ESTIMATES FOR GENERAL COERCIVE BOUNDARY PROBLEMS ON A HALF-SPACE FOR A CLASS OF ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

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Introduction. In recent years elliptic boundary value problems have been studied in great detail; see, for example, Agmon **(1)**, Agmon, Douglis, and Nirenberg **(2)**, Browder **(4)**, Hörmander **(7)**, Schechter **(10; 11; 12)**, Agranovich and Dynin **(3)**. In all these cases the boundary problems considered were local or semilocal, i.e. the boundary operators involved are differential operators possibly having singular integral operators for coefficients (cf. **(3)**). The basic tool in these investigations is the following coercive inequality

$$||u||_m \leqslant C(||Au||_0 + \textstyle\sum_j \langle B_j u\rangle_{k_j} + ||u||_0)$$

for $u \in C_0^\infty(\Omega)$, where $\langle \ldots \rangle_{k_j}$ denotes the boundary Hilbert space norms and Ω is the basic domain.

In 1964 Browder **(6)** defined a non-local elliptic boundary problem on a bounded domain. His boundary operators are general continuous operators defined on the boundary Hilbert spaces and the problem is supposed to satisfy a coercive inequality.

Earlier, M. I. Višik **(15)** gave a complete treatment of general boundary value problems for second-order elliptic partial differential operators on bounded domains. In fact, Višik's work includes non-coercive boundary problems as well.

In this paper we study general, local as well as non-local coercive boundary problems on a half-space $\overline{R_n^+}$. The following definition is Browder's, adapted for the unbounded domain $\overline{R_n^+}$.

Definition 1. Under a general coercive boundary problem on a half-space $\overline{R_n^+}$ we understand the triple (A, B, Γ), where A is an elliptic partial differential operator on $\overline{R_n^+}$ and B is a continuous operator from $W^{m,2}(\overline{R_n^+})$ into some Hilbert space Γ, such that B: $W_0^{m,2}(\overline{R_n^+}) \to 0$ and

$$||u||_m \leqslant C(||Au||_0 + ||Bu||_\Gamma + ||u||_0) \tag{1}$$

for $u \in C_0^\infty(\overline{R_n^+})$.

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We study a subclass of elliptic partial differential operators which we call simple; for precise definitions see §1. Without any a priori assumptions on the form of B and Γ we obtain a sufficient condition, Condition K, for (A, B, Γ) to be coercive on $\overline{R_n^+}$ if A is simple. Our method is an adaption of a recent proof Schechter gave of the usual local coercive estimates (see **(12)**). We also show that the usual local elliptic boundary problems are included in our result. Finally we study an inequality of the form

$$|u|_m \leqslant C(||Au||_0 + ||Bu||_\Gamma).$$

This is satisfied by differential elliptic boundary value problems. We show that Condition K is actually necessary if we want (A, B, Γ) to satisfy such an inequality.

1. Notation and terminology. Let R_n, $n \geqslant 2$, denote the Euclidean n-space, R_n^+ the open half-space

$$R_n^+ = \{(y_1, \ldots, y_n) | \ y_n > 0\},$$

and $\overline{R_n^+}$ its closure. For the sake of convenience, points of $\overline{R_n^+}$ will be written as (x, t), where

$$x = (y_1, \ldots, y_{n-1}), \qquad t = y_n.$$

$R_{n-1}^{(K)}$ for some $K > 0$ will denote the domain $\{\xi | \ \xi \in R_{n-1}, |\xi| > K\}$.

We need partial Fourier transforms with respect to the variables x and t given by

$$f_1(\xi, t) = (2\pi)^{-(n-1)/2} \int_{R_{n-1}} f(x, t) e^{-ix\cdot\xi}\, dx = F_x f(\xi, t),$$

$$f^{\wedge}(x, \tau) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x, t) e^{-it\tau}\, dt = F_t f(x, \tau).$$

The Fourier transform with respect to both x and t is denoted by $f_1\hat{}(\xi, \tau)$ The inverse Fourier transform is defined in the usual fashion. Let

$$\mu = (\mu_1, \ldots, \mu_n)$$

be a multi-index of non-negative integers with length $|\mu| = \mu_1 + \ldots + \mu_n$. Let $D_j = \partial/j\partial x_j$, $1 \leqslant j \leqslant n$, and set

$$D_x = (D_1, \ldots, D_{n-1}), \qquad D_t = D_n.$$

A partial differential operator of order m is denoted by

$$A(y, D) = \sum_{|\mu| \leqslant m} a_\mu(x, t) D_1^{\mu_1} \ldots D_{n-1}^{\mu_{n-1}} D_t^{\mu_n},$$

where the variable coefficients $a_\mu(x, t)$ are defined in $\overline{R_n^+}$. For each fixed $(x, t) \in \overline{R_n^+}$ we associate a polynomial $A_{x,t}^{(p)}(\xi, \tau)$ with $A(y, D)$ given by

$$A_{x,t}^{(p)}(\xi, \tau) = \sum_{|\mu|=m} a_\mu(x, t) \xi_1^{\mu_1} \ldots \xi_{n-1}^{\mu_{n-1}} \tau^{\mu_n}.$$

For our purposes it is sufficient to assume that $a_\mu(x, t)$, $|\mu| = m$ are constants, that is

$$A_{x,t}^{(p)}(\xi, \tau) = A^{(p)}(\xi, \tau).$$

A partial differential operator $A(y, D)$ whose principal part has constant coefficients is said to be elliptic if

$$|A^{(p)}(\xi, \tau)| \geqslant C(|\xi|^2 + |\tau|^2)^{m/2}$$

for some $C > 0$ and all $(\xi, \tau) \in R_n$. An elliptic partial differential operator is always of even order $m = 2r$ for $n > 2$, and for $n = 2$ we assume that A has even order. We shall always assume that an elliptic partial differential operator satisfies the root condition, i.e. for each $\xi \neq 0$, $A^{(p)}(\xi, \tau) = 0$ has exactly $r = m/2$ roots with positive imaginary part as a function of τ. This condition is only necessary for $n = 2$ since for $n > 2$ it is automatically satisfied (cf. **(10)**).

We shall call a homogeneous elliptic partial differential operator of order $m = 2r$ with constant coefficients simple if for each $|\xi| = 1$, $A(\xi, \tau) = 0$ has simple roots. For example, $D_1^{2r} + \ldots + D_n^{2r}$ is simple for every positive r. Similarly, we call an elliptic differential operator with variable coefficients simple if its principal part is simple.

To define a boundary problem we need some Hilbert spaces of distributions; for details the reader should consult the comprehensive treatise of Hörmander **(8)**. Denote by $W^{\alpha,2}(R_n)$ the Hilbert space of distributions u in R_n whose Fourier transform $\tilde{u}$ is a function which satisfies

$$\int_{R_n} (1 + |\xi|^2)^\alpha |\tilde{u}(\xi)|^2 d\xi < \infty.$$

The norm is given by

$$\langle u\rangle_\alpha^2 = \int_{R_n} (1 + |\xi|^2)^\alpha |\tilde{u}(\xi)|^2 d\xi.$$

If α is a positive integer k, then $W^{k,2}(R_n)$ consists of all functions u whose distribution derivatives $D^\alpha u$ for $|\alpha| \leqslant k$ belong to $L^2(R_n)$. Using this notion one defines the Hilbert space of functions $W^{k,2}(\overline{R_n^+})$ for k a positive integer. In this case the norm is defined by

$$||u||_k^2 = \sum_{|\alpha| \leqslant k} ||D^\alpha u||_0^2 = \sum_{j \leqslant k} |u_j|^2,$$

$$|u|_j^2 = \sum_{|\alpha| = j} ||D^\alpha u||_0^2.$$

It is well known that the mapping

$$d: u(x, t) \to (u(x, 0), \ldots, D_t^{m-1}u(x, 0)),$$

where $u \in C_0^\infty(\overline{R_n^+})$ can be extended to a continuous map

$$d: W^{m,2}(\overline{R_n^+}) \to \sum_{j=0}^{m-1} \oplus W^{m-j-1/2,2}(R_{n-1})$$

which is onto with kernel $W_0^{m,2}(\overline{R_n^+})$, i.e. functions of $W^{m,2}(\overline{R_n^+})$ whose $0, 1, \ldots, m-1$ derivatives vanish at $t = 0$. This allows us to treat $W^{m,2}(\overline{R_n^+})$ as the space

$$W_0^{m,2}(\overline{R_n^+}) \oplus \sum_{j=0}^{m-1} \oplus W^{m-j-1/2,2}(R_{n-1}).$$

On the domain $\overline{R_n^+}$ a general boundary operator B associated with an mth order elliptic partial differential operator A is a continuous mapping

$$B: W^{m,2}(\overline{R_n^+}) \to \Gamma$$

into some Hilbert space Γ such that $BW_0^{m,2}(\overline{R_n^+}) = 0$, that is

$$B: \sum_{j=0}^{m-1} \oplus W^{m-j-1/2,2}(R_{n-1}) \to \Gamma$$

defines B. To be more precise, let $u \in C_0^\infty(\overline{R_n^+})$ and set

$$d_k: u \to D_t^k u(x, 0).$$

Extend it to all of $W^{m,2}(\overline{R_n^+})$. Then we set

$$B = \sum_{k=0}^{m-1} B_k d_k,$$

where B_k is a continuous mapping

$$B_k: W^{m-k-1/2,2}(R_{n-1}) \to \Gamma.$$

Assuming $A^{(p)}(D_x, D_t)$ has constant coefficients,

$$A^{(p)}(\xi, \tau) = A_+(\xi, \tau) A_-(\xi, \tau),$$

where $A^{(p)}(\xi, \tau)$ is considered as a polynomial in τ for each fixed ξ and where $A_+(\xi, \tau)$ contains the product of all zeros of $A^{(p)}(\xi, \tau)$ with positive imaginary part.

Finally, C will denote a general constant which might be different for different formulas.

2. Let $A(D)$ be a homogeneous simple elliptic partial differential operator of order $m = 2r$. The main result of this section (Theorem 1) is the derivation of some sufficient condition (see Definition 2) for the boundary problem (A, B, Γ) to be coercive. As a corollary we give a routine extension of this result to simple elliptic partial differential operators with variable coefficients, i.e. to operators of the form $A(y, D) = A(D) + P(y, D)$, where $A(D)$ is simple and $P(y, D)$ is a partial differential operator of order at most $m - 1$ with continuous coefficients uniformly bounded on $\overline{R_n^+}$. Finally in Theorem 2 we show that the usual local elliptic boundary conditions are included in our result.

The proof of Theorem 1 will be given in the partially Fourier transformed space $F_x W^{m,2}(\overline{R_n^+})$. For this we need a certain amount of preparation. By definition, for each $|\xi| = 1$, $A(\xi, \tau)$ has m simple roots, $\tau_1(\xi), \ldots, \tau_m(\xi)$, $r = m/2$ having positive and the rest negative imaginary parts. We might as well assume that $\operatorname{Im} \tau_k(\xi) > 0$ for $k = 1, \ldots, r$. Set

$$A_k(\xi, \tau) = \frac{A(\xi, \tau)}{\tau - \tau_k(\xi)} \tag{2}$$

for $\xi \neq 0$. Using this notation, define

$$W_k u = i(2\pi)^{-1/2} A_k(\xi, D_t) u_1(\xi, 0)$$

for $k = 1, \ldots, m$ and $u \in C_0^\infty(\overline{R_n^+})$. It is clear that $\tau_k(\xi)$, $k = 1, \ldots, m$, are homogeneous of degree one. Hence $A_k(\xi, \tau)$ is homogeneous of degree $m - 1$ for $k = 1, \ldots, m$. Therefore the mapping

$$u \to (W_1 u, \ldots, W_r u) = Wu$$

defined for $u \in C_0^\infty(\overline{R_n^+})$ can be extended to a continuous map

$$W^{m,2}(\overline{R_n^+}) \to \sum \oplus F_x W^{1/2,2}(R_{n-1}) \tag{3}$$

where the summation is r times. Let

$$B = \sum_{j=0}^{m-1} B_j d_j$$

be the boundary operator associated with $A(D)$. For $u \in C_0^\infty(\overline{R_n^+})$ set

$$g(x, t) = \begin{cases} A(D_x, D_t) u(x, t), & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

Now $B_j d_j$ can be written as $(B_j F_\xi^{-1})(F_x d_j)$. Since for $u \in C_0^\infty(\overline{R_n^+})$, $F_x d_j u = d_j F_x u$ for such u, we have

$$B_j d_j u = (B_j F_\xi^{-1})(\gamma_j F_x) u = (B_j F_\xi^{-1}) \gamma_j u_1,$$

where $\gamma_j = d_j$ applied to $u_1(\xi, t)$. Define the operator γ_{j-} by

$$\gamma_{j-} u_1(\xi) = \int_{-\infty}^{\infty} \frac{\tau^j}{A(\xi, \tau)} g_1{}^\wedge(\xi, \tau)\, d\tau$$

for $u(x, t) \in C_0^\infty(\overline{R_n^+})$ and set

$$\gamma_{j-}^{(K)} u_1(\xi) = \begin{cases} \gamma_{j-} u_1(\xi), & |\xi| \geqslant K, \\ 0, & |\xi| < K. \end{cases}$$

Now

(4) $$\langle F_\xi^{-1}\gamma_{j-}^{(K)}u_1\rangle^2_{m-j-1/2}$$
$$= \int_{|\xi|>K}(1+|\xi|^2)^{m-j-1/2}\left|\int_{-\infty}^{\infty}\frac{\tau^j}{A(\xi,\tau)}g_1{}^\wedge(\xi,\tau)d\tau\right|^2 d\xi$$
$$\leqslant C\int_{|\xi|>K}d\xi\left|\int_{-\infty}^{\infty}\frac{|\xi|^{m-j-1/2}\tau^j}{A(\xi,\tau)}g_1{}^\wedge(\xi,\tau)\,d\tau\right|^2$$
$$\leqslant C\int_{|\xi|>K}d\xi\left(\int_{-\infty}^{\infty}\frac{|\xi|^{2m-2j-1}|\tau|^{2j}}{|A(\xi,\tau)|^2}d\tau\int_{-\infty}^{\infty}|g_1{}^\wedge(\xi,\tau)|^2d\tau\right)$$
$$\leqslant C||g||_0^2 \leqslant C||u||_m^2$$

since $A(\xi,\tau)$ is elliptic and homogeneous of order m as a consequence of which

$$\int_{-\infty}^{\infty}\frac{|\xi|^{2m-2j-1}|\tau|^{2j}}{|A(\xi,\tau)|^2}d\tau < C < \infty$$

for some $C > 0$ independent of ξ. This shows that $F_\xi^{-1}\gamma_{j-}^{(K)}F_x$ is a continuous operator on $C_0^\infty(\overline{R_n^+})$ in the $W^{m,2}(\overline{R_n^+})$ norm. Therefore $F_\xi^{-1}\gamma_{j-}^{(K)}F_x$ can be extended by continuity to all of $W^{m,2}(\overline{R_n^+})$. Thus

$$B_-^{(K)} = \sum_{j=0}^{m-1}B_j'\gamma_{j-}^{(K)}F_x = \sum_{j=0}^{m-1}B_{j-}^{(K)}$$

is a continuous operator on $W^{m,2}(\overline{R_n^+})$ where we set

$$B_{j-}^{(K)} = B_j'\gamma_{j-}^{(K)}F_x \quad\text{and}\quad B_j' = B_j\,F_\xi^{-1}.$$

Furthermore, (4) gives the estimate

(5) $$||B_-^{(K)}u||_\Gamma \leqslant C||g||_0, \qquad u \in C_0^\infty(\overline{R_n^+}).$$

For $u \in C_0^\infty(\overline{R_n^+})$ set

$$\gamma_j^{(K)}u_1(\xi) = \begin{cases}\gamma_j u_1(\xi), & |\xi| \geqslant K,\\ 0, & |\xi| < K,\end{cases}$$

$$u_1^{(K)}(\xi,t) = \begin{cases}u_1(\xi,t), & |\xi| \geqslant K,\\ 0, & |\xi| < 0,\end{cases}$$

and set

$$B^{(K)}u = \sum_{j=0}^{m-1}B_j'\gamma_j^{(K)}u_1.$$

Finally, define $B_+^{(K)}$ by

$$B_+^{(K)}u = B^{(K)}u - B_-^{(K)}u, \qquad u \in C_0^\infty(\overline{R_n^+}).$$

Definition 2. Let $A = A(D)$ be a homogeneous simple elliptic partial differential operator of order $m = 2r$. We say the boundary problem (A, B, Γ) satisfies Condition K for some $K > 0$ if

$$\sum_{k=1}^{r}\int_{|\xi|>K}|\xi||W_k u|^2d\xi \leqslant C||B_+^{(K)}u||_\Gamma^2, \qquad u \in C_0^\infty(\overline{R_n^+}).$$

THEOREM 1. *Let $A(D)$ be a homogeneous simple elliptic partial differential operator of order $m = 2r$ defined on $\overline{R_n^+}$. Let B be a boundary operator such that (A, B, Γ) satisfies Condition K for some $K > 0$. Then for all $u \in W^{m,2}(\overline{R_n^+})$ we have the coercive estimate*

$$||u||_m \leqslant C(||Au||_0 + ||Bu||_\Gamma + ||u||_0).$$

COROLLARY. *Let $A = A(y, D) = A(D) + P(y, D)$ where $A^{(p)} = A(D)$ is homogeneous and simple of order $m = 2r$ and $P(y, D)$ is of order at most $m - 1$ with uniformly bounded continuous coefficients in $\overline{R_n^+}$. If $(A^{(p)}, B, \Gamma)$ satisfies Condition K for some $K > 0$, then (A, B, Γ) is a coercive boundary problem on $\overline{R_n^+}$.*

Proof. The inequality

$$||u||_j \leqslant \epsilon||u||_m + K(\epsilon)||u||_0 \tag{6}$$

is well known to hold for all $\epsilon > 0$, $j = 0, 1, \ldots, m - 1$ and $u \in W^{m,2}(\overline{R_n^+})$ (cf. (**5**, p. 39)). Employing (6), we obtain

$$\begin{aligned}||A^{(p)}u||_0 &\leqslant ||Au||_0 + ||Pu||_0\\ &\leqslant ||Au||_0 + C_1(\epsilon||u||_m + K(\epsilon)||u||_0).\end{aligned}$$

By the hypothesis, Theorem 1 can be applied if $A = A^{(p)}$. Hence

$$\begin{aligned}||u||_m &\leqslant C_2(||A^{(p)}u||_0 + ||Bu||_\Gamma + ||u||_0)\\ &\leqslant C_2(||Au||_0 + ||Pu||_0 + ||Bu||_\Gamma + ||u||_0)\\ &\leqslant C_3(\epsilon)(||Au||_0 + ||Bu||_\Gamma + ||u||_0) + C_1\, C_2\, \epsilon||u||_m.\end{aligned}$$

Choose $\epsilon > 0$ such that $C_1\, C_2\, \epsilon = 1/2$. Then

$$||u||_m \leqslant 2C_3(\epsilon)(||Au||_0 + ||Bu||_\Gamma + ||u||_0).$$

This proves the corollary.

In the rest of §2 we shall assume that $A(D_x, D_t)$ is a homogeneous simple elliptic partial differential operator of order $m = 2r$, i.e. $A = A^{(p)}$. To simplify the proof of Theorem 1 we present some simple lemmas.

LEMMA 1. *To prove Theorem 1 it suffices to show that for $u \in C_0^\infty(\overline{R_n^+})$*

$$||Ru||_0 \leqslant C(||Au||_0 + ||Bu||_\Gamma + ||u||_0), \tag{7}$$

where $R(D_x, D_t)$ is any homogeneous partial differential operator with constant coefficients of order $m = 2r$.

Proof. Same as the proof of the corollary of Theorem 1.

LEMMA 2. *It is sufficient to prove (7) if $R(D_x, D_t)$ is of order at most $m - 1$ in D_t.*

Proof. Suppose that we proved (7) if R is of degree $\leqslant m - 1$ in D_t. Since $A(D_x, D_t)$ is elliptic,

$$D_t^m u = aA(D_x, D_t)u + R(D_x, D_t)u,$$

where $R(D_x, D_t)$ is of order $\leqslant m-1$ with respect to D_t. Therefore

$$\begin{aligned}||D_t^m u||_0 &\leqslant C||Au||_0 + ||Ru||_0 \\ &\leqslant C(||Au||_0 + ||Bu||_\Gamma + ||u||_0)\end{aligned}$$

for $u \in C_0^\infty(\overline{R_n^+})$. This proves Lemma 2.

From now on $R(\xi, \tau)$ denotes a homogeneous polynomial of order $m = 2r$ in (ξ, τ) which is of order at most $m-1$ in τ. Inequality (7) can be written as

$$\int_{R_{n-1}} d\xi \int_0^\infty |R(\xi, D_t)u_1(\xi, t)|^2 dt \leqslant C(||Au||_0 + ||Bu||_\Gamma + ||u||_0) \tag{8}$$

for $u \in C_0^\infty(\overline{R_n^+})$. The following inequality can be proved easily (cf. **(9; 11)**)

$$\int_0^\infty |R(\xi, D_t)u_1(\xi, t)|^2 dt \leqslant C_K \int_0^\infty |A(\xi, D_t)u_1(\xi, t)|^2 dt + \int_0^\infty |u_1(\xi, t)|^2 dt$$

for all $|\xi| < K$ where C_K depends only on the coefficients of D_t^j, $j = 0, 1, \ldots, m-1$ in $R(\xi, D_t)$ and $A(\xi, D_t)$ for $|\xi| < K$. K denotes the fixed number occurring in Condition K. From this we have

$$\int_{|\xi|<K} d\xi \int_0^\infty |R(\xi, D_t)u_1(\xi, t)|^2 dt \leqslant C_K(||Au||_0^2 + ||u||_0^2). \tag{9}$$

LEMMA 3. *To prove Theorem* 1 *it suffices to prove the following inequality*:

$$\int_{|\xi|>K} d\xi \int_0^\infty |R(\xi, D_t)u_1(\xi, t)|^2 dt \leqslant C(||g||_0^2 + ||Bu||_\Gamma^2 + ||u||_0^2),$$

where $g(x, t) = A(D_x, D_t)u(x, t)$ *for* $t \geqslant 0$ *and otherwise* $g = 0$.

LEMMA 4. *Let* $u \in C_0^\infty(\overline{R_n^+})$. *Then for* $0 \leqslant j \leqslant m-1$,

$$\int_{R_n} (1 + |\xi|^2)^{m-j-1/2} |D_t^j(u_1 - u_1^{(K)})(\xi, 0)|^2 d\xi \leqslant C_K(||Au||_0^2 + ||u||_0^2).$$

Proof. It is easily seen that for $j = 0, 1, \ldots, m-1$

$$|D_t^j(u_1 - u_1^{(K)})(\xi, 0)|^2 \leqslant C_m \sum_{j=0}^m \int_0^\infty |D_t^j(u_1 - u_1^{(K)})(\xi, t)|^2 dt;$$

see, for example, Agmon **(1**, p. 198**)**. Therefore

$$\begin{aligned}\int_{R_n} & (1 + |\xi|^2)^{m-j-1/2} |D_t^j(u_1 - u_1^{(K)})(\xi, 0)|^2 d\xi \\ &= \int_{|\xi|<K} (1 + |\xi|^2)^{m-j-1/2} |D_t^j u_1(\xi, 0)|^2 d\xi \\ &\leqslant C_K \int_{|\xi|<K} |D_t^j u_1(\xi, 0)|^2 d\xi \\ &\leqslant C_K' \sum_{j=0}^m \int_{|\xi|<K} d\xi \int_0^\infty |D_t^j u_1(\xi, t)|^2 dt.\end{aligned}$$

Now we have the well-known inequality

$$\sum_{k=0}^{m} \int_0^\infty |D_t^k u_1(\xi, t)|^2 \, dt \leqslant C_K'' \int_0^\infty |A(\xi, D_t)u_1(\xi, t)|^2 \, dt + \int_0^\infty |u_1(\xi, t)|^2 \, dt,$$

where C_K'' depends only on the coefficients of D_t^k in $A(\xi, D_t)$ for $|\xi| < K$; see, for example, Schechter (**12**, p. 265). Now integrating this over the domain $|\xi| < K$ proves Lemma 4.

Proof of Theorem 1. Let $u(x, t)$ be a fixed function in $C_0^\infty(R_n^+)$. Using the property of the Fourier transform

$$[D_t u_1(\xi, t)]^\wedge = \tau u_1^\wedge(\xi, \tau) + i(2\pi)^{-1/2} u_1(\xi, 0),$$

one easily derives the identity

$$\begin{aligned} (10) \qquad g_1^\wedge(\xi, t) &= (\tau - \tau_k)[A_k(\xi, D_t)u_1(\xi, t)]^\wedge + i(2\pi)^{-1/2} A_k(\xi, D_t)u_1(\xi, 0) \\ &= (\tau - \tau_k)[A_k(\xi, D_t)u_1(\xi, t)]^\wedge + W_k u \end{aligned}$$

for $k = 1, 2, \ldots, m$. Since $R(\xi, \tau)$ is of order $\leqslant m - 1$ in τ, we can expand it in partial fractions with respect to $A(\xi, \tau)$:

$$\frac{R(\xi, \tau)}{A(\xi, \tau)} = \sum_{k=1}^{m} \frac{e_k(\xi)}{\tau - \tau_k(\xi)}, \qquad \xi \neq 0,$$

where

$$e_k(\xi) = [R(\xi, \tau)]_{\tau=\tau_k} \Big/ \left[\frac{\partial A(\xi, \tau)}{\partial \tau}\right]_{\tau=\tau_k}.$$

Clearly $e_k(\xi)$, $k = 1, \ldots, m$, is homogeneous of degree one. Similarly $\tau_k(\xi)$, $k = 1, \ldots, m$, is homogeneous of degree one and nowhere real. Thus

$$|e_k(\xi)| \leqslant C|\xi|,$$

$$C^{-1}|\xi| \leqslant |\operatorname{Im} \tau_k(\xi)| \leqslant C|\xi|$$

for $k = 1, \ldots, m$. Now

$$R(\xi, \tau) = \sum_{k=1}^{m} e_k(\xi) A_k(\xi),$$

$$\begin{aligned} [R(\xi, D_t)u_1(\xi, t)]^\wedge &= \sum_{k=1}^{m} e_k(\xi)[A_k(\xi, D_t)u_1(\xi, t)]^\wedge \\ &= \sum_{k=1}^{m} e_k(\xi) \frac{g_1^\wedge(\xi, \tau) - W_k u}{\tau - \tau_k(\xi)} \\ &= \frac{R(\xi, \tau)}{A(\xi, \tau)} g_1^\wedge(\xi, \tau) - \sum_{k=1}^{m} e_k(\xi) \frac{W_k u}{\tau - \tau_k(\xi)}. \end{aligned}$$

Therefore

$$|[R(\xi, D_t)u_1(\xi, t)]^\wedge| \leqslant C\left\{|g_1^\wedge(\xi, \tau)| + |\xi| \sum_{k=1}^{m} \frac{|W_k u|}{|\tau - \tau_k(\xi)|}\right\}.$$

Squaring this inequality and integrating with respect to τ gives

$$\int_0^\infty |R(\xi, D_t)u_1(\xi, t)|^2 dt$$

$$\leqslant C\left\{\int_{-\infty}^\infty |g_1{}^\wedge(\xi,\tau)|^2 d\tau + |\xi|^2 \sum_{k=1}^m |W_k u|^2 \int_{-\infty}^\infty \frac{d\tau}{|\tau - \tau_k(\xi)|^2}\right\}$$

$$\leqslant C\left\{\int_{-\infty}^\infty |g_1{}^\wedge(\xi,\tau)|^2 d\tau + |\xi| \sum_{k=1}^m |W_k u|^2\right\}.$$

Now we employ identity (10):

$$(11)\qquad \int_{-\infty}^\infty \frac{g_1{}^\wedge(\xi,\tau)}{\tau - \tau_k(\xi)}\, d\tau = \int_{-\infty}^\infty [A_k(\xi, D_t)u_1(\xi, t)]^\wedge dt$$

$$+ i(2\pi)^{-1/2} A_k(\xi, D_t)u_1(\xi, 0) \int_{-\infty}^\infty \frac{d\tau}{\tau - \tau_k(\xi)}$$

$$= (\pi/2)^{1/2} A_k(\xi, D_t)u_1(\xi, 0)$$

$$+ i(2\pi)^{-1/2} [A_k(\xi, D_t)u_1](\xi, 0)(\pi i \operatorname{sgn} \operatorname{Im} \tau_k)$$

$$= \begin{cases} -2\pi i W_k u, & k = r+1, \ldots, m, \\ 0, & k = 1, \ldots, r, \end{cases}$$

where we used the well-known formula (cf. (**14**, p. 25))

$$u_1(\xi, 0) = (2/\pi)^{1/2} \text{P.V.} \int_{-\infty}^\infty u_1{}^\wedge(\xi, \tau)\, d\tau.$$

Therefore for $k = r + 1, \ldots, m$

$$|\xi||W_k u|^2 \leqslant C|\xi| \left| \int_{-\infty}^\infty \frac{g_1{}^\wedge(\xi,\tau)}{\tau - \tau_k(\xi)}\, d\tau \right|^2$$

$$\leqslant C|\xi| \int_{-\infty}^\infty \frac{d\tau}{|\tau - \tau_k(\xi)|^2} \int_{-\infty}^\infty |g_1{}^\wedge(\xi,\tau)|^2 d\tau$$

$$\leqslant C \int_{-\infty}^\infty |g_1{}^\wedge(\xi,\tau)|^2 d\tau.$$

Hence for $|\xi| > K$

$$\int_0^\infty |R(\xi, D_t)u_1(\xi, t)|^2 dt \leqslant C\left(\int_{-\infty}^\infty |g_1{}^\wedge(\xi,\tau)|^2 d\tau + |\xi| \sum_{k=1}^r |W_k u|^2\right).$$

Recall Condition K,

$$\sum_{k=1}^r \int_{|\xi|>K} |\xi||W_k u|^2 d\xi \leqslant C||B_+{}^{(K)} u||_\Gamma^2,$$

which implies that

$$\int_{|\xi|>K} d\xi \int_0^\infty |R(\xi, D_t)u_1(\xi, t)|^2 dt$$

$$\leqslant C\left(\int_{|\xi|>K} d\xi \int_{-\infty}^\infty |g_1{}^\wedge(\xi,\tau)|^2 d\tau + ||B_+{}^{(K)}u||_\Gamma^2\right)$$

$$\leqslant C(||g||_0^2 + ||B_+{}^{(K)}u||_\Gamma^2).$$

Now recall (5), that is

$$\|B_-^{(K)}u\|_\Gamma^2 \leqslant C\|g\|_0^2.$$

Finally, using (9), we obtain

$$\begin{aligned}\|Ru\|_0^2 &\leqslant C(\|g\|_0^2 + \|B_+^{(K)}u\|_\Gamma^2 + \|u\|_0^2)\\ &= C(\|g\|_0^2 + \|(B^{(K)} - B_-^{(K)})u\|_\Gamma^2 + \|u\|_0^2)\\ &\leqslant C(\|g\|_0^2 + \|B^{(K)}u\|_\Gamma^2 + \|u\|_0^2)\\ &\leqslant C(\|g\|_0^2 + \|Bu\|_\Gamma^2 + \|u\|_0^2),\end{aligned}$$

where the last inequality follows from Lemma 4, since

$$\begin{aligned}\|B^{(K)}u\|_\Gamma &= \left\|\sum_{j=0}^{m-1} B_j'\gamma_j^{(K)}u_1\right\|_\Gamma\\ &\leqslant \|Bu\|_\Gamma + \left\|\sum_{j=0}^{m-1} B_j'(\gamma_j - \gamma_j^{(K)})u_1\right\|_\Gamma\end{aligned}$$

and

$$\begin{aligned}\left\|\sum_{j=0}^{m-1} B_j'(\gamma_j - \gamma_j^{(K)})u_1\right\|_\Gamma &\leqslant \sum_{j=0}^{m-1} \|B_j'(\gamma_j - \gamma_j^{(K)})u_1\|_\Gamma\\ &\leqslant C\sum_{j=0}^{m-1} \langle F_\xi^{-1}D_t^{\,j}(u_1 - u_1^{(K)})_{t=0}\rangle_{m-j-1/2}\\ &\leqslant C_K(\|Au\|_0 + \|u\|_0).\end{aligned}$$

This proves Theorem 1.

To show that this result is meaningful, we prove that the usual differential elliptic boundary-value problems are included in Theorem 1. For this we need the definition of covering.

Definition 3 (Schechter **(11)**). Let $A(D_x, D_t)$ be a homogeneous elliptic partial differential operator of order $m = 2r$ with constant coefficients. Suppose that for all ξ, $|\xi| = 1$, $A(\xi, \tau) = 0$ has r roots $\tau_1, \ldots, \tau_r$ with positive imaginary parts. Let $B_1, \ldots, B_r$ be a set of homogeneous differential operators of orders $m_j < m$, $j = 1, \ldots, r$, with constant coefficients. Consider (A, B) on $\overline{R_n^+}$. The system $B = (B_1, \ldots, B_r)$ is said to cover A if for all $\xi \neq 0$, $B_1(\xi, \tau), \ldots, B_r(\xi, \tau)$ are independent modulo $A_+(\xi, \tau)$.

THEOREM 2. *Let $A(D_x, D_t)$ be a homogeneous simple elliptic partial differential operator of order $m = 2r$. Let $B_1, \ldots, B_r$ be homogeneous partial differential operators with constant coefficients of orders $m_1, \ldots, m_r < m$ such that $B_1, \ldots, B_r$ cover A on $\overline{R_n^+}$. Set*

$$\Gamma = \sum_{j=1}^{r} \oplus W^{m-m_j-1/2,2}(R_{n-1}).$$

Then (A, B, Γ) satisfies Condition K for all $K > 0$.

Proof. Clearly

$$B^{(K)} = (B_1^{(K)}, \ldots, B_r^{(K)}).$$

Let $v(x, t)$ be an extension of $u(x, t)$ to all of R_n such that $v(x, t)$ is m times continuously differentiable with compact support. Set

$$B_j''(\xi, \tau) = |\xi|^{m-1-m_j} B_j(\xi, \tau),$$

$j = 1, \ldots, r$. It is easy to see that the set of polynomials $B_j''(\xi, \tau), j = 1, \ldots, r$, is also linearly independent modulo $A_+(\xi, \tau)$ for all $\xi \neq 0$.

Set

$$\frac{B_j''(\xi, \tau)}{A(\xi, \tau)} = \sum_{k=1}^{m} \frac{q_{jk}(\xi)}{\tau - \tau_k(\xi)}.$$

Then

$$\begin{aligned}
\sum_{k=1}^{m} -2\pi i q_{jk}(\xi) W_k u(\xi) &= \sum_{k=1}^{m} q_{jk}(\xi) \int_{-\infty}^{\infty} A_k(\xi, \tau) v_1{}^{\wedge}(\xi, \tau) d\tau \\
&= \sum_{k=1}^{m} q_{jk}(\xi) \int_{-\infty}^{\infty} \frac{A(\xi, \tau)}{\tau - \tau_k(\xi)} v_1{}^{\wedge}(\xi, \tau) d\tau \\
&= \int_{-\infty}^{\infty} B_j''(\xi, \tau) v_1{}^{\wedge}(\xi, \tau) d\tau \\
&= (2\pi)^{1/2} B_j''(\xi, D_t) v_1(\xi, 0) \\
&= (2\pi)^{1/2} B_j''(\xi, D_t) u_1(\xi, 0).
\end{aligned}$$

If $|\xi| > K$ for some $K > 0$, using (11), we obtain

$$\begin{aligned}
B_{j-}'' u_1(\xi) &= \int_{-\infty}^{\infty} \frac{B_j''(\xi, \tau)}{A(\xi, \tau)} g_1{}^{\wedge}(\xi, \tau) d\tau \\
&= \sum_{k=1}^{m} q_{jk}(\xi) \int_{-\infty}^{\infty} \frac{g_1{}^{\wedge}(\xi, \tau)}{\tau - \tau_k(\xi)} d\tau \\
&= \sum_{k=r+1}^{m} -2\pi i q_{jk}(\xi) W_k u(\xi).
\end{aligned}$$

Hence

$$i(2\pi)^{-1/2} B_{j+}''(\xi, D_t) u_1(\xi) = \sum_{k=1}^{r} q_{jk}(\xi) W_k u(\xi).$$

Using Schechter's **(12)** argument, now it is easy to prove Theorem 2. For any r-tuple of numbers $(w_1, \ldots, w_r)$,

$$\sum_{k=1}^{r} q_{jk} w_k = 0, \qquad 1 \leqslant j \leqslant r \Rightarrow w_1 = \ldots = w_r = 0.$$

Otherwise there exist numbers $(\lambda_1, \ldots, \lambda_r)$ such that

$$\sum_{k=1}^{r} \lambda_j q_{jk} = 0, \qquad k = 1, \ldots, r.$$

Then

$$A^{-1}\sum_{j=1}^{r} \lambda_j B_j'' = \sum_{j=1}^{r} \sum_{k=1}^{m} \frac{\lambda_j q_{jk}}{\tau - \tau_k}$$
$$= \sum_{k=r+1}^{m} \sum_{j=1}^{r} \frac{\lambda_j q_{jk}}{\tau - \tau_k},$$

which implies that

$$\sum_{j=1}^{r} \lambda_j B_j''$$

is a multiple of $A_+(\xi, \tau)$. This contradicts the assumption that $B_j, j = 1, \ldots, r$, covers A. Therefore

$$\sum_{j=1}^{r} \left| \sum_{k=1}^{r} q_{jk}(\xi) w_k \right| > C > 0$$

for all $|\xi| = 1$ and

$$\sum_{k=1}^{r} |w_k|^2 = 1.$$

Hence, using the homogeneity, we obtain

$$\sum_{k=1}^{r} |w_k|^2 \leqslant C_K \sum_{j=1}^{r} \left| \sum_{k=1}^{r} q_{jk}(\xi) w_k \right|^2$$

for all $|\xi| > K > 0$ and $(w_1, \ldots, w_r)$. Setting $w_k = W_k(\xi) = W_k\, u(\xi)$, $k = 1, \ldots, r$,

$$\sum_{k=1}^{r} |W_k(\xi)|^2 \leqslant C_K \sum_{j=1}^{r} \left| \sum_{k=1}^{r} q_{jk}(\xi) W_k(\xi) \right|^2,$$

i.e.,

$$\begin{aligned}
&\sum_{k=1}^{r} \int_{|\xi|>K} |\xi| |W_k(\xi)|^2 \, d\xi \\
&\leqslant C_K \sum_{j=1}^{r} \int_{|\xi|>K} |\xi| \left| \sum_{k=1}^{r} q_{jk}(\xi) W_k(\xi) \right|^2 d\xi \\
&= C_K \sum_{j=1}^{r} \int_{|\xi|>K} |\xi| |B_{j+}''(\xi, D_t) u_1(\xi, 0)|^2 \, d\xi \\
&= C_K \sum_{j=1}^{r} \int_{|\xi|>K} |\xi|^{2m-2m_j-1} |B_{j+}(\xi, D_t) u_1(\xi, 0)|^2 \, d\xi \\
&\leqslant C_K \sum_{j=1}^{r} \int_{|\xi|>K} (1 + |\xi|^2)^{m-m_j-1/2} |B_{j+}(\xi, D_t) u_1(\xi, 0)|^2 d\xi \\
&= C_K ||B_+^{(K)} u||_\Gamma^2,
\end{aligned}$$

where $B_+^{(K)} = (B_{1+}^{(K)}, \ldots, B_{r+}^{(K)})$. This proves Theorem 2.

3. All through this section we assume that A is a homogeneous simple

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elliptic partial differential operator. In proving Theorem 1 we showed that for all functions $u(x, t) \in C_0^\infty(\overline{R_n^+})$

$$|u^{(K)}|_m \leqslant C||(Au||_0 + ||B_+^{(K)}u||_\Gamma).$$

It is easy to see that $u_1^{(K)}(\xi, t) = 0$ for $|\xi| < K$ implies

$$||u^{(K)}||_m \leqslant C|u^{(K)}|_m, \qquad u \in C_0^\infty(\overline{R_n^+}),$$

that is

$$||u^{(K)}||_m \leqslant C(||Au||_0 + ||B_+^{(K)}u||_\Gamma) \tag{12}$$

for all $u \in C_0^\infty(\overline{R_n^+})$.

In this section we discuss Condition K. We show (Theorem 6) that Condition K is actually necessary if we want (A, B, Γ) to satisfy (12) for some $K > 0$. The idea involved is simple and follows from the formal correspondence

$$F_x[A(D_x, D_t)u] = A(\xi, D_t)u_1(\xi, t), \qquad u \in W^{m,2}(\overline{R_n^+}).$$

First we note that the mapping

$$T_K\colon C_0^\infty(\overline{R_n^+}) \to W^{m,2}(\overline{R_n^+})$$

given by

$$T_K\, u(x, t) = F_\xi^{-1}(u_1^{(K)})(x, t) = u^{(K)}(x, t)$$

is continuous in $W^{m,2}(\overline{R_n^+})$ norm. Therefore T_K can be extended by continuity to all of $W^{m,2}(\overline{R_n^+})$. Since A and $B_+^{(K)}$ are continuous maps defined on $W^{m,2}(\overline{R_n^+})$, we see that (12) holds for all $u \in W^{m,2}(\overline{R_n^+})$. Denote by $W_K^{m,2}(\overline{R_n^+})$ the closure of the subspace $\{u^{(K)}(x, t) | u \in C_0^\infty(\overline{R_n^+})\}$ in $W^{m,2}(\overline{R_n^+})$. We can write (12) as

$$||T_K\, u||_m \leqslant C(||Au||_0 + ||B_+^{(K)}u||_\Gamma), \tag{13}$$

where T_K is the projection map onto $W_K^{m,2}(\overline{R_n^+})$. Set

$$v_k(\xi, t) = V_k(\xi)\exp\{it\tau_k(\xi)\},$$

where $V_k(\xi) \in C_0^\infty(R_{n-1}^{(K)})$, $k = 1, \ldots, r$, and

$$u_k(x, t) = (2\pi)^{-(n-1)/2} \int_{R_{n-1}} \exp(ix.\xi)\, V_k(\xi) \exp\{it\tau_k(\xi)\} d\xi.$$

It is clear that $u_k(\xi, t) \in W^{m,2}(\overline{R_n^+})$. Therefore there is a sequence of functions $\phi_n(x, t) \in C_0^\infty(\overline{R_n^+})$ such that

$$||u_k - \phi_n||_m \to 0, \qquad k = 1, \ldots, r,$$

as $n \to \infty$. By taking partial Fourier transforms with respect to x we easily see that

$$||u_k - \phi_n^{(K)}||_m \leqslant ||u_k - \phi_n||_m, \qquad u = 1, 2, \ldots.$$

Hence $u_k(x, t) \in W_K^{m,2}(\overline{R_n^+})$ and we have

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$$\|u_k\|_m \leqslant C(\|Au_k\|_0 + \|B_+^{(K)}u_k\|_\Gamma), \tag{14}$$

$k = 1, \ldots, r$. Naturally we can replace u_k by

$$u = \sum_{k=1}^{r} u_k$$

in (14). Furthermore

$$A\left(\sum_{k=1}^{r} u_k\right) = 0.$$

Thus

$$\left\|\sum_{k=1}^{r} u_k\right\|_m \leqslant C\left\|B_+^{(K)}\left(\sum_{k=1}^{r} u_k\right)\right\|_\Gamma. \tag{15}$$

We also have the estimate

$$\|V_k(\xi)\exp\{it\tau_k(\xi)\}\|_m \leqslant C\langle V_k\rangle_{m-1/2}. \tag{16}$$

Let us take (16) for granted for now; we shall return to it at the end of this section. Now (16) implies that we can extend (15) by continuity to all functions

$$u = \sum_{k=1}^{r} u_k,$$

where

$$u_k(x, t) = F_\xi^{-1}[V_k(\xi)\exp\{it\tau_k(\xi)\}], \tag{17}$$

where $V_k \in W^{m-1/2,2}(R_{n-1}^{(K)})$, $k = 1, \ldots, r$. Thus we have proved

THEOREM 3. *Let* (A, B, Γ) *be a coercive boundary problem for which* (12) *holds for all* $u \in C_0^\infty(\overline{R_n^+})$. *Then*

$$\|u\|_m \leqslant C\|B_+^{(K)}u\|_\Gamma$$

for all

$$u = \sum_{k=1}^{r} u_k,$$

where $u_k(x, t)$ *is defined by* (17).

Let $u \in C_0^\infty(\overline{R_n^+})$. Consider

$$v_k^{(K)}(\xi, t) = \frac{\sqrt{2\pi}}{i} \frac{W_k u^{(K)}(\xi)}{A_k(\tau_k)} \exp\{it\tau_k(\xi)\}. \tag{18}$$

Clearly

$$\frac{W_k u^{(K)}(\xi)}{A_k(\tau_k)} \in W^{m-1/2,2}(R_{n-1}^{(K)})$$

and we have

$$W_k v_k^{(K)}(\xi) = W_k u^{(K)}(\xi).$$

Furthermore (16) implies that

$$\begin{aligned}||F_\xi^{-1} v_k^{(K)}||_m &\leqslant C \left\langle \frac{W_k u^{(K)}}{A_k(\tau_k)} \right\rangle_{m-1/2} \\ &\leqslant C \int_{|\xi|>K} |\xi|^{2m-1} |A_k(\tau_k)|^{-2} |W_k u|^2 \, d\xi \\ &\leqslant C \int_{|\xi|>K} |\xi| |W_k u|^2 \, d\xi \\ &\leqslant C ||u||_m^2,\end{aligned}$$

where we used (3). Now the above estimates imply

THEOREM 4. *The space M_K spanned by*

$$Wu^{(K)} = (W_1 u^{(K)}, \ldots, W_r u^{(K)}), \qquad u \in W^{m,2}(\overline{R_n^+}),$$

for any $K > 0$ is also spanned by

$$(W_1 u_1, \ldots, W_r u_r),$$

where

$$u_k(x, t) = F_\xi^{-1}[V_k(\xi) \exp\{it\tau_k(\xi)\}]$$

with $V_k \in W^{m-1/2,2}(R_{n-1}^{(K)})$ and the correspondence is given by (18). *Therefore we have*

$$M_K = \sum \oplus W^{1/2,2}(R_{n-1}^{(K)})$$

where the sum has r terms.

Now by the continuity of the mapping W we have

$$\langle Wu \rangle_{1/2}^2 \equiv \sum_{k=1}^{r} \langle W_k u \rangle_{1/2}^2 \leqslant C ||u||_m^2, \qquad u \in W^{m,2}(\overline{R_n^+}).$$

THEOREM 5. *The mapping W is an isomorphism of the subspace of $W^{m,2}(\overline{R_n^+})$ containing all functions*

$$u = \sum_{k=1}^{r} u_k$$

onto M_K, where u_k, $k = 1, \ldots, r$, is defined in Theorem 4.

Proof. Theorem 5 follows from formal properties of the partial Fourier transform F_x. On the other hand, since our maps are defined by extension by continuity from a dense subspace, we have to be more careful. Theorem 5 will follow from

LEMMA 5. *Let*

$$u = \sum_{k=1}^{r} u_k,$$

u_k defined in Theorem 4. *Then*

$$||u||_m \leqslant C \langle Wu \rangle_{1/2}. \tag{19}$$

Proof of Lemma 5. It suffices to prove (19) for

$$u(x, t) = u_k(x, t) = F_\xi^{-1}[V_k(\xi)\exp\{it\tau_k(\xi)\}]$$

where $V_k(\xi) \in C_0^\infty(R_{n-1}^{(K)})$, $k = 1, \ldots, r$. Now

$$W_k(V_k(\xi)\exp\{it\tau_k(\xi)\}) = i(2\pi)^{-1/2}V_k(\xi)A_k(\xi, D_t)[\exp\{it\tau_k(\xi)\}]_{t=0}$$
$$= i(2\pi)^{-1/2}V_k(\xi)A_k(\xi, \tau_k).$$

Similarly

$$\xi^\alpha D_t^{m-|\alpha|}[V_k(\xi)\exp\{it\tau_k(\xi)\}] = \xi^\alpha V_k(\xi)[\tau_k(\xi)]^{m-|\alpha|}\exp\{it\tau_k(\xi)\},$$

$$\int_0^\infty |\exp\{it\tau_k(\xi)\}|^2 dt = 2[\operatorname{Im} \tau_k(\xi)]^{-1}.$$

Therefore

$$\int_0^\infty |\xi^\alpha D_t^{m-|\alpha|}[V_k(\xi) \exp\{it\tau_k(\xi)\}]|^2\, dt$$
$$= 2|\xi|^{2|\alpha|}|V_k(\xi)|^2|\tau_k(\xi)|^{2m-2|\alpha|}[\operatorname{Im} \tau_k(\xi)]^{-1}$$
$$\leqslant C|\xi||A_k(\tau_k)|^2|V_k(\xi)|^2 \leqslant C|\xi||W_k[V_k(\xi) \exp\{it\tau_k(\xi)\}]|^2,$$

for $0 \leqslant |\alpha| \leqslant m$.

Thus integrating with respect to ξ for $|\xi| > K$ and summing over all $0 \leqslant |\alpha| \leqslant m$, we obtain

$$|u_k|_m^2 \leqslant C\langle W_k u_k\rangle_{1/2}^2.$$

Since $V_k(\xi) = 0$ for $|\xi| < K$, we have

$$||u_k||_m^2 \leqslant C|u_k|_m^2.$$

This proves Lemma 5.

Remark. Theorem 5 implies Theorem 4 with the exception of relation (18), which is quite interesting by itself.

Theorems 3 and 5 show that Condition K is actually necessary if we require that the boundary problem (A, B, Γ) satisfy (12). To be more precise we have

THEOREM 6. *Let (A, B, Γ) be a coercive boundary problem for which* (12) *holds for all $u \in C_0^\infty(\overline{R_n^+})$. Then*

$$\langle Wu\rangle_{1/2} \leqslant C||B_+^{(K)}u||_\Gamma \tag{20}$$

for all

$$u = \sum_{k=1}^r u_k,$$

where u_k, $k = 1, \ldots, r$, is defined by $u_k(x, t) = F_\xi^{-1} V_k(\xi)\exp\{it\tau_k(\xi)\}$ with $V_k(\xi) \in W^{m-1/2,2}(R_{n-1}^{(K)})$, $k = 1, \ldots, r$. Hence, by Theorem 5, (20) *holds for all $u \in C_0^\infty(\overline{R_n^+})$.*

We still have to prove (16), i.e.

$$||V_k(\xi)\exp\{it\tau_k(\xi)\}||_m \leqslant C\langle V_k\rangle_{m-1/2}$$

for $V_k \in C_0^\infty(R_{n-1}^{(K)})$. Consider

$$D_x^\alpha D_t^{m-|\alpha|}u_k(x, t) = (2\pi)^{-(n-1)/2}D_x^\alpha D_t^{m-|\alpha|}\int_{R_{n-1}} \exp(ix\cdot\xi)\, V_k(\xi)\, \exp\{it\tau_k(\xi)\}d\xi$$

$$= (2\pi)^{-(n-1)/2}\int_{R_{n-1}} \exp(ix\cdot\xi)\,(\xi^\alpha[\tau_k(\xi)]^{m-|\alpha|}\, V_k(\xi)\, \exp\{it\tau_k(\xi)\})d\xi.$$

Therefore

$$\int_0^\infty dt \int_{R_{n-1}} |D_x^\alpha D_t^{m-|\alpha|}u_k(x, t)|^2\, dx$$

$$\leqslant C\int_{R_{n-1}} |\xi|^{2|\alpha|}|\tau_k(\xi)|^{2m-2|\alpha|}|V_k(\xi)|^2\, d\xi \int_0^\infty \exp\{-2t \operatorname{Im} \tau_k(\xi)\}dt$$

$$\leqslant C\int_{R_{n-1}} |\xi|^{2|\alpha|}|\tau_k(\xi)|^{2m-2|\alpha|}|\operatorname{Im} \tau_k(\xi)|^{-1}|V_k(\xi)|^2\, d\xi$$

$$\leqslant C\langle V_k\rangle_{m-1/2}^2,$$

where we used Parseval's formula and interchanged the order of integration. Thus we have

$$|u_k|_m \leqslant C\langle V_k\rangle_{m-1/2}$$

and using $||u_k||_m \leqslant C|u_k|_m$, which is true since $V_k(\xi) \in C_0^\infty(R_{n-1}^{(K)})$, we immediately obtain (16).

Remark. When $A(\xi, \tau) = 0$ has multiple roots $\tau_k(\xi)$, the situation becomes slightly more complicated. We hope to return to that problem elsewhere.

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