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ASYMPTOTIC RESULTS FOR THE NUMBER OF PATHS IN A GRID

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Abstract

In two recent papers, Albrecht and White ['Counting paths in a grid', *Austral. Math. Soc. Gaz.* **35** (2008), 43–48] and Hirschhorn ['Comment on "Counting paths in a grid"', *Austral. Math. Soc. Gaz.* **36** (2009), 50–52] considered the problem of counting the total number $P_{m,n}$ of certain restricted lattice paths in an $m \times n$ grid of cells, which appeared in the context of counting train paths through a rail network. Here we give a precise study of the asymptotic behaviour of these numbers for the square grid, extending the results of Hirschhorn, and furthermore provide an asymptotic equivalent of these numbers for a rectangular grid with a constant proportion $\alpha = m/n$ between the side lengths.

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1. Introduction

In the recent paper [2], Albrecht and White studied the problem of counting the number $P_{m,n}$ of lattice paths in an $m \times n$ rectangular grid starting at $(1, p)$ and ending at (m, q), for some p, q with $1 \le p \le q \le n$, where the permissible moves from (i, j) are to $(i, j + 1)$, $(i + 1, j)$ or $(i + 1, j + 1)$; throughout this paper we thus refer to 'permissible lattice paths'. These numbers arise in connection with a scheduling problem for train paths in a rail network (see [2]), where, as the authors mention, the order of magnitude of the numbers $P_{m,n}$ for large values of m and n would be of interest. An example visualizing all such paths for $m = 2$ and $n = 3$ is given in Figure 1.

By solving a bivariate recurrence via generating functions, in [2] an explicit formula for $P_{m,n}$ was obtained. Recently, Hirschhorn [6] obtained the following simpler expression for $P_{m,n}$:

$$P_{m,n} = \sum_{k\ge0} 2^k \binom{m-1}{k}\binom{n+1}{k+2}. \tag{1.1}$$

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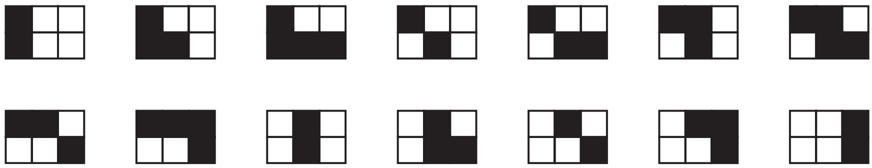

Figure 1. A visualization of all 14 permissible lattice paths for a grid of $m = 2$ rows and $n = 3$ columns; thus $P_{2,3} = 14$.

Using formula (1.1) and earlier results from [5], Hirschhorn [6] was able to describe the asymptotic behaviour of $P_{m,n}$ for the particular case $m = n$, that is, the diagonal elements, as follows:

$$P_{m,m} \sim \frac{1}{\sqrt{16\pi\sqrt{2}m}}(\sqrt{2}+1)^{2m+1} \quad \text{for } m \to \infty.$$

Based on numerical computations, he also made the following conjecture about the second-order term in the asymptotic expansion of $P_{m,m}$:

$$P_{m,m} \sim \frac{1}{\sqrt{16\pi\sqrt{2}m}}(\sqrt{2}+1)^{2m+1} \cdot \Big(1 - \frac{c_1}{m} + o(m^{-1})\Big) \quad \text{for } m \to \infty, \tag{1.2}$$

with a constant $c_1 \approx 0.824524$.

The aim of this paper is to give a more detailed study of the asymptotic behaviour of the numbers $P_{m,n}$ for large values of m and n, as it is of interest here. First, we demonstrate how to prove the conjecture (1.2) and to identify the constant c_1 for the diagonal elements $P_{m,m}$. The method applied would, at least in principle, even allow us to obtain asymptotic expansions for $P_{m,m}$ of arbitrary high order; here we restrict ourselves to stating the first three terms in the asymptotic expansion, but one could easily go further. Secondly, we provide results for the asymptotic behaviour of $P_{m,n}$ for $m, n \to \infty$, if $m = \alpha n$ with a positive constant $\alpha \in \mathbb{R}^+$, which covers the most important growth range of m and n. One could also obtain refined results and extensions to other growth ranges of m and n, but we restrict ourselves to this case, since it seems to be of most interest and we wish to avoid running into further technicalities.

We remark that there are relations to the problem of counting the number $r_{m,m}$ of ways a king can cross an $m \times m$ chessboard from the lower left-hand corner to the upper right-hand corner by using only moves to a neighbouring square either to the right or upwards or diagonally upwards to the right, which was studied by Hirschhorn in [5]. As a consequence of our computations we also obtain a refinement on the corresponding asymptotic results stated in [5].

Our results are obtained by applying complex analytic techniques, namely the so-called diagonalization method [4] and the saddle point method (see, for example, [3]), respectively, and use as a starting point the following explicit formula for the bivariate

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generating function $P(x, z) := \sum_{m,n\geq 1} P_{m,n}x^m z^n$, which was computed in [2, 6]:

$$P(x, z) = \frac{xz}{(1 - z)^2(1 - x - z - xz)}. \tag{1.3}$$

2. Asymptotic results for the diagonal elements

We apply a method introduced by Hautus and Klarner [4] in combination with singularity analysis of generating functions to obtain precise results concerning the asymptotic behaviour of the diagonal elements $P_{m,m}$, that is, the number of permissible lattice paths in an $m \times m$ square grid, for $m \to \infty$.

Let us assume that the bivariate generating function $F(x, z) := \sum_{m,n\geq 0} F_{m,n}x^m z^n$ of a sequence $F_{m,n}$ converges for all x and z such that $|z| < A$ and $|x| < B$, for arbitrary $A, B > 0$. Then it has been shown in [4] that, for all complex t with $|t| < AB$, the generating function $\hat{F}(t) := \sum_{n\geq 0} F_{n,n}t^n$ of the diagonal can be computed via the following contour integral:

$$\hat{F}(t) = \frac{1}{2\pi i}\int_C \frac{F(\frac{t}{z}, z)}{z}\,dz,$$

where the contour C is a simple closed positively oriented curve around the origin staying in the annulus $\{z \in \mathbb{C} : |t|/B < |z| < A\}$.

Considering the bivariate generating function $P(x, z)$ of the numbers $P_{m,n}$ as given in (1.3), one can immediately see that the series certainly converges for all x, z with $|x|, |z| < \frac{1}{3}$. Thus, for all complex t with $|t| < \frac{1}{12}$, the generating function $\hat{P}(t) := \sum_{m\geq 1} P_{m,m}t^m$ of the diagonal elements can be obtained by the contour integral

$$\hat{P}(t) = \frac{1}{2\pi i}\int_C \frac{P(\frac{t}{z}, z)}{z}\,dz,$$

where we can always choose as contour C a positively oriented circle around the origin with radius $\frac{3}{10}$. Plugging in (1.3), simple manipulations lead to

$$\hat{P}(t) = \frac{1}{2\pi i}\int_C \frac{P(\frac{t}{z}, z)}{z}\,dz = -\frac{1}{2\pi i}\int_C \frac{t}{(1 - z)^2(z^2 - (1 - t)z + t)}\,dz. \tag{2.1}$$

The solutions of the equation $z^2 - (1 - t)z + t = 0$ are given by

$$z_1(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2} \quad \text{and} \quad z_2(t) = \frac{1 - t + \sqrt{1 - 6t + t^2}}{2}.$$

Then $z_1(t) \to 0$ and $z_2(t) \to 1$, for $t \to 0$. Thus, for all t in a complex neighbourhood of $t = 0$, we have that in the contour integral (2.1) the only singularity enclosed by the circle C of radius $\frac{3}{10}$ is a simple pole at $z = z_1(t)$. Thus we can evaluate the contour integral (2.1) by an application of the residue theorem and obtain the following

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representation of the generating function $\hat{P}(t)$ of the diagonal elements $P_{m,m}$ valid *a priori* in a complex neighbourhood of $t = 0$, but which is uniquely given in a much larger complex domain due to analytic continuation:

$$\begin{aligned}
\hat{P}(t) &= -\operatorname*{Res}_{z=z_1(t)} \frac{t}{(1-z)^2(z-z_1(t))(z-z_2(t))} \\
&= -\frac{t}{(1-z_1(t))^2(z_1(t)-z_2(t))} \\
&= \frac{4t}{(1+t+\sqrt{1-6t+t^2})^2\sqrt{1-6t+t^2}} \\
&= \frac{2t}{(1-t)^2+(1+t)\sqrt{1-6t+t^2}} \cdot \frac{1}{\sqrt{1-6t+t^2}} \\
&= \frac{1}{\sqrt{1-6t+t^2}} \cdot \frac{(1-t)^2-(1+t)\sqrt{1-6t+t^2}}{8t} \\
&= \frac{(1-t)^2}{8t\sqrt{1-6t+t^2}} - \frac{1+t}{8t}.
\end{aligned}$$

As we have remarked earlier, there are relations between the problem considered and counting the number $r_{m,m}$ of ways a king can cross an $m \times m$ chessboard from one corner of the board to the opposite one, where only the three kinds of 'forward moves' are permissible; we now make this relation precise. It has been stated in [5, Equation (2)] that the generating function $R(t) := \sum_{m\geq 0} r_{m+1,m+1}t^m$ is given by

$$R(t) = \frac{1}{\sqrt{1-6t+t^2}},$$

which implies that $8t\hat{P}(t) = (1-t)^2R(t) - (1+t)$. Extracting coefficients gives the relation

$$P_{m,m} = \tfrac{1}{8}\Delta^2 r_{m,m} \quad \text{for } m \geq 1, \tag{2.2}$$

where, as usual, Δ denotes the forward difference operator, $\Delta f(m) := f(m+1) - f(m)$, for an arbitrary function f. This implies also that $\Delta^2 r_{m,m}$ is divisible by 8.

Since it might be of independent interest, we first deduce from $R(t)$ an asymptotic expansion for the numbers $r_{m+1,m+1}$ and then use (2.2) to show a corresponding one for $P_{m,m}$. We get $1-6t+t^2 = (t-\rho)(t-\overline{\rho}) = (1-\overline{\rho}t)(1-\rho t)$, with $\rho := 3+2\sqrt{2}$ and $\overline{\rho} := 3-2\sqrt{2}$. Thus the unique dominant singularity, that is, the singularity of smallest modulus, of $R(t)$ is at $t = \overline{\rho}$. According to singularity analysis (see [3]), the asymptotic behaviour of the coefficients of

$$R(t) = \frac{1}{\sqrt{1-\rho t}\sqrt{1-\overline{\rho}t}}$$

is determined by the local behaviour of $R(t)$ in a complex neighbourhood of the dominant singularity $t = \overline{\rho}$. Thus we expand $R(t)$ around $t = \overline{\rho}$ (that is, in terms

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of $(1-\rho t)$), where we restrict ourselves to determining the first three terms in the asymptotic expansion. We obtain

$$\begin{aligned} R(t) &= \frac{\rho}{\sqrt{1-\rho t}\sqrt{\rho^2-\rho t}} = \frac{\rho}{\sqrt{1-\rho t}\sqrt{\rho^2-1+1-\rho t}} \\ &= \frac{\rho}{\sqrt{\rho^2-1}\sqrt{1-\rho t}} \cdot \frac{1}{\sqrt{1+\frac{1-\rho t}{\rho^2-1}}} \\ &= \frac{\rho}{\sqrt{\rho^2-1}} \cdot \frac{1}{\sqrt{1-\rho t}} \cdot \Big(1 - \frac{1-\rho t}{2(\rho^2-1)} + \frac{3}{8}\frac{(1-\rho t)^2}{(\rho^2-1)^2} + O((1-\rho t)^3)\Big) \qquad (2.3) \\ &= \frac{\rho}{\sqrt{\rho^2-1}}(1-\rho t)^{-\frac{1}{2}} - \frac{1}{2}\frac{\rho}{(\rho^2-1)^{\frac{3}{2}}}(1-\rho t)^{\frac{1}{2}} + \frac{3}{8}\frac{\rho}{(\rho^2-1)^{\frac{5}{2}}}(1-\rho t)^{\frac{3}{2}} \\ &\quad + O((1-\rho t)^{\frac{5}{2}}). \end{aligned}$$

By extracting coefficients from the binomial series and applying singularity analysis, respectively, we immediately obtain from (2.3) that

$$r_{m+1,m+1} = [t^m]R(t) = \rho^{m+1}\bigg(\frac{\binom{m-\frac{1}{2}}{m}}{\sqrt{\rho^2-1}} - \frac{\binom{m-\frac{3}{2}}{m}}{2(\rho^2-1)^{\frac{3}{2}}} + \frac{3\binom{m-\frac{5}{2}}{m}}{8(\rho^2-1)^{\frac{5}{2}}} + O(m^{-\frac{7}{2}})\bigg), \qquad (2.4)$$

with

$$\rho = 3+2\sqrt{2} \quad \text{and} \quad \binom{\alpha}{m} := \frac{\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdots(\alpha-m+1)}{m!}$$

the common definition of the binomial coefficient for α real and m a nonnegative integer.

To get a final result we require an asymptotic expansion of binomial expressions $\binom{m+s}{m}$, with $s\in\mathbb{R}$ fixed, for $m\to\infty$, which can be obtained easily by using Stirling's formula for the factorials (see, for example, [1] and note that modern computer algebra systems 'know' these expansions); one gets

$$\binom{m+s}{m} = \frac{m^s}{\Gamma(s+1)}\cdot\Big(1+\frac{s(s+1)}{2m} + \frac{s(s+1)(s-1)(3s+2)}{24m^2} + O(m^{-3})\Big). \qquad (2.5)$$

The following theorem easily follows from (2.4) and (2.5) and evaluations of the Γ-function.

THEOREM 2.1. *The number $r_{m+1,m+1}$ of ways a king can cross an $(m+1)\times(m+1)$ chessboard from one corner of the board to the opposite one, where only the three kinds of 'forward moves' are permissible, admits, for $m\to\infty$, the following asymptotic expansion:*

$$r_{m+1,m+1} = \frac{(\sqrt{2}+1)^{2m+1}}{2\cdot 2^{\frac{1}{4}}\sqrt{\pi}\sqrt{m}}\cdot\Big(1+\frac{3\sqrt{2}-8}{32m} + \frac{113-72\sqrt{2}}{1024m^2} + O(m^{-3})\Big).$$

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Finally, using (2.2), Theorem 2.1 also leads, after easy computations, to an asymptotic result concerning the numbers $P_{m,m}$.

COROLLARY 2.2. *The number $P_{m,m}$ of permissible lattice paths in an $m \times m$ square grid admits, for $m \to \infty$, the following asymptotic expansion:*

$$P_{m,m} = \frac{(\sqrt{2}+1)^{2m+1}}{4 \cdot 2^{\frac{1}{4}} \sqrt{\pi}\sqrt{m}} \cdot \left(1 - \frac{c_1}{m} + \frac{c_2}{m^2} + O(m^{-3})\right),$$

with

$$c_1 = \frac{8+13\sqrt{2}}{32} \approx 0.824524 \quad \textit{and} \quad c_2 = \frac{401+312\sqrt{2}}{1024} \approx 0.822494.$$

3. Asymptotic results for rectangular grids with a constant proportion between the side lengths

We determine the asymptotic behaviour of the number of permissible paths $P_{m,n}$ in an $m \times n$ rectangular grid with a constant proportion $\alpha = m/n > 0$ between the side lengths m and n, for $n \to \infty$, via the saddle point method. From the bivariate generating function $P(x, z)$ as given in (1.3) we first obtain:

$$\begin{aligned} P_{m,n} &= [x^m z^n]P(x,z) = [x^{m-1}z^{n-1}]\frac{1}{(1-z)^2(1-z-x(1+z))} \\ &= [x^{m-1}z^{n-1}]\frac{1}{(1-z)^3(1-x\frac{1+z}{1-z})} = [z^{n-1}]\frac{(1+z)^{m-1}}{(1-z)^{m+2}}. \end{aligned}$$

Due to Cauchy's integral formula, we can write this expression as a contour integral:

$$P_{m,n} = \frac{1}{2\pi i}\int_C \frac{(1+z)^{m-1}}{z^n(1-z)^{m+2}}\,dz =: I,$$

where C is a positively oriented simple curve around the origin within a suitable complex domain, for example, within the punctured unit disc $\{z \in \mathbb{C} : 0 < |z| < 1\}$. To evaluate the integral expression I asymptotically, we choose the contour C in such a way that it passes through the saddle point $z = r'$ located on the positive real axis. If we denote the integrand of the contour integral by

$$g(z) := \frac{(1+z)^{m-1}}{z^n(1-z)^{m+2}},$$

then the saddle point satisfies $g'(r') = 0$. The resulting equation

$$g'(z) = -\frac{n(1+z)^{m-1}}{z^{n+1}(1-z)^{m+2}} + \frac{(m-1)(1+z)^{m-2}}{z^n(1-z)^{m+2}} + \frac{(m+2)(1+z)^{m-1}}{z^n(1-z)^{m+3}} = 0$$

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has the solutions

$$z_{1,2} = \frac{-(2m+1) \pm \sqrt{(2m+1)^2 + 4n(n+3)}}{2(n+3)}.$$

Thus the saddle point of interest is given by

$$r' = \frac{-(2m+1) + \sqrt{(2m+1)^2 + 4n(n+3)}}{2(n+3)}. \tag{3.1}$$

Since we have assumed that $m = \alpha n$, with a constant $\alpha > 0$, we can plug this into (3.1). After some easy computations, one gets that r' has the following asymptotic behaviour, for $n \to \infty$:

$$r' = \sqrt{1+\alpha^2} - \alpha + O(n^{-1}).$$

In the present problem it suffices that the contour C does not really pass through the saddle point r', but just passes close by. Since it simplifies the computations, we thus choose as our contour C a positively oriented circle around the origin with radius $r := \sqrt{1+\alpha^2} - \alpha$, that is,

$$C = \{z \in \mathbb{C} : z = re^{i\varphi}, 0 \le \varphi \le 2\pi\} \quad \text{with } r = \sqrt{1+\alpha^2} - \alpha.$$

The idea of the saddle point method is that the main contribution of the contour integral comes from the curve in a small neighbourhood of the saddle point. Therefore we write the integral expression as $I = I_1 + I_2$ with $I_1 := (1/2\pi i)\int_{C_1} g(z)\,dz$ and $I_2 := (1/2\pi i)\int_{C_2} g(z)\,dz$, where we split the contour C into the following two parts:

$$\begin{aligned} C_1 &:= \{z \in \mathbb{C} : z = re^{i\varphi}, -\varphi_0 \le \varphi \le \varphi_0\} \quad \text{with } \varphi_0 = \varphi_0(n) = n^{-\frac{1}{2}+\epsilon} \text{ and } \epsilon > 0,\\ C_2 &:= \{z \in \mathbb{C} : z = re^{i\varphi}, \varphi_0 < \varphi \le 2\pi - \varphi_0\}. \end{aligned}$$

We first evaluate I_1, which turns out to give the main contribution to I, whereas I_2 is asymptotically negligible. To do this we require an asymptotic expansion of $g(z)$, for $z \in C_1$, that is, for $z = re^{i\varphi}$ with $|\varphi|$ small. From $1 + z = 1 + re^{i\varphi} = r + 1 + ir\varphi - (r\varphi^2/2) + O(\varphi^3)$ we easily get

$$\begin{aligned} (1+z)^{m-1} &= (r+1)^{m-1}\Big(1 + \frac{r}{r+1}i\varphi - \frac{r\varphi^2}{2(r+1)} + O(\varphi^3)\Big)^m \cdot (1 + O(\varphi))\\ &= (r+1)^{m-1}\exp\Big(m\log\Big(1 + \frac{r}{r+1}i\varphi - \frac{r\varphi^2}{2(r+1)} + O(\varphi^3)\Big)\Big) \cdot (1 + O(\varphi))\\ &= (r+1)^{m-1}\exp\Big(m\Big(\frac{r}{r+1}i\varphi - \frac{r\varphi^2}{2(r+1)} + \frac{r^2\varphi^2}{2(r+1)^2} + O(\varphi^3)\Big)\Big) \cdot (1 + O(\varphi))\\ &= (r+1)^{m-1} \cdot \exp\Big(\frac{r}{r+1}im\varphi\Big) \cdot \exp\Big(-\frac{r}{2(r+1)^2}m\varphi^2\Big) \cdot (1 + O(\varphi) + O(m\varphi^3)). \end{aligned}$$

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Analogously one obtains from

$$1 - z = 1 - re^{i\varphi} = 1 - r - ir\varphi + \frac{r\varphi^2}{2} + O(\varphi^3)$$

the expansion

$$\frac{1}{(1-z)^{m+2}} = (1-r)^{-m-2} \cdot \exp\Big(\frac{r}{1-r} im\varphi\Big) \cdot \exp\Big(-\frac{r}{2(1-r)^2} m\varphi^2\Big) \cdot (1 + O(\varphi) + O(m\varphi^3)).$$

Together with

$$\frac{1}{z^n} = r^{-n} e^{-in\varphi},$$

we obtain the following local expansion of $g(z)$, for $z = re^{i\varphi}$ with $|\varphi|$ small:

$$\begin{aligned} g(z) = {} & \frac{(r+1)^{m-1}}{r^n (1-r)^{m+2}} \cdot \exp\Big(i\varphi\Big(\frac{r}{r+1} m + \frac{r}{1-r} m - n\Big)\Big) \\ & \cdot \exp\Big(-\frac{r}{2(r+1)^2} m\varphi^2 - \frac{r}{2(1-r)^2} m\varphi^2\Big) \cdot (1 + O(\varphi) + O(m\varphi^3)). \end{aligned}$$

Using $\alpha = m/n$ and $r = \sqrt{1+\alpha^2} - \alpha$, one can easily check that

$$\frac{r}{r+1} m + \frac{r}{1-r} m - n = 0.$$

This shows that for all $z = re^{i\varphi} \in C_1$ the following expansion holds:

$$g(z) = \frac{(r+1)^{m-1}}{r^n (1-r)^{m+2}} \cdot \exp\Big(-\frac{r}{2}\Big(\frac{1}{(r+1)^2} + \frac{1}{(1-r)^2}\Big) m\varphi^2\Big) \cdot (1 + O(n^{-\frac{1}{2}+3\epsilon})). \tag{3.2}$$

Equation (3.2) leads to the following asymptotic evaluation of the integral expression I_1, for $n \to \infty$:

$$I_1 = \frac{1}{2\pi i} \int_{C_1} g(z)\, dz \sim \frac{r}{2\pi} \frac{(r+1)^{m-1}}{r^n (1-r)^{m+2}} \int_{-\varphi_0}^{\varphi_0} \exp\Big(-\frac{r}{2}\Big(\frac{1}{(r+1)^2} + \frac{1}{(1-r)^2}\Big) m\varphi^2\Big)\, d\varphi$$

and, by using the substitution $\varphi = t/\sqrt{n}$, further to

$$\begin{aligned} I_1 & \sim \frac{1}{2\pi} \frac{(r+1)^{m-1}}{r^{n-1}(1-r)^{m+2}\sqrt{n}} \int_{-n^\epsilon}^{n^\epsilon} \exp\Big(-\frac{r}{2}\Big(\frac{1}{(r+1)^2} + \frac{1}{(1-r)^2}\Big) \frac{m}{n} t^2\Big)\, dt \\ & \sim \frac{1}{2\pi} \frac{(r+1)^{m-1}}{r^{n-1}(1-r)^{m+2}\sqrt{n}} \int_{-\infty}^{\infty} \exp\Big(-\frac{r}{2}\Big(\frac{1}{(r+1)^2} + \frac{1}{(1-r)^2}\Big) \frac{m}{n} t^2\Big)\, dt. \end{aligned} \tag{3.3}$$

The integral appearing in (3.3) can be evaluated easily by using

$$\int_{-\infty}^{\infty} e^{-qt^2}\, dt = \frac{\sqrt{\pi}}{\sqrt{q}} \quad \text{for } q > 0,$$

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leading to

$$I_1 \sim \frac{1}{2\sqrt{\pi}} \frac{(r+1)^{m-1}}{(1-r)^{m+2} r^{n-1} \sqrt{\frac{r}{2}\left(\frac{1}{(r+1)^2} + \frac{1}{(1-r)^2}\right)} \sqrt{m}}.$$

Eventually, simple manipulations yield the expression

$$I_1 \sim \frac{(r+1)^m}{2\sqrt{\pi}(1-r)^{m+1} r^{n-\frac{1}{2}} \sqrt{r^2+1}\sqrt{m}} \quad \text{for } n \to \infty.$$

To show that the remaining integral expression I_2 is asymptotically negligible one has to consider the integrand $g(z)$ for $z \in C_2$, that is, for $z = re^{i\varphi}$ with $\varphi_0 < \varphi \le 2\pi - \varphi_0$. It is easily seen that for $z \in C_2$ we get the following bound on $g(z)$:

$$|g(z)| \le \frac{|1 + re^{i\varphi_0}|^{m-1}}{r^n |1 - re^{i\varphi_0}|^{m+2}} = O\left(\frac{1}{r^n} \cdot \left|\frac{1 + re^{i\varphi_0}}{1 - re^{i\varphi_0}}\right|^m\right).$$

Via standard manipulations, which are omitted here, one can show that

$$\left|\frac{1 + re^{i\varphi_0}}{1 - re^{i\varphi_0}}\right|^m = O\left(\frac{(1+r)^m}{(1-r)^m} \cdot \exp\left(-m\varphi_0^2 \frac{r(1+r^2)}{(1-r^2)^2}\right)\right).$$

Since $\varphi_0 = n^{-\frac{1}{2}+\epsilon}$ and $m = \alpha n$, we thus obtain for all $z \in C_2$ the bound

$$|g(z)| = O\left(\frac{(1+r)^m}{r^n(1-r)^m} e^{-\beta n^{2\epsilon}}\right) \quad \text{with the constant } \beta = \frac{\alpha r(1+r^2)}{(1-r^2)^2} > 0.$$

Thus we have also

$$|I_2| = \left|\frac{1}{2\pi i} \int_{C_2} g(z)\, dz\right| = O\left(\frac{(1+r)^m}{r^n(1-r)^m} e^{-\beta n^{2\epsilon}}\right),$$

which is exponentially small compared to I_1. Therefore we get $P_{m,n} = I = I_1 + I_2 \sim I_1$, which proves the following theorem.

THEOREM 3.1. *The number $P_{m,n}$ of permissible lattice paths in an $m \times n$ rectangular grid admits, for $m = \alpha n$ with $\alpha > 0$ fixed and $n \to \infty$, the following asymptotic equivalent:*

$$P_{m,n} \sim \frac{(r+1)^m}{2\sqrt{\pi}(1-r)^{m+1} r^{n-\frac{1}{2}} \sqrt{r^2+1}\sqrt{m}}$$
$$= \frac{\sqrt{r}}{2\sqrt{\pi}\sqrt{\alpha}(1-r)\sqrt{r^2+1}\sqrt{n}} \cdot \left(\frac{(r+1)^\alpha}{(1-r)^\alpha r}\right)^n,$$

with $r = \sqrt{\alpha^2+1} - \alpha$.

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