A GENERALIZATION OF THE REGULAR MAPS OF TYPE $\{4, 4\}_{b, c}$ AND $\{3, 6\}_{b, c}$

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1. Introduction. In [1], Coxeter gave a complete enumeration of the regular maps on a torus. The maps consist of two families of type $\{4, 4\}_{b, c}$ and $\{3, 6\}_{b, c}$ (and their duals). b and c are non-negative integers, which determine the maps uniquely. The maps are irreflexible if and only if $bc(b - c) \neq 0$.

On surfaces of genus $h > 1$, irreflexible regular maps are rather exceptional. The simplest surface of negative characteristic which admits irreflexible regular maps is the orientable surface of genus 7. This was shown by the author [4, Theorem 3.1]. The corresponding map was discovered by J.R. Edmonds [2, p. 388].

In the present note we shall show that every regular map on a torus also leads to a family of regular maps on surfaces of higher genus. We shall denote the induced maps by $\{4(2n - 1), 4(2n - 1)\}_{b, c}$ and $\{3(2n - 1), 6(2n - 1)\}_{b, c}$ respectively. Irreflexibility of these maps will be completely characterized by the condition $bc(b - c) \neq 0$. As was shown in [4, Theorem 6.6], $N_2 = 5$ is the smallest number of polygons which is realized by irreflexible regular maps. Thus the family $\{4(2n - 1), 4(2n - 1)\}_{b, c}$ contains the simplest of all irreflexible regular maps.

Finally, F.A. Sherk [5] deduced a family of type $\{6, 6\}_{b, c}$ from the maps $\{3, 6\}_{b, c}$, forming the direct extension of the rotation group of $\{3, 6\}_{b, c}$ by a group of order 2. Using Sherk's method, we obtain regular maps of type $\{6(2n-1), 6(2n - 1)\}_{b, c}$.

2. Basic statements on regular maps. A detailed introduction of the concept of a regular map can be found in [2] or [3] (cf. also [4]). We need the following statement:

Let $[p, q]^+$ be the polyhedral group which is defined by

$$R^p = S^q = (RS)^2 = E \qquad (p, q \text{ integers } \geq 3). \tag{2.1}$$

The rotation group $\mathfrak{G}$ of a regular map $\{p, q\}$ on a closed orientable surface is a finite factor group of $[p, q]^+$. The corresponding normal subgroup $\mathfrak{h}$ of $[p, q]^+$ contains no elements of finite order. Conversely, every normal subgroup $\mathfrak{h}$ of finite index N in $[p, q]^+$, $\mathfrak{h}$ containing no elements of finite order, establishes a regular map $\{p, q\}$ on a closed orientable surface of genus

$$h = 1 + \frac{N}{2}\left(\frac{1}{2} - \frac{1}{p} - \frac{1}{q}\right). \tag{2.2}$$

The rotation group $\mathfrak{G}$ is isomorphic to $[p, q]^+/\mathfrak{h}$. Different normal subgroups lead to different regular maps.

A regular map is said to be reflexible if it has more than one vertex and more than one face and if its group of automorphisms contains an element which interchanges two vertices of an edge without interchanging the two incident faces. Otherwise the map is called irreflexible. It will be useful to translate the geometrical definition of reflexibility into a purely group-theoretical one.

LEMMA. A regular map is reflexible if and only if for every word $W(R, S) \in \mathfrak{h}$ simultaneously $W(R^{-1}, S^{-1}) \in \mathfrak{h}$.

Proof. Let $\tilde{R}$ and $\tilde{S}$ be the rotations which arise by reversal of the directions of R and S respectively. A regular map is reflexible if and only if the cosets, corresponding to R and S in $[p, q]^+/\mathfrak{h}$, fulfill the same relations as the cosets corresponding to $\tilde{R}$ and $\tilde{S}$ (Fig. 1). But we have $\tilde{R} = R^{-1}$ and $\tilde{S} = RS^{-1}R^{-1}$. This implies the assertion of the Lemma.

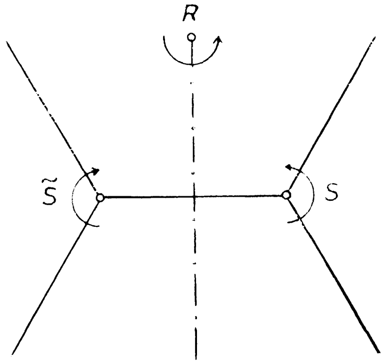

Fig. 1

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The regular maps of genus 1 are the maps $\{4, 4\}_{b, c}$ and $\{3, 6\}_{b, c}$ (and their duals). The corresponding rotation groups of order $m[b^2 + c^2 + (4 - \ell)bc]$ are given by

$$(2.3) \qquad R^{\ell} = S^m = (RS)^2 = (R^{-1}S^{\frac{m}{\ell}})^b (RS^{-\frac{m}{\ell}})^c = E$$

where b, c are non-negative integers, $(b, c) \neq (0, 0)$ and $l = 4$ or 3, $m = 4$ or 6, respectively. The maps are irreflexible if and only if $bc(b - c) \neq 0$ [3, pages 103-108].

3. <u>The regular maps</u> $\{4(2n - 1), 4(2n - 1)\}_{b, c}$, $\{3(2n - 1), 6(2n - 1)\}_{b, c}$, <u>and</u> $\{6(2n - 1), 6(2n - 1)\}_{b, c}$.

THEOREM. <u>The relations</u>

$$(3.1) \qquad \begin{aligned} R^{\ell(2n-1)} = S^{m(2n-1)} = (RS)^2 = R^m S^m = E, \\ R^{\ell n^{\ell-3}(b-c)} (R^{-1}S^{\frac{m}{\ell}})^b (RS^{-\frac{m}{\ell}})^c = E \end{aligned}$$

<u>with</u> b, c, ℓ, m <u>as in</u> (2.3) <u>and</u> n <u>a positive integer</u>, <u>define a family of regular maps</u> $\{\ell(2n - 1), m(2n - 1)\}_{b, c}$ <u>on orientable surfaces of genus</u> $h = 1 + \frac{m}{2}(n - 1)[b^2 + c^2 + (4 - \ell)bc]$. <u>The maps are irreflexible if and only if</u> $bc(b - c) \neq 0$.

<u>Proof</u>. (2.3) is a factor group of (3.1). The corresponding normal subgroup of (3.1) is cyclic and is generated by R^{ℓ}. Thus the order of the normal subgroup is at most $2n - 1$. Comparing the factor groups with respect to the commutator subgroups of (2.3) and (3.1) and using the isomorphism theorem, we see that the order of $\{R^{\ell}\}$ is exactly $2n - 1$. Hence the order of (3.1) is $m(2n - 1)[b^2 + c^2 + (4 - \ell)bc]$. Observing (2.2) and $2(\ell + m) = \ell m$, we get the first part of the assertion.

If $bc(b - c) = 0$, the condition of the Lemma is satisfied as can be seen immediately from (3.1). Hence the maps are reflexible. Let us assume that $\{\ell(2n - 1), m(2n - 1)\}_{b, c}$ is reflexible and $bc(b - c) \neq 0$ for a certain quintuple ℓ, m, n, b, c. According to the Lemma we can

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replace R and S in (3.1) by R^{-1} and S^{-1} respectively. Of course, this is also true for the factor group (2.3). But the Lemma again implies that $\{\ell, m\}_{b,c}$ is reflexible. This is a contradiction to $bc(b-c) \neq 0$.

Remark: One will look in vain for regular maps $\{\ell \cdot 2n, m \cdot 2n\}_{b,c}$ which exist for all triples b, c, n and have $2^{4-\ell}[b^2 + c^2 + (4-\ell)bc]$ faces. The case b = 2, c = 1, for instance, which is the simplest to yield an irreflexible map, cannot be realized for n = 1 as there exists no map {8, 8} of genus 6 [4, Section 7].

Using Sherk's method, as has been outlined in [5], we get two immediate consequences of the Theorem.

COROLLARY 1. The relations

$$R^{6(2n-1)} = S^{6(2n-1)} = (RS)^2 = R^6 S^6 = E,$$
(3.2)
$$(R^3, S) = (R^{-2}S^{-2})^b (R^2S^2)^c = E$$

define the rotation group of a regular map $\{6(2n-1), 6(2n-1)\}_{b,c}$ of genus $h = 1 + (6n-5)(b^2 + c^2 + bc)$. A necessary and sufficient condition for the irreflexibility of the map is $bc(b-c) \neq 0$.

COROLLARY 2. Only for (b, c) = (2, 0) and (b, c) = (1, 0) is it possible to deduce a regular map {6(2n - 1), 4} of genus $h = 1 + (6n-5)(b^2 + c^2 + bc)$ from $\{6(2n-1), 6(2n-1)\}_{b,c}$ by the process of truncation.

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