow F_3 \overset{\theta}{\twoheadrightarrow} M_2$ such that N does not contain any primitive elements of F_3.

1. **Introduction.** Let F_n and M_n denote the free group of rank n and the free metabelian group of rank n respectively.

An element x of F_n is said to be *primitive* if it can be included in some basis of F_n and similarly an element $y \in M_n$ is said to be *primitive* if it can be included in a basis of M_n. Observe that if P denotes the normal closure in M_n of the primitive element $p \in M_n$ then $M_n/P \cong M_{n-1}$. Let π denote the natural map $\pi: F_n \twoheadrightarrow F_n/F_n'' = M_n$ where we identify M_n with F_n/F_n'' in the usual way. We say that a primitive element p of M_n is *induced* if there exists a primitive element x of F_n such that $x\pi = p$. A well-known theorem of S. Bachmuth and H. Mochizuki [2] implies that, for $n \neq 3$, every primitive element of M_n is induced. In contrast, V. A. Roman'kov [7] has shown that there are primitive elements of M_3 which are not induced. However, his work establishes only the existence of such elements and, to the best of the author's knowledge, no specific primitive element of M_3 has been shown to be non-induced.

Following [3] we say that the presentation $N \hookrightarrow F_n \overset{\theta}{\twoheadrightarrow} G$ of a group G is essentially $(n-1)$-generator if there exist epimorphisms ψ and η that make the diagram

$$\begin{array}{ccccc} & & F_{n-1} & & \\ & \psi \nearrow & & \searrow \eta & \\ N \hookrightarrow F_n & & \overset{\theta}{\twoheadrightarrow} & & G \end{array}$$

commutative.

Equivalently (see [3]), $N \hookrightarrow F_n \overset{\theta}{\twoheadrightarrow} G$ is essentially $(n-1)$-generator if and only if N contains a primitive element of F_n.

Let $N \hookrightarrow F_n \overset{\theta}{\twoheadrightarrow} M_{n-1}$ be a presentation of M_{n-1}. Since $F_n'' \leq N$, there is an induced epimorphism $\gamma: M_n \twoheadrightarrow M_{n-1}$ such that $\theta = \pi\gamma$. It follows immediately from a recent result of C. K. Gupta, N. D. Gupta and G. A. Noskov ([4] Theorem 3.1) that $\ker\gamma$ is the normal closure of some primitive element g of M_n. Now if $n \neq 3$, g is an induced primitive element of M_n and consequently $\ker\theta$ contains a primitive element of F_n. Therefore

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every presentation of the form $N \hookrightarrow F_n \twoheadrightarrow M_{n-1}$, where $n \neq 3$, is essentially $(n-1)$-generator.

In this note we shall use the aforementioned result of Roman'kov on non-induced primitive elements of M_3 to prove the following theorem.

THEOREM A. *The free metabelian group of rank* 2 *has presentations on three generators that are not essentially* 2*-generator. More explicitly, let* $p \in M_3$ *be a primitive element of* M_3 *that is not induced and let* P *denote the normal closure of* p *in* M_3*. Let* $\rho: M_3 \twoheadrightarrow M_2$ *be an epimorphism that has kernel* P *and define* $\theta: F_3 \twoheadrightarrow M_2$ *by* $\theta = \pi\rho$ *where* $\pi: F_3 \twoheadrightarrow M_3$ *is the natural map. Then* $\ker\theta \hookrightarrow F_3 \xrightarrow{\theta} M_2$ *is not essentially* 2*-generator, i.e.* $\ker\theta$ *does not contain a primitive element of* F_3.

Our proof of Theorem A depends on the following result which is of independent interest.

THEOREM B. *Let* q *be a primitive element of* M_n *and suppose that* q *is contained in the normal closure in* M_n *of some element* $y \in M_n$*. Then* q *is conjugate to* y *or* y^{-1}.

2. **Proofs.**

THE PROOF OF THEOREM B. It is clear that the conclusion of Theorem B is correct for $n = 1$. Accordingly, let us fix $n \geq 2$, once and for all.

We shall use the Magnus embedding of M_n throughout. Thus, we view M_n as the group generated under formal matrix multiplication by

$$y_1 = \begin{pmatrix} a_1 & 0 \\ t_1 & 1 \end{pmatrix}, y_2 = \begin{pmatrix} a_2 & 0 \\ t_2 & 1 \end{pmatrix}, \ldots, y_n = \begin{pmatrix} a_n & 0 \\ t_n & 1 \end{pmatrix}$$

where $A_n = \langle a_1, a_2, \ldots, a_n \rangle$ is a free abelian group of rank n and $t_1, t_2, \ldots, t_n$ form a basis for a free (right) $\mathbb{Z}A_n$-module W of rank n. A result of S. Bachmuth [1] asserts that M_n consists of all matrices of the form

$$\begin{pmatrix} h & 0 \\ \sum_{i=1}^{n} t_i r_i & 1 \end{pmatrix}. \tag{1}$$

where $h \in A_n$, $r_1, \ldots, r_n \in \mathbb{Z}A_n$ and

$$\sum_{i=1}^{n} (a_i - 1) r_i = h - 1 \quad \text{(Bachmuth's Criterion)}. \tag{2}$$

The reader who is unfamiliar with the Magnus embedding is directed to the useful expository article by H. Mochizuki [6]. For typographical reasons it is convenient to define

$$L(a, m) = \begin{pmatrix} a & 0 \\ m & 1 \end{pmatrix} \quad \text{for all } \begin{pmatrix} a & 0 \\ m & 1 \end{pmatrix} \in M_n.$$

In this notation we have that $L(a, m)\,L(a', m') = L(aa', ma' + m')$ for all $L(a, m), L(a', m') \in M_n$.

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We write ϕ for the ring homomorphism $\phi\colon \mathbb{Z}A_n \twoheadrightarrow \mathbb{Z}A_{n-1}$ given by $a_1\phi = a_1, a_2\phi = a_2, \ldots, a_{n-1}\phi = a_{n-1}, a_n\phi = 1$ and note that $\ker\phi = (a_n - 1)\mathbb{Z}A_n$. Also, we write ϵ for the augmentation map $\epsilon\colon \mathbb{Z}A_n \longrightarrow \mathbb{Z}$ with $a_i\epsilon = 1$ for $i = 1, 2, \ldots, n$.

Since $\operatorname{Aut} M_n$ is transitive on the set of primitive elements of M_n it suffices to assume that y_n is contained in the normal closure of some element $y \in M_n$ and prove that y_n is conjugate to y or y^{-1}. So suppose that $y_n = L(a_n, t_n)$ is in the normal closure of $y = L(a, w) \in M_n$. By abelianizing, it is easy to see that $a = a_n^{\pm 1}$ and so, on replacing y with y^{-1} if necessary, we may assume that y_n is in the normal closure of $y = L(a_n, w) \in M_n$ and prove Theorem B by showing that y_n is conjugate to y. The proof is broken-down into several steps. We write $w = t_n + m$ where $m = \sum_{i=1}^{n} t_i\alpha_i$ so that $y = L(a_n, t_n + m)$ and, by Bachmuth's criterion, $\sum_{i=1}^{n}(a_i - 1)\alpha_i = 0$.

STEP 1. For $i = 1, \ldots, n-1$ we have that $\alpha_i = (a_n - 1)\alpha_i'$ for some $\alpha_i' \in \mathbb{Z}A_n$.

Note that $y_n^{-1}y = L(1, m)$ so that $L(1, m)$ is contained in the normal closure $y^{M_n} = \langle y, [y, M_n]\rangle$. It follows easily that $L(1, m)$ is contained in $[y, M_n]$. Direct calculation shows that, for each $L(h, m_h) \in M_n$, we have that

$$[y, L(h, m_h)] = \left[\begin{pmatrix} a_n & 0 \\ t_n + m & 1 \end{pmatrix}, \begin{pmatrix} h & 0 \\ m_h & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ (t_n + m)(h - 1) + m_h(1 - a_n) & 1 \end{pmatrix}$$

so that $L(1, m)$ lies in the subgroup $[y, M_n]$ generated by elements of this form. Noting that the product of two lower unitriangular matrices is given by $L(1, v)L(1, u) = L(1, v + u)$ we see that

$$m = (t_n + m)\left(\sum_{h\in A_n}(h - 1)z_h\right) + \gamma(1 - a_n) \tag{3}$$

where z_h is an integer depending on h, z_h is zero for almost all h and γ is some element of W, the precise form of which need not concern us here.

Recall that $m = \sum_{i=1}^{n} t_i\alpha_i$ and that W is a free $\mathbb{Z}A_n$-module with basis $\{t_1, t_2, \ldots, t_n\}$. Let $\gamma = \sum_{i=1}^{n} t_i\gamma_i$. From (3) we have that

$$\left(\sum_{i=1}^{n} t_i\alpha_i\right)\left(1 - \sum_{h\in A_n}(h - 1)z_h\right) = t_n\left(\sum_{h\in A_n}(h - 1)z_h\right) + \sum_{i=1}^{n} t_i\gamma_i(1 - a_n)$$

and equating coefficients of t_i on both sides of this equation yields

$$\alpha_i\left(1 - \sum_{h\in A_n}(h - 1)z_h\right) = \gamma_i(1 - a_n) \quad \text{for } i = 1, 2, \ldots, n-1.$$

Thus, $\alpha_i\left(1 - \sum_{h\in A_n}(h - 1)z_h\right)$ is contained in the prime ideal I of $\mathbb{Z}A_n$ generated by $a_n - 1$. Now $1 - \sum_{h\in A_n}(h - 1)z_h$ has augmentation 1 and so it cannot be contained in I. It follows that, for each i, $i = 1, \ldots, n-1$, we have that $\alpha_i \in I$ and so $\alpha_i = (a_n - 1)\alpha_i'$ for some $\alpha_i' \in \mathbb{Z}A_n$. Step 1 is complete.

STEP 2. $\alpha_n + 1 = b + \alpha_n'(a_n - 1)$ for some $\alpha_n' \in \mathbb{Z}A_n$ and some $b \in \pm A_{n-1}$.

Here A_{n-1} denotes the free abelian group generated by $a_1, a_2, \ldots, a_{n-1}$. It is clear that $M_n = \langle y^{M_n}, y_1, y_2, \ldots, y_{n-1} \rangle$. Moreover,

$$L(h, m_h)^{-1} y L(h, m_h) = \begin{pmatrix} h & 0 \\ m_h & 1 \end{pmatrix}^{-1} y \begin{pmatrix} h & 0 \\ m_h & 1 \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ m_h(1-a_n) + (t_n + m)h & 1 \end{pmatrix}$$

for all $L(h, m_h) \in M_n$ so that W is generated as a $\mathbb{Z}A_n$-module by $t_1, t_2, \ldots, t_{n-1}$ together with elements of the form $m_h(1-a_n) + (t_n + m)h$.

Now the t_n component of $m_h(1-a_n) + (t_n + m)h$ has the form $t_n\big(\beta_h(1-a_n) - (1+\alpha_n)h\big)$ for some $\beta_h \in \mathbb{Z}A_n$ and consequently $t_n\mathbb{Z}A_n$ is generated by elements of the form $t_n\big(\beta_h(1-a_n) + (1+\alpha_n)h\big)$. In particular we have that $t_n = t_n\big(\beta(1-a_n) + \sum_{h \in A_n}(1+\alpha_n)hc_h\big)$ for some $\beta \in \mathbb{Z}A_r$ and some $c_h \in \mathbb{Z}A_n$ such that $c_h = 0$ for almost all $h \in A_n$. Therefore, with these β and c_h, we have that

$$\beta(1-a_n) + \sum_{h \in A_n} (1+\alpha_n)hc_h = 1.$$

Applying the ring homomorphism $\phi\colon \mathbb{Z}A_n \twoheadrightarrow \mathbb{Z}A_{n-1}$ to this equation we obtain that $\big(\sum_{h \in A_n}(1+\alpha_n)hc_h\big)\phi = 1$ and so $(1+\alpha_n)\phi\,(\sum_{h \in A_n} hc_h)\phi = 1$. Therefore $(1+\alpha_n)\phi$ is a unit of $\mathbb{Z}A_{n-1}$. It is well-known [5] that the units in $\mathbb{Z}A_{n-1}$ are of the form $\pm g$ for $g \in A_{n-1}$. It follows that $1 + \alpha_n \equiv b$ modulo $\ker\phi$ for some $b \in \pm A_{n-1}$. Now $\ker\phi = (a_n - 1)\mathbb{Z}A_n$ so $1 + \alpha_n = b + \alpha_n'(a_n - 1)$ for some $\alpha_n' \in \mathbb{Z}A_n$ and $b \in \pm A_{n-1}$ as required.

STEP 3. Conjugacy of $y_n = L(a_n, t_n)$ and $y = L(a_n, t_n + m)$.

Recall that $m = \sum_{i=1}^n t_i\alpha_i$ where $\alpha_i = \alpha_i'(a_n - 1)$ for $i = 1, \ldots, n-1$ and $1 + \alpha_n = b + \alpha_n'(a_n - 1)$. Since Bachmuth's criterion shows that $\sum_{i=1}^n (a_i - 1)\alpha_i = 0$, we have that $\sum_{i=1}^{n-1}(a_i - 1)\alpha_i + (a_n - 1)(1 + \alpha_n) = (a_n - 1)$. Therefore $\sum_{i=1}^{n-1}(a_i - 1)\alpha_i'(a_n - 1) + (a_n - 1)\big(b + \alpha_n'(a_n - 1)\big) = a_n - 1$ and, since $\mathbb{Z}A_n$ is a domain, we deduce that

$$\sum_{i=1}^{n-1}(a_i - 1)\alpha_i' + b + \alpha_n'(a_n - 1) = 1. \tag{4}$$

Applying the augmentation map $\epsilon\colon \mathbb{Z}A_n \twoheadrightarrow \mathbb{Z}$ to this equation we find that $b\epsilon = 1$ and, since $b \in \pm A_{n-1}$ we obtain that $b \in A_{n-1}$. On rewriting (4), we get that $\sum_{i=1}^n (a_i - 1)\alpha_i' = 1 - b$ so that $\sum_{i=1}^n (a_i - 1)b^{-1}\alpha_i' = b^{-1} - 1$, and it now follows from Bachmuth's criterion that

$$B = \begin{pmatrix} b^{-1} & 0 \\ \sum_{i=1}^n t_i b^{-1}\alpha_i' & 1 \end{pmatrix} \in M_n.$$

A routine calculation using the facts that $\alpha_i = (a_n - 1)\alpha_i'$ for $i = 1, \ldots, n-1$ and $\alpha_n + 1 = b + (a_n - 1)\alpha_n'$ shows that $B^{-1}yB = \begin{pmatrix} a_n & 0 \\ t_n & 1 \end{pmatrix} = y_n$. Therefore y is conjugate to y_n in M_n and the proof is complete.

PROOF OF THEOREM A. Since Roman'kov [7] has proved the existence of non-induced primitive elements of M_3, it suffices to prove the more explicit part of the theorem.

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Suppose that $x \in \ker\theta$ is a primitive element of F_3. Then $x\pi$ is an induced primitive element of M_3. Moreover $x\pi$ is contained in P, the normal closure of p. Theorem B implies that $x\pi$ is conjugate to p or p^{-1}. Therefore p is conjugate to an induced primitive element of M_3 and it follows immediately that p is itself an induced primitive element of M_3. This contradiction completes the proof.

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