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TWO FAMILIES OF ASSOCIATED WILSON POLYNOMIALS

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ABSTRACT. Two families of associated Wilson polynomials are introduced. Both families are birth and death process polynomials, satisfying the same recurrence relation but having different initial conditions. Contiguous relations for generalized hypergeometric functions of the type ${}_7F_6$ are derived and used to find explicit representations for the polynomials and to compute the corresponding continued fractions. The absolutely continuous components of the orthogonality measures of both families are computed. Generating functions are also given.

1. Introduction and Notation. The Wilson polynomials, introduced by James A. Wilson in his Ph.D. thesis [**39**], are the most general polynomials known of hypergeometric type. All of the classical, and many other polynomials as well, can be expressed as special or limiting cases of these polynomials. Wilson's account of some of his thesis work on the Wilson polynomials is in [**40**], and further work can be found in Askey and Wilson [**3**]. Complete references and further results are in the article of Andrews and Askey [**1**]. A q-analogue of the Wilson polynomials are called the Askey-Wilson polynomials or the ${}_4\phi_3$ polynomials. Andrews and Askey [**1**] define a sequence of polynomials to be classical if and only if it is a special case or a limiting case of the Askey-Wilson polynomials.

The Wilson polynomials are a four parameter set which we denote by

$$P_n(a,\ b,\ c,\ d;\ x) \equiv P_n(x). \tag{1.1}$$

They satisfy the three term recurrence relation

$$\begin{aligned} -xy_n &= \lambda_n y_{n+1} + \mu_n y_{n-1} - (\lambda_n + \mu_n)y_n, \\ x &:= a^2 - t^2, y_{-1} = 0,\ y_0 = 1 \\ \lambda_n &:= \frac{(n+a+b)(n+a+c)(n+a+d)(n+s-1)}{(2n+s)(2n+s-1)}, \\ \mu_n &:= \frac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+s-2)(2n+s-1)}, \\ s &:= a+b+c+d. \end{aligned} \tag{1.2}$$

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The Wilson polynomials are birth and death process polynomials with birth rates λ_n and death rates μ_n. Birth and death processes are treated in Bailey [**6**]. The connection between birth and death processes and polynomials orthogonal on a subset of $[0, \infty)$ was pointed out by Karlin and McGregor in [**19**].

Wilson [**39**] gave a closed-form expression for these polynomials,

$$(1.3) \qquad P_n(a,\ b,\ c,\ d;\ x) = {}_4F_3\left(\begin{matrix} -n, n+s-1, a-t, a+t \\ a+b, a+c, a+d \end{matrix} ;1\right)$$

and showed that when, a, b, c, d were real and positive, for example, the polynomials constituted an orthogonal set on $(-\infty,\ \infty)$ with respect to the weight function

$$(1.4) \qquad |\Gamma(a+it)\Gamma(b+it)\Gamma(c+it)\Gamma(d+it)/\Gamma(2it)|^2.$$

We view it as unlikely that the Wilson polynomials can be further generalized by the straightforward process of sticking more numerator or denominator parameters into the ${}_4F_3$ in (1.3) for, as the work of Lewanowicz, [**22**], [**23**], implies, such a hypergeometric function would undoubtedly satisfy a recurrence relation of minimal order greater than two. There are, however, other procedures available for extending the Wilson polynomials. One is to discuss q-analogues of the polynomials, and the other is to discuss what we call associated Wilson polynomials.

Askey and Wilson, see [**4**], employed the former approach and we employ the latter approach in this paper. Recall that if a system of polynomials $\{p_n(x)\}$ satisfies the three-term recurrence relation

$$(1.5) \qquad \begin{gathered} -xy_n = A_n y_{n+1} + B_n y_{n-1} - C_n y_n, \\ n = 0,\ 1,\ 2,\ \ldots,\ p_{-1} = 0,\ p_0 = 1, \end{gathered}$$

where A_t, B_t, C_t are defined for all $t > 0$, then for any $\gamma \geqq 0$, we may define the associated polynomials by the recurrence relation

$$(1.6) \qquad \begin{gathered} -xy_n = A_{n+\gamma} y_{n+1} + B_{n+\gamma} y_{n-1} - C_{n+\gamma} y_n, \\ n = 0,\ 1,\ 2,\ \ldots,\ p_{-1} = 0,\ p_0 = 1. \end{gathered}$$

The associated polynomials are always more than a laboratory curiosity. They are often vital for discussing the distribution function and spectrum of the original polynomials $p_n(x)$ and for analyzing the nature of the Padé approximants based on the continued fraction defined by (1.5). However, it is usually quite difficult to provide an explicit representation for the associated polynomials $p_n(x;\ \gamma)$ or to construct their distribution function. Even for the Jacobi polynomials, this task is formidable, see Wimp [**41**]. As the reader can well guess, the task for the Wilson polynomials, which generalize Jacobi polynomials, is even more daunting. The analysis is extremely technical, and rests on a remarkable body of nearly forgotten work associated with such names as Bailey, Orr, Whipple, all British analysts who over forty year period starting late in the last century discovered many intriguing properties of generalized hypergeometric series, particularly series of the type ${}_7F_6$ with argument 1. Their work enables us both to construct an expression for the associated Wilson polynomials and to compute the continuous component of their distribution function.

For most of our special functions, we employ the notation of Erdélyi [**11**] and Slater [**35**] and the terminology of asymptotic analysis of Olver [**28**]. We denote the associated Wilson polynomials by $P_n(a,\ b,\ c,\ d;\ x;\ \gamma) = P_n(x;\ \gamma)$. The hypergeometric function

$$ {}_{p+1}F_p \left(\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{matrix} ; 1 \right) $$

is meromorphic in its parameters. We use the above to signify the analytic continuation of the function in its parameter space in C^{2p+1}, see Section 2.

The purpose of this work is to analyze the associated Wilson polynomials. In Section 2 we derive and study bases of solutions to the associated Wilson recurrence relation. This is achieved by investigating the invariance of the associated Wilson recurrence relation. It turns out that there are two families of associated Wilson polynomials. Sections 3 and 4 contain explicit representations for both families and the corresponding weight functions. Section 5 is devoted to a general discussion about the nature of generating functions of polynomials $\{p_n(x)\}$ generated by a three term recurrence relation with coefficients which are certain rational functions of n. This is an extension of the works of Letessier and Valent [**20**], [**21**] who developed the generating function method when the coefficients in the three term recurrence relation are polynomials of degree 2. In Section 6 we apply the methods of Section 5 to derive two different generating functions for a family of associated Wilson polynomials.

The associated Askey-Wilson polynomials are currently under investigation [**17**]. The methodology of [**17**] overlaps with this work in the sense that both use contiguous relations. However [**17**] does not contain any generating functions for the associated Askey-Wilson polynomials. On the other hand [**17**] contains some representations that exhibit the polynomial character of the polynomials under consideration. The associated Askey-Wilson polynomials are orthogonal on a bounded set where general theorems exclude non-isolated discrete masses except possibly at the end points of the continuous spectrum. The situation in the case of the associated Wilson polynomials is far more complicated because of the unboundedness of the spectrum.

This work motivated Masson [**26**] to find an alternate approach to compute the weight function of one of the two families of associated Wilson polynomials considered in this paper.

2. Solutions of The Associated Wilson Recursion. For the moment let a, b, c, d be fixed positive parameters, z a complex variable. We find it convenient to use the following notation.

$$(2.1) \qquad s := a+b+c+d, \quad A(z) := \frac{(2z+s)(2z+s+1)}{(z+1)(z+b+c)(z+b+d)(z+c+d)},$$
$$C(z) := \frac{(z+s)(z+a+b+1)(z+a+c+1)(z+a+d+1)(2z+s)}{(z+1)(2z+s+2)(z+b+c)(z+b+d)(z+c+d)},$$
$$B(z) := -1 - C(z).$$

It is easily verified that the following algebraic equation is an identity in the complex variable z:

$$(2.2) \qquad A(z)\left\{\frac{k(k+2a)(-z+k-2)(z+s+k-1)}{(z+2)(z+s-1)} - \frac{k(a+b+k-1)(a+c+k-1)(a+d+k-1)}{(z+2)(z+s-1)}\right\}$$
$$- B(z)\frac{(-z+k-2)(z+s+k-1)}{(z+2)(z+s-1)} + C(z)\frac{(z+s+k)(z+s+k-1)}{(z+s-1)(z+s)}$$
$$+ \frac{(-z+k-1)(-z+k-2)}{(z+1)(z+2)} = 0.$$

Let

$$(2.3) \qquad f(z) := {}_4F_3\left(\begin{matrix} -z, z+s-1, a+t, a-t \\ a+b, a+c, a+d \end{matrix}; 1\right),$$

$$(2.4) \qquad h_k(t) := (a-t)_k(a+t)_k.$$

We first derive a second order difference equation satisfied by $f(z)$. The result is stated as Theorem 1 below. Note that

$$(2.5) \qquad h_{k+1}(t) = (a^2-t^2)h_k(t) + k(k+2a)h_k(t).$$

We multiply equation (2.2) by

$$(2.6) \qquad \frac{(-z-2)_k(z+s-1)_k}{k!(a+b)_k(a+c)_k(a+d)_k} h_k(t)$$

and sum over k from $k = 0$ to ∞. The second, third, and fourth sums converge, and give

$$(2.7) \qquad f(z) + B(z)f(z+1) + C(z)f(z+2).$$

For the first sum, we get

$$(2.8)\quad A(z)\lim_{N\to\infty}\left\{\sum_{k=0}^{N}\frac{(-z-1)_{k-1}(z+s)_{k-1}}{\Gamma(k)(a+b)_{k-1}(a+c)_{k-1}(a+d)_{k-1}}h_k(t)\right.$$
$$\left.-\sum_{k=0}^{N}\frac{k(k+2a)(-z-1)_k(z+s)_k}{k!(a+b)_k(a+c)_k(a+d)_k}h_k(t)\right\}$$
$$=A(z)\lim_{N\to\infty}\left\{\sum_{k=0}^{N-1}\frac{(-z-1)_k(z+s)_kh_{k+1}(t)}{k!(a+b)_k(a+c)_k(a+d)_k}\right.$$
$$\left.-\sum_{k=0}^{N}\frac{k(k+2a)(-z-1)_k(z+s)_kh_k(t)}{k!(a+b)_k(a+c)_k(a+d)_k}\right\}$$
$$=A(z)\lim_{N\to\infty}\left\{\sum_{k=0}^{N-1}\frac{(-z-1)_k(z+s)_k[h_{k+1}(t)-k(k+2a)h_k(t)]}{k!(a+b)_k(a+c)_k(a+d)_k}\right.$$
$$\left.-\frac{N(N+2a)(-z-1)_N(z+s)_Nh_N(t)}{N!(a+b)_N(a+c)_N(a+d)_N}\right\}.$$

Using Stirling's formula on the last term and the relationship (2.4) in the sum gives

$$(2.9)\quad A(z)(a^2-t^2)f(z+1)-A(z)\Gamma\left[\begin{matrix}a+b,a+c,a+d\\ -z-1,z+s,a+t,a-t\end{matrix}\right].$$

In (2.9) we used Slater's notation for products of gamma functions. We have thus proved the following theorem.

THEOREM 1. *For all but isolated values of a, b, c, d, z, the function f(z) in* (2.3) *satisfies the second order non-homogeneous difference equation*

$$(2.10)\quad f(z)+[A(z)(a^2-t^2)+B(z)]f(z+1)+C(z)f(z+2)=G(z),$$
$$G(z):=\frac{-(2z+s)(2z+s-1)}{(b+c+z)(b+d+z)(c+d+z)}\Gamma\left[\begin{matrix}a+b,a+c,a+d\\ -z,z+s,a+t,a-t\end{matrix}\right].$$

Observe that when $z=n$, a non-negative integer, $G(z)=0$ and the recurrence relation is precisely the one satisfied by the Wilson polynomials. However, when $z=n+\gamma$ and γ is not a non-negative integer, then (2.10) does not yield the homogeneous recurrence relation for the associated Wilson polynomials. Our task (and it is not a trivial one) in this section is, from the above considerations, to find a basis of solutions for the latter recurrence relation.

It will simplify matters to introduce the functions

$$(2.11)\quad F(z):=\phi(z)f(z),\ \phi(z):=(a+b)_z(a+c)_z(a+d)_z/(z+s-1)_z$$

where

$$(2.12)\qquad (u)_\alpha := \Gamma(u+\alpha)/\Gamma(u).$$

Therefore $F(z)$ satisfies the non-homogeneous recurrence relation

$$(2.13)\qquad \begin{aligned} &\mathcal{L}[F(z)] = H(z),\ H(z) := \phi(z)G(z),\\ &\mathcal{L}[F(z)] := E^0 + [\alpha(z)(a^2-t^2)+\beta(z)]E^1 + \gamma(z)E^2, \end{aligned}$$

where E^k is the shift operator and

$$(2.14)\qquad \begin{aligned} \alpha(z) &:= (2z+s-1)_2 A(z)/[(z+s-1)(a+b+z)(a+c+z)(a+d+z)]\\ \beta(z) &:= (2z+s-1)_2 B(z)/[(z+s-1)(a+b+z)(a+c+z)(a+d+z)]\\ \gamma(z) &:= (2z+s-1)_4 C(z)/[(z+s-1)_2(a+b+z)_2\\ &\qquad (a+c+z)_2(a+d+z)_2] \end{aligned}$$

Now let $J(z)$ be any function of z, a, b, c, d, and define

$$(2.15)\qquad \hat{J}(z) = J(z) \text{ with } a,\ b, \text{ and } z \text{ replaced by } 1-b,\ 1-a, \text{ and } a+b+z-1, \text{ respectively.}$$

Thus "^" defines an operator. A straightforward (but very tedious) computation shows that $\mathcal{L}$ is invariant under the operator ^, that is

$$(2.16)\qquad \hat{\mathcal{L}} = \mathcal{L}.$$

Applying ^ to both sides of the equation in (2.13) gives

$$(2.17)\qquad \hat{\mathcal{L}}[\hat{F}(z)] = \hat{H}(z),$$

$$(2.18)\qquad \mathcal{L}[\hat{F}(z)] = -\frac{\sin[\pi(a+b+z)]\Gamma(a+t)\Gamma(a-t)}{\sin(\pi z)\Gamma(1-b+t)\Gamma(1-b-t)}H(z).$$

Since the function $m(z)$

$$(2.19)\qquad m(z) := \sin[\pi(a+b+z)]/\sin(\pi z)$$

has period 1, we may move $1/m(z)$ inside the $\mathcal{L}$ operator to obtain

$$(2.20)\qquad \mathcal{L}\left\{\frac{\sin(\pi z)\Gamma(1-b+t)\Gamma(1-b-t)}{\sin[\pi(a+b+z)]\Gamma(a+t)\Gamma(a-t)}\hat{F}(z)\right\} = -H(z).$$

Adding equations (2.13) and (2.20) gives

$$(2.21)\qquad \mathcal{L}[Y(z)] = 0,\ Y(z) := F(z) + \frac{\sin(\pi z)\Gamma(1-b+t)\Gamma(1-b-t)}{\sin[\pi(a+b+z)]\Gamma(a+t)\Gamma(a-t)}\hat{F}(z).$$

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It turns out that, by using known transformations and connecting formulas for ${}_7F_6$'s and ${}_4F_3$'s, $Y(z)$ may be expressed in terms of a ${}_7F_6$.

Here and in what follows, we use the notation

$$(2.22)\quad W(a;\, c,\, d,\, e, f,\, g) = {}_7F_6\left(\begin{matrix} a, 1+a/2, c, d, e, f, g \\ a/2, 1+a-c, 1+a-d, 1+a-e, 1+a-f, 1+a-g \end{matrix} ; 1\right)$$

for a very well-poised ${}_7F_6$. The W notation, due to Bailey [**7**], has many advantages. It is easier to type and print and exhibits the essential parameters in the very well-poised ${}_7F_6$ series. Using the formula (4) on page 29 of Bailey [**7**], we find $Y(z)$ may be expressed in terms of a single function, $W(2a+c+d-1+z;\ a+c+z,\ a+d+z,\ a+t,\ a-t,\ z+s-1)$ if we make the identification

$$(2.23)\quad A = 2a+c+d-1+z,\ C = a+c+z,\ D = a+d+z,$$
$$E = a+t,\ F = a-t,\ G = z+s-1.$$

(We will always use capitals to denote Bailey's parameters.)

Referring back to the original equation (2.10) we see that

$$(2.24)\quad f_1(z) := \Gamma\left[\begin{matrix} 2a+c+d+z, c+d+z \\ a+c+d+z+t, a+c+d-t \end{matrix}\right] W(2a+c+d-1+z;$$
$$a+c+z,\ a+d+z,\ z+s-1,\ a+t,\ a-t)$$

is a solution of the difference equation

$$(2.25)\quad f(z) + [A(z)(a^2-t^2) + B(z)]f(z+1) + C(z)f(z+2) = 0.$$

Note that if the ${}_7F_6$ in (2.24) is interpreted as a series, then W will converge only when Re $z < 1-$ Re $(a+b)$. However, a transformation of Bailey [**7**, (1), p. 62] with $A = 2a+c+d+z-1,\ C = a+t,\ D = a-t,\ E = a+c+z,\ F = a+d+z,$ produces a ${}_7F_6$ convergent for Re $z > -$ Re $(c+d)$. We do not necessarily assume any ${}_{p+1}F_p(1)$ makes sense as a convergent series, rather, we use ${}_{p+1}F_p(1)$ to indicate the analytic continuation of that function in its parameter space. This continuation will exist provided that none of the denominator parameters nor the quantity

$$(2.26)\quad \sum \text{(denominator parameters)} - \sum \text{(numerator parameters)},$$

is a negative integer or zero, see Wimp [**42**] .

Other linearly independent solutions of the equation (2.25) may be obtained by interchanging b and c or b and d in $f_1(z)$.

By letting $z = n+\gamma$, we arrive at Theorem 2.

THEOREM 2. *The associated Wilson recurrence relation*

$$(2.27)\quad -xy_n = \lambda_n y_{n+1} + \mu_n y_{n-1} - (\lambda_n + \mu_n) y_n,\ x := a^2 - t^2,$$

$$\lambda_n := \frac{(n+\gamma+a+b)(n+\gamma+a+c)(n+\gamma+a+d)(n+\gamma+s-1)}{(2n+2\gamma+s-1)(2n+2\gamma+s)},$$

$$\mu_n := \frac{(n+\gamma)(n+\gamma+b+c-1)(n+\gamma+b+d-1)(n+\gamma+c+d-1)}{(2n+2\gamma+s-1)(2n+2\gamma+s-2)},$$

has the pairwise linearly independent solutions

$$(2.28)\quad u_n := \Gamma\left[\begin{matrix}2a+c+d+n+\gamma,\ c+d+n+\gamma,\ a+b,\ 1-b+t,\ 1-b-t\\ a+c+d+n+\gamma+t,\ a+c+d+n+\gamma-t,\ a+1-b\end{matrix}\right]$$
$$\times \frac{\sin(\pi(a+b))}{\pi}$$
$$\times W(2a+c+d+n+\gamma-1;\ a+c+n+\gamma,\ a+d+n+\gamma,\ n+\gamma+s-1,\ a+t,\ a-t)$$

$$v_n := u_n\big|_{b\leftrightarrow c},\ w_n := u_n\big|_{b\leftrightarrow d}.$$

The associated Wilson polynomials $P_n(a, b, c, d; x; \gamma) \equiv P_n(x; \gamma)$ satisfy the above recurrence with

$$(2.29)\quad P_{-1}(x; \gamma) = 0,\ P_0(x; \gamma) = 1.$$

We shall demonstrate the linear independence of the functions u_n, v_n, w_n later.

It will be useful to write out u_n explicitly as a sum of ${}_4F_3$'s that is

$$(2.30)\quad u_n = {}_4F_3\left(\begin{matrix}-n-\gamma,\ n+\gamma+s-1,\ a+t,\ a-t\\ a+b,\ a+c,\ a+d\end{matrix};1\right)$$
$$+\frac{\sin(\pi\gamma)}{\sin(\pi(a+b+\gamma))}$$
$$\times\Gamma\left[\begin{matrix}n+\gamma+1,\ 1-b+t,\ 1-b-t,\ a+b,\ a+c,\ a+d,\ n+\gamma+c+d\\ a+t,\ a-t,\ n+\gamma+a+b,\ 2-a-b,\ n+\gamma+s-1,\ 1-b+c,\ 1-b+d\end{matrix}\right]$$
$$\times {}_4F_3\left(\begin{matrix}1-n-\gamma-a-b,\ n+\gamma+c+d,\ 1-b+t,\ 1-b-t\\ 2-a-b,\ 1-b+c,\ 1-b+d\end{matrix};1\right).$$

The associated Wilson polynomials may then be written

$$(2.31)\quad P_n(a, b, c, d; x; \gamma) = \frac{u_{-1}v_n - v_{-1}u_n}{u_{-1}v_0 - v_{-1}u_0}.$$

For certain values of the parameter γ, the above polynomials reduce to Wilson polynomials, $P_n(a, b, c, d; x)$. First, obviously, we have

$$(2.32)\quad P_n(a, b, c, d; x; 0) = P_n(a, b, c, d; x).$$

Next consider the case when $\gamma \longrightarrow 1 - a - b$. In this case u_{-1} contains a factor with a denominator of $\sin(\pi(a+b+\gamma))\Gamma(\gamma+a+b-1)$, which is not zero. Multiplying both numerators and denominators of (2.31) by $\sin(\pi(a+b+\gamma))$ and taking the limit gives

$$(2.33)\quad P_n(a, b, c, d; x; 1-a-b) = \frac{(c+d+1-a-b)_n(2-a-b)_n}{n!(c+d)_n} \times P_n(1-b, 1-a, c, d; y),$$
$$x = \sqrt{a^2 - t^2},\ y = \sqrt{(1-b)^2 - t^2}.$$

By interchanging b and c or b and d, one obtains similar expressions in the cases $\gamma = 1 - a - c$, and $\gamma = 1 - a - d$. Observe the $P_n(a, b, c, d; x; \gamma)$ is symmetric in b, c, d but it is not fully symmetric in the four parameters a, b, c, d. However the polynomials

$$(2.34)\quad (a+b+\gamma)_n(a+c+\gamma)_n(a+d+\gamma)_n P_n(a, b, c, d; x; \gamma)$$

are fully symmetric in a, b, c, and d. Using this fact provides an evaluation of $P_n(a, b, c, d; x; \gamma)$ for $\gamma = 1-b-c,\ 1-b-d$ or $1-c-d$.

The next step is to determine the asymptotic behavior of the associated Wilson polynomials for large n and fixed x in order to determine the weight function. Unexpectedly, determining the asymptotic behavior of u_n, v_n, and w_n turned out to be a trivial job. The asymptotics follow from (3) on page 62 in Bailey [**7**] with the following choices.

$$(2.35)\quad A = 2a+c+d+n+\gamma-1,\ C = n+\gamma+s-1,\ D = a+c+n+\gamma,$$
$$E = a+d+n+\gamma,\ F = a-t,\ G = a+t.$$

The ${}_4F_3$'s which arise have only two denominator parameters which depend on n. Both denominator parameters are $O(n)$ as $n \longrightarrow \infty$. Thus we get

$$(2.36)\quad u_n = \frac{\sin(\pi(a+b))\sin(\pi(\gamma+b+t))}{\pi\sin(\pi(a+b+\gamma))} \times \Gamma\left[\begin{matrix} a+b,\ a+c,\ a+d,\ 1-b-t,\ 2t,\ c+d+n+\gamma,\ b+t+n+\gamma \\ a+t,\ c+t,\ d+t,\ a+b+n+\gamma,\ a+c+d+n+\gamma-t \end{matrix}\right]$$
$$\times \{1 + O(n^{-2})\} - (\text{the same quantity with } t \text{ and } -t \text{ interchanged}),$$

and similarly for v_n and w_n.

Note that these asymptotics agree with those predicted by the Birkhoff-Trjitzinsky theory, see Wimp [**43**], which gives

$$(2.37)\quad K_1 n^{-2a+2t}\left[1+\frac{C_1}{n}+\frac{C_2}{n^2}+\cdots\right]+K_2 n^{-2a-2t}\left[1+\frac{D_1}{n}+\frac{D_2}{n^2}+\cdots\right],$$

as the asymptotic form of any solution of (2.27).

3. Explicit Representations for P_n and the Weight function. Let

$$(3.1)\quad \Delta_n := u_n v_{n+1} - v_n u_{n+1}, \quad n=-1,\ 0,\ 1,\ 2,\ \ldots .$$

Then (2.31) reads

$$(3.2)\quad P_n(x;\ \gamma) = (u_{-1}v_n - v_{-1}u_n)/\Delta_{-1}, \quad n=-1,\ 0,\ 1,\ 2,\ \ldots .$$

Note that

$$(3.3)\quad u_{n-1}(\gamma+1) = u_n(\gamma) \quad \text{and } v_{n-1}(\gamma+1) = v_n(\gamma).$$

We wish to calculate Δ_n and determine its asymptotic behavior as $n \longrightarrow \infty$. Rewriting equation (2.27) with $y_n = u_n$ and $y_n = v_n$ gives

$$(3.4)\quad \begin{aligned} -xu_n &= \lambda_n u_{n+1} + \mu_n u_{n-1} - (\lambda_n+\mu_n)u_n \quad \text{and} \\ -xv_n &= \lambda_n v_{n+1} + \mu_n v_{n-1} - (\lambda_n+\mu_n)v_n. \end{aligned}$$

Multiplying the first equation in (3.4) by v_n, the second by u_n and subtracting shows that

$$(3.5)\quad \Delta_n/\Delta_{n-1} = \mu_n/\lambda_n, \quad n=0,\ 1,\ \ldots ,$$

or

$$(3.6)\quad \begin{aligned} \Delta_n &= \Delta_0 \prod_{j=0}^{n-1} \frac{\mu_{j+1}}{\lambda_{j+1}} \\ &= \frac{\Delta_0(\gamma+1)_n(\gamma+b+c)_n(\gamma+b+d)_n(\gamma+c+d)_n(\gamma+1+s/2)_n}{(\gamma+a+b+1)_n(\gamma+a+c+1)_n(\gamma+a+d+1)_n(\gamma+s)_n(\gamma+s/2)_n}. \end{aligned}$$

Thus we have established

$$(3.7)\quad \begin{aligned} \Delta_n &= \Delta_0 E n^{-4a-1}[1+O(n^{-1})], \quad \text{as } n \longrightarrow \infty, \\ E &= (\gamma+s/2)^{-1}\Gamma\left[\begin{matrix} \gamma+a+b+1,\ \gamma+a+c+1,\ \gamma+a+d+1,\ \gamma+s \\ \gamma+1,\ \gamma+b+c,\ \gamma+b+d,\ \gamma+c+d \end{matrix}\right]. \end{aligned}$$

We now put the asymptotic estimates (2.36) for u_n and v_n into the expression (3.1) for Δ_n and compare with (3.7) to terms $O(n^{-4a-1})$ using

$$(3.8)\qquad \frac{\Gamma(\alpha+n)}{\Gamma(\beta+n)} = n^{\alpha-\beta}\left\{1+\frac{(\alpha-\beta)(\alpha+\beta-1)}{2n}+O(n^{-2})\right\}, \text{ as } n\longrightarrow\infty.$$

This gives, after a very tiresome manipulation of trigonometric identities, the value

$$(3.9)\qquad \Delta_n = \frac{2\pi\sin(\pi(a+c))\sin(\pi\gamma)\sin(\pi(b+c+\gamma)) \times\sin(\pi(b-c))\sin(\pi(a+b))(n+\gamma+s/2)}{\sin(\pi(b+t))\sin(\pi(b-t))\sin(\pi(c+t)) \times\sin(\pi(c-t))\sin(\pi(a+b+\gamma))\sin(\pi(a+c+\gamma))}$$

$$\times\,\Gamma\left[\begin{matrix} a+b,\ a+b,\ a+c,\ a+c,\ a+d,\ a+d, \\ n+\gamma+1,\ n+\gamma+b+c,\ n+\gamma+b+d,\ n+\gamma+c+d \\ a+t,\ a-t,\ b+t,\ b-t,\ c+t,\ c-t,\ d+t,\ d-t,\ n+\gamma \\ +a+b+1, n+\gamma+a+c+1,\ n+\gamma+a+d+1,\ n+\gamma+s \end{matrix}\right].$$

Thus for general t, u_n and v_n are linearly independent solutions of the associated Wilson recurrence relation provided that none of the following are integers

$$(3.10)\qquad \gamma,\ b-c,\ b+c+\gamma.$$

(Note that if any of $a+b$, $a+c$, $a+b+\gamma$ or $a+c+\gamma$ is an integer then u_n and v_n may be multiplied by suitable constants to obtain linearly independent solutions.)

Let us look at the moment problem related to $P_n(x;\ \gamma)$. The recurrence relation satisfied by the monic associated Wilson polynomials $\{f_n(x)\}$ is

$$(3.11)\qquad f_{n+1}(x) = [x-(\lambda_n+\mu_n)]f_n(x) - \mu_n\lambda_{n-1}f_{n-1}(x).$$

The coefficients $\lambda_n+\mu_n$ and $\lambda_{n-1}\mu_n$ in the above recursion do not form bounded sequences. Thus the spectrum is unbounded, Chihara [**10**, Theorem 2.2 p. 109]. Using the notation of Erdélyi et al [**11**, vol. 2, 10.3] we let k_n denote the coefficient of x^n in $P_n(x;\ \gamma)$, and h_n denote the normalization constant, that is

$$(3.12)\qquad \phi_n(x;\ \gamma) := P_n(x;\ \gamma)/\sqrt{h_n}$$

is an orthonormal set. The relations in [**11**, vol. 2, 10.3] give

$$(3.13)\qquad \frac{k_{n+1}}{k_n} = -\frac{1}{\lambda_n},\quad \frac{h_n}{h_{n-1}} = \frac{\mu_n}{\lambda_{n-1}} = \frac{\lambda_n\Delta_n}{\lambda_{n-1}\Delta_{n-1}},\qquad n=1,\ 2,\ \ldots,$$

so

$$(3.14)\qquad h_n = \frac{\Delta_n\lambda_n}{\Delta_0\lambda_0} = \frac{\lambda_n}{\lambda_0}\prod_{j=1}^{n}\frac{\mu_j}{\lambda_j},\qquad n=0,\ 1,\ 2,\ \ldots.$$

Therefore

$$(3.15) \quad h_n \approx \text{(non-zero constant) } n^{1-4a}, \text{ as } n \longrightarrow \infty.$$

The recurrence relation satisfied by $\phi_n(x;\gamma)$ is

$$(3.16) \quad x\phi_n(x;\gamma) = \sqrt{\lambda_n\mu_{n+1}}\phi_{n+1}(x;\gamma) + \sqrt{\lambda_{n-1}\mu_n}\phi_{n-1}(x;\gamma) - (\lambda_n+\mu_n)\phi_n(x;\gamma), \quad n = 0, 1, 2, \dots .$$

The moment problem corresponding to the polynomials $\{\phi_n(x;\gamma)\}$ is determined if for some value of x, real or complex,

$$(3.17) \quad \sum_{n=0}^{\infty} |\phi_n(x;\gamma)|^2 = \infty,$$

see Shohat and Tamarkin [**34**, p. 50]. Now $P_n(0;\gamma)$ satisfies

$$(3.18) \quad P_{n+1}(0;\gamma) = (1+\mu_n/\lambda_n)P_n(0;\gamma) - (\mu_n/\lambda_n)P_{n-1}(0;\gamma), \quad n = 0, 1, 2, \dots .$$

By induction we find

$$(3.19) \quad P_n(0;\gamma) = \sum_{j=0}^{n}\prod_{r=0}^{j-1}\frac{\mu_r}{\lambda_r}, \quad n = 0, 1, 2, \dots .$$

Here we follow the usual convention that empty sum is zero but an empty product is 1. The sequence $\{P_n(0;\gamma)\}$ converges since

$$(3.20) \quad \prod_{r=0}^{j-1}\{\mu_r/\lambda_r\} = O(j^{-4a-1}), \text{ as } j \longrightarrow \infty$$

and the constant in O in (3.20) is positive. This and (3.12) and (3.15) show that

$$(3.21) \quad |\phi_n(0;\gamma)|^2 \approx \text{(positive constant)} n^{4a-1}, \quad \text{as } n \longrightarrow \infty.$$

Thus the sum (3.17) diverges for $x = 0$. Therefore the Hamburger moment problem is determined and the distribution function $d\psi(x;\gamma)$ may be found by inverting the Stieltjes transform

$$(3.22) \quad F(z) = \int_{-\infty}^{\infty}\frac{d\psi(x;\gamma)}{z-x}, \ z \notin (-\infty, \infty),$$

where

$$F(z) = \lim_{n\to\infty} \frac{P_{n-1}(z;\ \gamma+1)}{P_n(z;\ \gamma)}. \tag{3.23}$$

The numerator and denominator inside the limit in (3.23) are the numerator and denominator convergents of the related continued fraction, see [**11**, vol. 2, 10.5].

We set

$$t = (a^2 - x)^{1/2}. \tag{3.24}$$

The branch of the square root in (3.24) is chosen in the following manner.

$$\sqrt{a^2 - x} = |a^2 - x|^{1/2} \exp\left(\frac{i}{2}(\phi - \pi)\right),\ 0 < \phi < 2\pi. \tag{3.25}$$

Thus for x off the cut $[a^2,\ \infty)$, we have

$$\operatorname{Re}\{(a^2 - x)^{1/2}\} > 0 \tag{3.26}$$

and the dominant contribution to both u_n and v_n is of the order n^{2t-2a}. By an application of (3.3) and (3.5) we establish the relationship (3.27).

$$\begin{aligned} \frac{P_{n-1}(z;\ \gamma+1)}{P_n(z;\ \gamma)} &= \frac{\Delta_{-1}(\gamma)[u_{-1}(\gamma+1)v_{n-1}(\gamma+1) - v_{-1}(\gamma+1)u_{n-1}(\gamma+1)]}{\Delta_{-1}(\gamma+1)[u_{-1}(\gamma)v_n(\gamma) - v_{-1}(\gamma)u_n(\gamma)]} \\ &= \frac{\lambda_0}{\mu_0}\left\{\frac{u_0(\gamma)v_n(\gamma) - v_0(\gamma)u_n(\gamma)}{u_{-1}(\gamma)v_n(\gamma) - v_{-1}(\gamma)u_n(\gamma)}\right\}. \end{aligned} \tag{3.27}$$

The coefficient of n^{2t-2a} in the asymptotic development of the denominator of (3.27) can be determined from (2.36). It is a difference of ${}_7F_6$'s, and may be written as a single ${}_7F_6$. To see this it is necessary to use some very technical formulas due to Whipple, [**37**]. The relations needed connect series what Whipple calls $G_p(0)$, $G_p(1)$, $G_p(2)$. Specifically, we use (3.4), on page 341 in Whipple [**37**] with the choices

$$\begin{aligned} A &= 1/2 - 3x_0 + x_1 + x_2 + x_3 + x_4 + x_5, \\ B &= 1/2 - x_0 - x_1 + x_2 + x_3 + x_4 + x_5, \\ C &= 1/2 - x_0 + x_1 - x_2 + x_3 + x_4 + x_5, \\ D &= 1/2 - x_0 + x_1 + x_2 - x_3 + x_4 + x_5, \\ E &= 1/2 - x_0 + x_1 + x_2 + x_3 - x_4 + x_5, \\ F &= 1/2 - x_0 + x_1 + x_2 + x_3 + x_4 - x_5, \end{aligned} \tag{3.28}$$

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where

$$(3.29)\quad \begin{aligned} &A = a + c - b - t,\ B = a - t, C = a + c + \gamma - 1,\ D = c - t,\\ &E = 2 - b - d - \gamma,\ F = 1 - b - t,\ S = 1 - d - t, \end{aligned}$$

and we made the choices

$$(3.30)\quad \begin{aligned} G_p(0) &= \psi(B + C - S,\ B,\ C,\ 1 + A - E - F,\\ &\qquad 1 + A - D - F,\ 1 + A - D - E),\\ G_p(1) &= \psi(2B + C - A - S,\ B,\ B + C - A,\\ &\qquad 1 + A - E - F, 1 + A - D - F,\ 1 + A - D - E),\\ G_p(2) &= \psi(1 + C - B - D,\ 1 - D,\ 1 - B,\\ &\qquad C + E - A,\ C + F - A,\ 1 + A - B - D). \end{aligned}$$

The ψ functions are W functions multiplied by suitable gamma functions, see Whipple [37]. The result is that the coefficient of n^{2t-2a} is

$$(3.31)\quad \begin{aligned} &\frac{M}{2t}\Gamma\left[\begin{matrix} 1 - b + t,\ 1 - b - t,\ 1 - c + t,\ 1 - c - t \\ a + t,\ a - t,\ d + t,\ d - t,\ b + t + \gamma,\ c + t + \gamma,\ d + t + \gamma \end{matrix}\right]\\ &\qquad \times W(\gamma + 2t;\ \gamma,\ 1 - a + t,\ 1 - b + t,\ 1 - c + t,\ 1 - d + t), \end{aligned}$$

where M is independent of t. The coefficient of n^{2t-2a} in the numerator of (3.27) is the above expression (3.31) with γ replaced by $\gamma+1$, because of the periodicity in γ of the sine functions occurring in (2.36). The final result is the following limiting relation.

$$(3.32)\quad \lim_{n\to\infty} \frac{P_{n-1}(z;\ \gamma + 1)}{P_n(z;\ \gamma)} := F(z),$$

where

$$(3.33)\quad \begin{aligned} F(z) &= \frac{(\gamma + 2t + 1)}{(a + \gamma + t)(b + \gamma + t)(c + \gamma + t)(d + \gamma + t)}\\ &\qquad \times \frac{W(\gamma + 2t;\ \gamma,\ 1 - a + t,\ 1 - b + t,\ 1 - c + t,\ 1 - d + t)}{W(\gamma + 2t - 1;\ \gamma - 1,\ 1 - a + t,\ 1 - b + t,\ 1 - c + t,\ 1 - d + t)},\\ z &= a^2 - t^2. \end{aligned}$$

First we compute the absolutely continuous component of the measure $d\psi$. Using the inversion formula for the Stieltjes transform, see [**34**], we obtain

$$(3.34)\quad \psi'(x;\ \gamma) = \frac{1}{2\pi i}\,[F^- - F^+],$$

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where F^+ and F^- are the respective values of F above and below the cut $[a^2, \infty)$. This implies

$$(3.35)\quad \begin{aligned}\psi'(x;\gamma) &= \frac{i}{2\pi}\Bigg[\frac{(1+\gamma+2i\tau)}{(a+\gamma+i\tau)(b+\gamma+i\tau)(c+\gamma+i\tau)(d+\gamma+i\tau)} \\ &\times \frac{W(\gamma+2i\tau,\ \gamma,\ 1-a+i\tau,\ 1-b+i\tau,\ 1-c+i\tau,\ 1-d+i\tau)}{W(\gamma-1+2i\tau;\ \gamma-1,\ 1-a+i\tau,\ 1-b+i\tau,\ 1-c+i\tau,\ 1-d+i\tau)} \\ &- (\text{the same with } \tau \text{ replaced by } -\tau)\Bigg],\end{aligned}$$

where

$$\tau := \sqrt{x-a^2}.$$

Now consider the functions

$$(3.36)\quad r_n(t) := \Gamma\begin{bmatrix} n+\gamma,\ n+\gamma+1+2t,\ n+\gamma+b+c-1, \\ n+\gamma+b+d-1,\ n+\gamma+c+d-1 \\ n+\gamma+s-2,\ n+\gamma+a+t,\ n+\gamma+b+t, \\ n+\gamma+c+t,\ n+\gamma+d+t\end{bmatrix} \times W(n+\gamma+2t;\ n+\gamma,\ 1-a+t,\ 1-b+t,\ 1-c+t,\ 1-d+t).$$

Because of the symmetry of n and γ in equation (2.27) and the relation of $r_n(t)$ to $u_{-1}(\gamma)$, $r_n(t)$ satisfies the equation

$$(3.37)\quad (t^2-a^2)r_n(t) = \lambda_{n-1}r_{n+1}(t) + \mu_{n-1}r_{n-1}(t) - (\lambda_{n-1}+\mu_{n-1})r_n(t).$$

Another linearly independent solution of this difference equation is $r_n(-t)$. Thus ψ' may be written in the form

$$(3.38)\quad \psi'(x;\gamma) = \frac{i}{2\pi}\frac{r_0(i\tau)r_1(-i\tau) - r_0(-i\tau)r_1(i\tau)}{|r_0(i\tau)|^2}.$$

Evaluating the Casorati determinant in the numerator by the same process used to compute Δ_n of (3.1) yields the following weight function for $P_n(x;\gamma)$.

$$(3.39)\quad \begin{aligned}\psi'(x;\gamma) &= \frac{(2\gamma+s-2)_2\sqrt{x-a^2}}{\pi(\gamma+s-2)} \\ &\times \Gamma\begin{bmatrix} \gamma+s-2 \\ 1+\gamma,\ a+b+\gamma,\ a+c+\gamma,\ a+d+\gamma, \\ b+c+\gamma,\ b+d+\gamma,\ c+d+\gamma\end{bmatrix} \\ &\times (2\gamma+s-2) \\ &\times \left|\frac{\Gamma(a+i\tau+\gamma)\Gamma(b+i\tau+\gamma)\Gamma(c+i\tau+\gamma)\times\Gamma(d+i\tau+\gamma)/\Gamma(1+2i\tau+\gamma)}{W(\gamma+2i\tau;\ \gamma,\ 1-a+i\tau,\ 1-b+i\tau,\ 1-c+i\tau,\ 1-d+i\tau)}\right|^2, x \geqq a^2,\end{aligned}$$

where τ is as defined following (3.35) and W is the ${}_7F_6$ defined in (2.22).

The orthogonality relation is

$$(3.40) \qquad \int_{\sqrt{a}}^{\infty} P_n(x;\,\gamma)P_m(x;\,\gamma)d\psi(x;\,\gamma) = \delta_{m,n}/\pi_n,$$

$$(3.41) \qquad \pi_0 := 1,\ \pi_n := \prod_{j=1}^{n}\{\lambda_{j-1}/\mu_j\},\ n > 0.$$

Note that ψ' is symmetric in a, b, c, and d. This reflects the fact that $P_n(x;\,\gamma)$, suitably normalized, as a function of τ, a, b, c, d, is fully symmetric in the parameters a, b, c, and d. To see this symmetry write the recurrence relation in the monic form (3.11) and note that $\mu_n\lambda_{n-1}$ is fully symmetric, and a tedious calculation shows that

$$(3.42) \qquad a^2 - \lambda_n - \mu_n$$

is also fully symmetric. Thus, by induction, $f_n(x)$ is fully symmetric. We have not found a way to prove this directly from the representation (3.2). Thus this symmetry gives transformation formulas for the function defined by the right hand side of (3.2).

The question of whether there are mass points on the negative real axis can be answered by the following theorem.

THEOREM 3. *The polynomials generated by*

$$(3.43) \qquad P_{-1}(x) = 0,\ P_0(x) = 1,\ P_{n+1}(x) = (x - C_n)P_n(x) - L_nP_{n-1}(x),$$
$$n \geqq 0,\ L_{n+1} > 0,\ C_n > 0,\ n \geqq 0,$$

are birth and death process polynomials if and only if the spectrum of the distribution function ψ *is contained in* $[0,\,\infty)$.

Proof. We write the general birth and death process polynomials in the monic form (3.11). We then prove the theorem by utilizing a theorem in Chihara [**10**, p. 108, Theorem 2.1], namely that the support of $d\psi$ is contained in $[0,\,\infty)$ if and only if $C_n > 0$ and

$$(3.44) \qquad L_n/(C_nC_{n-1}),\ n = 1,\,2,\,\dots\,,$$

is a chain sequence. First let (3.43) be birth and death process polynomials, i.e. let (3.43) be of the form (3.11). Then choose a parameter sequence $\{g_n\}$ as follows. Let $g_0 = 0$ and

$$(3.45) \qquad g_n := \frac{r_nD_{n-1}}{(1 + r_n)D_n},\ n = 1,\,2,\,\dots \text{ with}$$
$$D_n := 1 + \sum_{j=1}^{n}\prod_{k=0}^{j-1} r_k,\ n = 1,\,2,\,\dots\,,$$

where

(3.46) $r_n := \mu_n/\lambda_n,\ n = 0,\ 1,\ \ldots .$

Obviously $0 < g_n < 1,\ n = 1,\ 2,\ \ldots$, and we have

$$(3.47)\quad g_n(1 - g_{n-1}) = \frac{r_n}{(1 + r_n)(1 + r_{n-1})} = \frac{\lambda_{n-1}\mu_n}{(\lambda_{n-1} + \mu_{n-1})(\lambda_n + \mu_n)},$$

so $\{L_n/(C_n C_{n-1}) : n = 1,\ 2,\ \ldots\}$ is a chain sequence and the support of the measure $d\psi$ is contained in $[0,\ \infty)$.

Conversely, let the support of the $d\psi$ be contained in $[0,\ \infty)$. Then there is a parameter sequence $\{g_n\}$ with $0 \leqq g_0 < 1,\ 0 < g_n < 1, n > 0$, and

$$(3.48)\quad \frac{L_n}{C_n C_{n-1}} = (1 - g_{n-1})g_n,\ n = 1,\ 2,\ \ldots .$$

We define r_n recursively by $r_0 = 1$,

$$(3.49)\quad r_n = \left[\frac{1 + r_{n-1}}{g_n(1 - g_{n-1})} - 1\right]^{-1},\ n = 1,\ 2,\ \ldots,$$

and then define λ_n and μ_n by

$$(3.50)\quad \lambda_{-1} = 1,\ \lambda_n = C_n/(1 + r_n),\ \mu_n = L_n/\lambda_{n-1},\ n = 0,\ 1,\ 2,\ \ldots .$$

This puts the recurrence (3.43) in the form (3.11).

As an application of Theorem 3 we find that if all of

$$(3.51)\quad \begin{gathered}\gamma,\ \gamma + s - 1,\ 2\gamma + s - 2,\ \gamma + a + b,\ \gamma + a + c,\\ \gamma + a + d,\ \gamma + b + c - 1,\ \gamma + b + d - 1,\ \gamma + c + d - 1\end{gathered}$$

are positive, the distribution function for the associated Wilson polynomials has no mass points in $(-\infty,\ 0)$.

Recall that the moment problem is determinate, see the argument following (3.20). To show that $x = \xi$ supports a mass we need to show that the series (3.17), that is

$$(3.52)\quad \sum_{n=1}^{\infty} \pi_n\{P_n(\xi;\ \gamma)\}^2$$

converges, [**34**, p. 50]. The relationship (3.19) shows that $P_n(0;\ \gamma) = D_n$, where D_n is as in (3.45). It is clear that the series in (3.52) diverges when $\xi = 0$.

On the other hand, determining whether there are mass points (jumps of $\psi(x)$) in $[0,\ \infty)$ is a much less tractable problem. One way to do this is to show that the denominator of $F(z)$, as given by (3.33), does not vanish for $z \in (0, \infty)$. This we have not been able to do.

There are several interesting special cases of formula (3.39).

First, $\gamma \longrightarrow 0$ gives the weight function for the Wilson polynomials, as is to be expected.

Secondly, the limiting case $d \longrightarrow \infty$ gives the weight function for the associated continuous dual Hahn polynomials discussed in [**15**].

Thirdly, replace c, d, and τ by $\alpha + i\lambda$, $\alpha - i\lambda$, and $\tau\lambda$, respectively, and express the ${}_7F_6$ as a difference of two ${}_4F_3$'s using (4), on page 29 in Bailey [**7**], then choose

$$(3.53) \quad A = \gamma + 2i\tau,\ C = 1 - a + i\lambda\tau,\ D = 1 - b + i\lambda\tau,$$
$$E = 1 - \alpha - i\lambda + i\lambda\tau,\ F = 1 - \alpha + i\lambda + i\lambda\tau,\ a = \gamma.$$

As $\lambda \longrightarrow \infty$ the two ${}_4F_3$'s become ${}_2F_1$'s. If, in those ${}_2F_1$'s, we let

$$(3.54) \quad \gamma = c,\ \alpha = (\beta + 1)/2,\ 2\alpha + a + b - 1 = \gamma,\ 1 - \tau^2 = x$$

and employ a Pfaff-Kummer transformation on the ${}_2F_1$'s we get the weight function for the associated Jacobi polynomials given in Wimp [**41**].

4. Zero-related Polynomials. Consider the general three-term (associated) recurrence relation

$$(4.1) \quad xy_n = \lambda_{n+\gamma}y_{n+1} + \mu_{n+\gamma}y_{n-1} + (\lambda_{n+\gamma} + \mu_{n+\gamma})y_n,\ n = 0,\ 1,\ \ldots .$$

Let $\{p_n(x;\ \gamma)\}$ denote the polynomial solution of (4.1) satisfying the initial conditions

$$(4.2) \quad p_{-1}(x;\ \gamma) = 0,\ p_0(x;\ \gamma) = 1.$$

Clearly

$$(4.3) \quad p_1(x;\ \gamma) = (x + \lambda_\gamma + \mu_\gamma)/\lambda_\gamma.$$

Let $u_n(\gamma)$ and $v_n(\gamma)$ be linearly independent solutions of (4.1) having the property

$$(4.4) \quad u_n(\gamma + 1) = u_n(\gamma);\ v_n(\gamma + 1) = v_n(\gamma).$$

Such solutions can be constructed easily from any given set of linearly independent solutions. Then

$$(4.5) \quad p_n(x;\ \gamma) = \frac{u_{-1}(\gamma)v_n(\gamma) - u_n(\gamma)v_{-1}(\gamma)}{\Delta(\gamma)},$$
$$\Delta(\gamma) := u_{-1}(\gamma)v_0(\gamma) - u_0(\gamma)v_{-1}(\gamma).$$

The convergents of the corresponding continued J-fraction are given by

$$(4.6)\qquad \frac{p_{n-1}(x;\ \gamma+1)}{p_n(x;\ \gamma)} = \frac{\lambda_\gamma[u_0(\gamma)v_n(\gamma) - u_n(\gamma)v_0(\gamma)]}{\mu_\gamma[u_{-1}(\gamma)v_n(\gamma) - u_n(\gamma)v_{-1}(\gamma)]}.$$

Let, as usual,

$$(4.7)\qquad \lim_{n\to\infty} \frac{p_{n-1}(z;\ \gamma+1)}{p_n(z;\ \gamma)} := F(z).$$

Ismail, Letessier, and Valent [**14**], [**15**] observed that when the recurrence relation (4.1) is modified by requiring $\mu_\gamma = 0$, then, at least in the special cases they studied, the resulting orthogonal polynomials are closely related to the original polynomials. Here we show that this is true in general, and explore the consequences for the associated Wilson polynomials.

Consider the polynomials $\{q_n(x;\ \gamma)\}$ defined by

$$(4.8)\qquad \begin{aligned} xq_n(x;\ \gamma) &= \lambda_{n+\gamma}q_{n+1}(x;\ \gamma) + \tilde{\mu}_{n+\gamma}q_{n-1}(x;\ \gamma) \\ &\qquad - (\lambda_{n+\gamma} + \tilde{\mu}_{n+\gamma})q_n(x;\ \gamma),\ n = 0,\ 1,\ \ldots, \\ &\tilde{\mu}_\gamma = 0, \quad \text{and } \tilde{\mu}_{n+\gamma} = \mu_{n+\gamma} \text{ for } n > 0. \end{aligned}$$

We will call these q_n's the zero-related polynomials.

It is easy to see that $q_n(x;\ \gamma)$ may be written

$$(4.9)\qquad q_n(x;\ \gamma) = p_n(x;\ \gamma) - (\mu_\gamma/\lambda_\gamma)p_{n-1}(x;\ \gamma+1),\ n \geqq 0.$$

Note that

$$(4.10)\qquad q_1(x;\ \gamma) = (x + \lambda_\gamma)/\mu_\gamma.$$

Thus we may think of $p_n(x;\ \gamma)$ and $q_n(x;\ \gamma)$ as satisfying the same recurrence relation for $n = 1,\ 2,\ 3,\ \ldots$ but with different conditions on $p_1(x;\ \gamma)$ and $q_1(x;\ \gamma)$.

We now construct the numerator polynomials $\{q_n^*(x;\ \gamma)\}$ for the continued J-fraction related to the q_n recurrence, (4.8), see 10.5 in volume 2 of Erdélyi et al [**11**]. The q^*'s also satisfy the difference equation (4.8) for $n > 0$ and are given initially by

$$(4.11)\qquad q_0^*(x;\ \gamma) = 0,\ q_1^*(x;\ \gamma) = 1.$$

Obviously

$$(4.12)\qquad q_n^*(x;\ \gamma) = p_{n-1}(x;\ \gamma+1).$$

We wish to compute

$$(4.13)\qquad F_q(z) := \lim_{n\to\infty} \frac{q_n^*(x;\ \gamma)}{q_n(x;\ \gamma)}.$$

Thus

$$(4.14)\quad F_q(z) = \lim_{n\to\infty} \frac{p_{n-1}(x;\ \gamma+1)}{p_n(x;\ \gamma) - (\mu_\gamma/\lambda_\gamma)p_{n-1}(x;\ \gamma+1)}.$$

or

$$(4.15)\quad F_q(z) = F(z)/[1 - \mu_\gamma F(z)/\lambda_\gamma].$$

When the moment problem is determinate then $F_q(z)$ is the Stieltjes transform of the measure $d_q\psi(x;\ \gamma)$ with respect to which the q_n's are orthogonal. It is clear that

$$(4.16)\quad F_q(x-i0) - F_q(x+i0) = \frac{F(x-i0) - F(x+i0)}{|1 - \mu_\gamma F(x+i0)/\lambda_\gamma|^2},$$

where "0" means "0+". We then use

$$(4.17)\quad 2\pi i\psi'(x;\ \gamma) = F(x-i0) - F(x+i0)$$

and obtain

$$(4.18)\quad \psi_q'(x;\ \gamma) = \psi'(x;\ \gamma)\left|1 - \frac{\mu_\gamma}{\lambda_\gamma}F(x+i0)\right|^{-2}.$$

This seems to be about as far as one can go in general, there being no obvious way of simplifying the denominator. But it is a striking fact that for the associated Wilson polynomials both $\psi'(x;\ \gamma)$ and $\psi_q'(x;\ \gamma)$ may be encompassed in the same formula. In fact, this is what Ismail, Letessier and Valent did for associated Laguerre, Meixner and continuous dual Hahn polynomials. However the computation in the case of associated Wilson polynomials requires a new recursion formula for the related ${}_7F_6$'s, (4.23).

We start with the identity

$$(4.19)\quad \frac{(g)_n}{(1+a-g)_n} - \frac{(g-1)_n}{(2+a-g)_n} = \frac{n(n+a)(g)_{n-1}}{(1+a-g)(2+a-g)(3+a-g)_{n-1}}, n \geqq 1.$$

We then multiply both sides of (4.19) by

$$(4.20)\quad \frac{1}{n!}\frac{(a)_n(1+a/2)_n(c)_n(d)_n(e)_n(f)_n}{(1+a-c)_n(1+a-d)_n(1+a-e)_n(1+a-f)_n}$$

and use on the right hand side the identity

$$(4.21)\quad (\alpha)_n = \alpha(\alpha+1)_{n-1}$$

to get

$$(4.22)\quad \frac{(a)_n(1+a/2)_n(c)_n(d)_n(e)_n(f)_n}{n!(1+a-c)_n(1+a-d)_n(1+a-e)_n(1+a-f)_n} \times \left\{\frac{(g)_n}{(1+a-g)_n} - \frac{(g-1)_n}{(2+a-g)_n}\right\} \Big/ (a/2)_n$$
$$= \frac{cdef(1+a)(2+a)(2+a)_{n-1}(2+a/2)_{n-1}}{(1+a-g)_2(1+a-c)(1+a-d)(1+a-e)(1+a-f)} \times \frac{(e+1)_{n-1}(f+1)_{n-1}(g+1)_{n-1}}{(n-1)!(1+a/2)_{n-1}(2+a-c)_{n-1}(2+a-d)_{n-1}} \times \frac{(c+1)_{n-1}(d+1)_{n-1}}{(2+a-e)_{n-1}(2+a-f)_{n-1}(3+a-g)_{n-1}}.$$

Summing from $n = 1$ to ∞ gives

$$(4.23)\quad W(a;\ c,\ d,\ e,\ f,\ g) - W(a;\ c,\ d,\ e,\ f,\ g-1) = \frac{cdef(1+a)(2+a)}{(1+a-g)_2(1+a-c)(1+a-d)(1+a-e)(1+a-f)} \times W(a+2;\ c+1,\ d+1,\ e+1,\ f+1,\ g).$$

We now get

$$(4.24)\quad 1 - \frac{\mu_\gamma}{\lambda_\gamma}F(z) = \frac{(2\gamma+s-2)(a+i\tau)W(\gamma+2i\tau;\ \gamma,\ -a+i\tau,\ 1-b+i\tau,\ 1-c+i\tau,\ 1-d+i\tau)}{(\gamma+s-2)(\gamma+a+i\tau)W(\gamma+2i\tau;\ \gamma,\ 1-a+i\tau,\ 1-b+i\tau,\ 1-c+i\tau,\ 1-d+i\tau)}$$

Putting this into (4.18) gives the following formula for the derivative of ψ

$$(4.25)\quad \psi'(x;\ \gamma) = \frac{(2\gamma+s-1)\tau}{\pi x} \times \Gamma\left[\begin{matrix}\gamma+s-1\\ \gamma+1, \gamma+a+b, \gamma+a+c, \gamma+a+d, \gamma+b+c, \gamma+b+d, \gamma+c+d\end{matrix}\right] \times \left|\frac{\Gamma\left[\begin{matrix}1+\gamma+a+i\tau, \gamma+b+i\tau, \gamma+c+i\tau, \gamma+d+i\tau\\ 1+\gamma+2i\tau\end{matrix}\right]}{W(\gamma+2i\tau, \gamma, -a+i\tau, 1-b+i\tau, 1-c+i\tau, 1-d+i\tau)}\right|^2,$$

$$\tau = \sqrt{x-a^2},\ x \geqq a^2.$$

The normalization relationship for the zero-related Wilson polynomials $\{Q_n(x;\ \gamma)\}$ is the same as for $\{P_n(x;\ \gamma)\}$.

We note that this weight function is not fully symmetric in the parameters a, b, c, and d. In fact neither are the polynomials $\{Q_n(x;\ \gamma)\}$, even when renormalized. It is easily verified that the monic polynomial $Q_1(x;\ \gamma)$ is not symmetric in a and b.

5. Associated Recurrences and Generating Functions. Let $\mu(t;\ \gamma)$ be a distribution function, i.e. $d\mu(t;\ \gamma)$ is a positive measure supported on an infinite subset of the real line and has finite moments of all orders. Assume that $d\mu(t;\ \gamma)$ depends on a real parameter $\gamma \geqq 0$. Let $\{p_n(t;\ \gamma)\}$ be polynomials orthogonal with respect to $d\mu(t;\ \gamma)$ and satisfy the initial conditions $p_{-1}(t;\ \gamma) = 0$, $p_0(t;\ \gamma) = 1$, for all $\gamma \geqq 0$. We call $\mu(t;\ \gamma)$ an associated distribution function if the numerator polynomials $\{q_n(t)\}$ of the corresponding J-fraction are given by the formula

$$q_n(t) = p_{n-1}(t;\ \gamma + 1), \quad n = 0,\ 1,\ 2,\ \dots . \tag{5.1}$$

We know of no simple conditions on a distribution function which will make it an associated distribution function, although by now many special cases of associated distributions have been found, see [**2**], [**5**], [**8**], [**14**], [**15**], [**17**], [**25**], [**26**], and the references given in these papers.

If $\mu(t;\ \gamma)$ is an associated distribution function, then the polynomials $\{p_n(t;\ \gamma)\}$ generated by it are called associated polynomials. Assume that the three-term recurrence relation satisfied by $\{p_n(t;\ \gamma)\}$ is

$$p_{n+1}(t;\ \gamma) = [tA_n(\gamma) + B_n(\gamma)]p_n(t;\ \gamma) - C_n(\gamma)p_{n-1}(t;\ \gamma), \quad n = 0,\ 1,\ 2,\ \dots . \tag{5.2}$$

Since the numerator polynomials $\{q_n(t;\ \gamma)\}$ satisfy the same recurrence relation, it is obvious that we must have

$$A_{n-1}(\gamma + 1) = A_n(\gamma), \tag{5.3}$$

and similarly for $B_n(\gamma)$ and $C_n(\gamma)$. Iterating (5.3) gives

$$A_n(\gamma) = A_0(n + \gamma) := A(n + \gamma). \tag{5.4}$$

Thus a three term recurrence relation satisfied by a set of associated polynomials must be of the form

$$p_{n+1}(t;\ \gamma) = [tA(n + \gamma) + B(n + \gamma)]p_n(t;\ \gamma) - C(n + \gamma)p_{n-1}(t;\ \gamma), \quad n = 0,\ 1,\ 2,\ \dots . \tag{5.5}$$

This condition is easily seen to be necessary and sufficient, and so, by application of the formulas [**11**, vol. 2, p. 159, (8)] one can obtain necessary and sufficient conditions for $\mu(t;\ \gamma)$ to be an associated distribution function in terms of the Gram determinants of the moments of $d\mu(t;\ \gamma)$,

$$G_n(\gamma) := \det|c_{i+j}(\gamma)|_{i,j=0,1,\dots,n},\ c_m := \int_{-\infty}^{\infty} t^m d\mu(t;\ \gamma). \tag{5.6}$$

The conditions one gets are not, however, very illuminating.

Some traditional families of orthogonal polynomials are associated in their natural setting, for instance, the Chebyshev polynomials, or the Pollaczek four parameter family of polynomials, [**30**]. Nevai [**27**] has shown that when γ is an integer, $d\mu(t;\ \gamma)$ may be expressed in terms of an integral involving $d\mu(t;\ 0)$.

One interesting condition on $d\mu(t;\ \gamma)$ may be expressed in terms of its Stieltjes transform

$$(5.7)\qquad F(t;\ \gamma) := \int_{-\infty}^{\infty} \frac{d\mu(w;\ \gamma)}{t-w}.$$

Let $\mu(t;\ \gamma)$ be an associated distribution function and let the moment problem for the recursion (5.2) be determined. Then one can choose a basis of solutions for the recurrence relation $u_n(t;\ \gamma) \equiv u_n(\gamma)$, $v_n(t;\ \gamma) \equiv v_n(\gamma)$ such that

$$(5.8)\qquad \lim_{n\to\infty} \frac{u_n(\gamma)}{v_n(\gamma)} = 0$$

and furthermore

$$(5.9)\qquad u_{n-1}(\gamma+1) = u_n(\gamma),\ v_{n-1}(\gamma+1) = v_n(\gamma).$$

We can write

$$(5.10)\quad p_n(t;\ \gamma) = [u_{-1}(\gamma)v_n(\gamma) - v_{-1}(\gamma)u_n(\gamma)]/\Delta_{-1}(\gamma),$$

where

$$(5.11)\quad \Delta_n(\gamma) = u_n(\gamma)v_{n+1}(\gamma) - v_n(\gamma)u_{n+1}(\gamma),\ n = -1,\ 0,\ 1,\ \dots .$$

Constructing the n*th* approximant (convergent) r_n to the related continued fraction in the usual way and employing (5.10) and the fact that

$$(5.12)\quad \Delta_{-1}(\gamma)/\Delta_0(\gamma) = 1/C(\gamma),$$

we find that

$$(5.13)\quad r_n = \frac{1}{C(\gamma)}\left[\frac{v_0(\gamma)u_n(\gamma) - u_0(\gamma)v_n(\gamma)}{v_{-1}(\gamma)u_n(\gamma) - u_{-1}(\gamma)v_n(\gamma)}\right].$$

Thus we have

$$(5.14)\quad F(t;\ \gamma) = \lim_{n\to\infty} r_n = \frac{u_0(\gamma)}{C(\gamma)u_{-1}(\gamma)}.$$

But

$$(5.15)\quad \frac{u_1(\gamma)}{u_0(\gamma)} = A(\gamma)t + B(\gamma) - C(\gamma)\frac{u_{-1}(\gamma)}{u_0(\gamma)}.$$

We thus find that if $\mu(t;\ \gamma)$ is an associated distribution function, its Stieltjes transform $F(t;\ \gamma)$ must satisfy the nonlinear difference equation

$$C(\gamma+1)F(t;\ \gamma+1) = [A(\gamma)t + B(\gamma)] - 1/F(t;\ \gamma), \qquad \gamma \geqq 0,\ t \notin \operatorname{Supp}\{d\mu(t;\ \gamma)\}. \tag{5.16}$$

A consequence of this is that the only constant (in γ) associated distribution function is that for the Chebyshev polynomials $U_n(x)$. In that case A, B, C and F are all constants and we get

$$CF^2(t;\ \gamma) - [tA + B]F(t;\ \gamma) + 1 = 0. \tag{5.17}$$

Solving this quadratic equation for F and inverting the Stieltjes transform gives the distribution function for the Chebyshev polynomials $U_n(x)$ (renormalized to some appropriate interval). This, of course, can also be inferred from the three-term recurrence relation.

One advantage of dealing with associated polynomials is, that when a nice formula is known for p_n, one has a nice formula for the Padé approximants to F. This was exploited in Wimp [**41**] to obtain a closed form expression for the $[(n-1)/n]$ Padé approximants to the continued fraction of Gauss. Another nice thing about associated polynomials is the extremely elegant character of their generating functions. In this section we will explore this idea further. Let us start with a general associated-type recurrence relation, which we will write in the form

$$a(n+\gamma+1)y_{n+1} + b(n+\gamma)y_n + c(n+\gamma-1)y_{n-1} = 0, \qquad n = 0,\ 1,\ 2,\ \dots . \tag{5.18}$$

We are interested in the solution $p_n(\gamma)$ with the property

$$p_{-1}(\gamma) = 0, \text{ and } p_0(\gamma) = 1. \tag{5.19}$$

Let

$$G_\gamma(x) := \sum_{n=0}^{\infty} w^n p_n(\gamma). \tag{5.20}$$

It might be thought that writing the above expression would involve problems of convergence. But this is not really the case, since everything we shall do, including the work on linear differential operators to follow, is easily justified by placing the discussion in the context of rings of formal series

$$w^\nu \sum_{n=0}^{\infty} \alpha_n w^n, \tag{5.21}$$

with coefficients α_n in a ring, or field. There are various manifestations of this theory, sometimes called the umbral calculus, see [**24**], [**44**], [**32**], [**33**].

We assume a, b, c of (5.18) are polynomials of the same degree, say σ

$$(5.22)\quad \begin{aligned} a(z) &= A\prod_{j=1}^{\sigma}(z+a_j) = A[z^\sigma + a^* z^{\sigma-1} + \cdots], \\ b(z) &= B\prod_{j=1}^{\sigma}(z+b_j) = B[z^\sigma + b^* z^{\sigma-1} + \cdots], \\ c(z) &= C\prod_{j=1}^{\sigma}(z+c_j) = C[z^\sigma + c^* z^{\sigma-1} + \cdots]. \end{aligned}$$

Others are best arrived at by confluence. We assume the a_j's are distinct. Let δ denote the operator

$$(5.23)\quad \delta = wD,\ D = \frac{d}{dw}.$$

Note that

$$(5.24)\quad \delta^k w^{r+\gamma} = (r+\gamma)^k w^{r+\gamma}.$$

Multiplying the recurrence reaction (5.18) by

$$(5.25)\quad w^{n+\gamma} p_n(\gamma)$$

and summing from $n = 0$ to ∞ gives

$$(5.26)\quad \mathcal{P}(\delta)\phi_\gamma(w) = a(\gamma)w^\gamma,\ \phi_\gamma(w) := w^\gamma G_\gamma(w),$$
$$\mathcal{P}(\delta) := A\prod_{j=1}^{\sigma}(\delta+a_j) + wB\prod_{j=1}^{\sigma}(\delta+b_j) + w^2 C\prod_{j=1}^{\sigma}(\delta+c_j).$$

This differential equation, which is of order σ, can be solved by variation of parameters if a basis of solutions can be determined for the corresponding homogeneous differential equation,

$$(5.27)\quad \mathcal{P}(\delta)y = 0.$$

Such a basis will be provided by the functions

$$(5.28)\quad \phi_{-a_j}(w).$$

In practice what usually happens is that explicitly available expressions for these quantities are insufficient to fill out a basis. This happens, for instance, with the associated Jacobi and associated Wilson polynomials. In both situations, one member of the basis is lacking.

Suppose, that we are lacking a full basis of the homogeneous equation (5.27), but that one solution, say y^*, is known and that y^* satisfies a linear differential equation with polynomial coefficients of order $\sigma - 1$

$$(5.29) \qquad \mathcal{G}(y) = \rho_1(w)y^{\sigma-1} + \rho_2(w)y^{(\sigma-2)} + \cdots = 0.$$

Then the equation (5.27) is called reducible. It is known that every solution of the so-called reduced equation (5.29), must be a solution of the original equation, so if we know a basis for (5.29), we have $\sigma - 1$ members of a basis for (5.27).

In rings of differential operators with polynomial coefficients, there is an Euclidean algorithm, see Forsyth [**12**, vol. 4, p. 223] which implies that we may factor $\mathcal{P}$ in the form

$$(5.30) \qquad \mathcal{P} = \mathcal{R}\mathcal{G}$$

where $\mathcal{R}$ is an operator of first order.

It is easily verified that

$$(5.31) \qquad \delta^n = w^n D^n + \frac{1}{2}n(n-1)D^{n-1} + \cdots, \qquad n = 1, 2, \ldots.$$

Write

$$(5.32) \qquad \mathcal{R} = \mu D + \nu I.$$

We have

$$(5.33) \qquad \begin{aligned} \mathcal{P} &= \mu\rho_1 D^\sigma = (\mu\rho_1' + \mu\rho_2 + \nu\rho_1)D^{\sigma-1} + \cdots \\ &= Sw^\sigma D^\sigma + \left[S\frac{\sigma(\sigma-1)}{2} + (Aa^* + wBb^* + w^2Cc^*)\right] w^{\sigma-1}D^{\sigma-1} \\ S &= S(w) := A + wB + w^2C. \end{aligned}$$

This establishes the relationships

$$(5.34) \qquad \begin{aligned} \mu\rho_1 &= Sw^\sigma, \\ \mu\rho_1' + \mu\rho_2 + \nu\rho_1 &= w^{\sigma-1}\left[S\frac{\sigma(\sigma-1)}{2} + Aa^* + wBb^* + w^2Cc^*\right]. \end{aligned}$$

This implies that as a condition for reducibility, there must exist polynomial solutions μ, ν of the above equations, for instance, ρ_1 must be a factor of Sw^σ, etc.

Dividing the first equation in (5.34) into the second equation in (5.34) gives

$$\frac{\rho_1'}{\rho_1}+\frac{\rho_1}{\rho_2}+\frac{\nu}{\mu}=\frac{A_1}{w}+\frac{A_2}{w-v_1}+\frac{A_3}{w-v_2} \tag{5.35}$$

where

$$v_1,\ v_2=\frac{1}{2C}[-B\pm\sqrt{B^2-4AC}\,], A_1=a^*+\frac{\sigma(\sigma-1)}{2}, \tag{5.36}$$
$$A_2=\frac{C}{\sqrt{B^2-4AC}}\left[\frac{A}{v_1}(a^*-c^*)+B(b^*-c^*)\right],$$
$$A_3=-\frac{C}{\sqrt{B^2-4AC}}\left[\frac{A}{v_1}(a^*-c^*)+B(b^*-c^*)\right].$$

If $B^2-4AC=0$, there are complications. These can be resolved, but, to keep things simple, we assume

$$B^2-4AC\neq 0, \tag{5.37}$$

which, in fact, is always the case in practice.

Integrating (5.35) yields

$$\ln|\rho_1|+\int\frac{\rho_2}{\rho_1}dw+\int\frac{\nu}{\mu}dw \tag{5.38}$$
$$=A_1\ \mathrm{In}\, w+A_2\ln\left(1-\frac{w}{v_1}\right)+A_3\ln\left(1-\frac{w}{v_2}\right)+M.$$

This shows that an integrating factor for the operator inverse to $\mathcal{R}$ is

$$\psi=\exp\left(\int\frac{\nu}{\mu}dw\right) \tag{5.39}$$
$$=w^{A_1}(1-w/v_1)^{A_2}(1-w/v_2)^{A_3}\exp\left(-\int\frac{\rho_2}{\rho_1}dw\right)/\rho_1.$$

By Abel's formula the quantity $\exp(-\int(\rho_2/\rho_1)dw)$ is proportional to the Wronskian, $W(w)$, of the reduced system. Thus we may take

$$\psi=\exp\left(\int\frac{\nu}{\mu}\,dw\right)=w^{A_1}(1-w/v_1)^{A_2}(1-w/v_2)^{A_3}W(w)/\rho_1 \tag{5.40}$$

as the integrating factor.

We now compute $\mathcal{R}^{-1}(g)$, which is equivalent to solving $(\mu D+\nu I)y=g$. The relationship (3.40) yields

$$\mathcal{R}^{-1}(g)=\frac{\rho_1}{AW(w)}w^{-A_1}(1-w/v_1)^{-A_2}(1-w/v_2)^{-A_3} \tag{5.41}$$
$$\times\int_0^w W(u)u^{A_1-\omega}(1-u/v_1)^{A_2-1}(1-u/v_2)^{A_3-1}g(u)du.$$

For our case, $g(u) = u^c$, we took the constant of integration to be zero. This will give the solution with the correct behavior as $t \longrightarrow 0$.

The particular solution which turns out to be the one required for

$$\mathcal{P}(\delta)y = Kw^\gamma$$

is

$$(5.42) \quad y = w^\gamma \mathcal{G}_\gamma(w) = \frac{K}{A}\sum_{k=1}^{\sigma-1} y_k(w) \int_0^w \frac{W_k(u)}{W(u)\rho_1(u)} \mathcal{R}^{-1}[u^\gamma]du,$$

see [**13**, p. 123], where $y_1, y_2, \ldots, y_k$ is a basis for (5.29) and W_k is the determinant formed from W by replacing the k*th* column by (0, 0, ..., 0, 1). Note that fragments of the solution to the homogeneous equation obviously cannot occur, since these will give rise to inappropriate powers of w.

Observe that when W is a polynomial $\mathcal{R}^{-1}(w^\gamma)$ can be written as a generalized hypergeometric function of several variables, the Lauricella function F_D, since

$$(5.43) \quad \int_0^w u^{\alpha-1} \prod_{j=1}^{p} (1 - \lambda_k u)^{\sigma_j} du$$
$$= \frac{w^\alpha}{\alpha} F_D(\alpha, -\sigma_1, -\sigma_2, \ldots, -\sigma_p, \alpha+1, \lambda_1 w, \lambda_2 w, \ldots, \lambda_p w),$$

see Slater [**35**, p. 228].

6. Generating Functions For Associated Wilson Polynomials. There are two generating functions for the Wilson polynomials. One, found by Wilson, is

$$(6.1) \quad {}_2F_1\left(\begin{matrix} a+t, b+t \\ a+b \end{matrix} ; w\right) {}_2F_1\left(\begin{matrix} c-t, d-t \\ d+c \end{matrix} ; w\right)$$
$$= \sum_{n=0}^{\infty} \frac{(a+c)_n(a+d)_n}{(c+d)_n n!} P_n(a, b, c, d; x)w^n, \quad x = a^2 - t^2.$$

This follows by taking the Cauchy product of the two series on the left, and applying the Whipple transformation, Bailey [**7**, (1) p. 56] on the ${}_4F_3$ which arises. A q-analogue of this generating function was proved by Ismail and Wilson in [**18**].

Another generating function is

$$(6.2) \quad \sum_{n=0}^{\infty} \frac{(s-1)_n}{n!} P_n(a, b, c, d; x)w^n$$
$$= (1-w)^{1-s} {}_4F_3\left(\begin{matrix} a+t, a-t, s/2, (s-1)/2 \\ a+b, a+c, a+d \end{matrix} ; \frac{-4w}{(1-w)^2}\right),$$
$$s = a+b+c+d.$$

This follows directly from the definition and rearrangement of series. The generating functions (6.1) and (6.2) seem to be the only generating functions of the Wilson polynomials that are known at this time.

The minimal order of the differential equation for any possible generating function for the associated Wilson polynomials will be five. We will show that the generating functions (6.1) and (6.2) can be made to yield 4 solutions of the differential equation (5.26).

We now derive an extension of (6.1) to the associated Wilson polynomials. Let us renormalize the associated Wilson polynomials as

$$(6.3)\qquad \begin{aligned} R_n &= R_n(a,\ b,\ c,\ d;\ x;\ \gamma) \\ &= \frac{(\gamma+a+c)_n(\gamma+a+d)_n}{(\gamma+1)_n(\gamma+c+d)_n} P_n(a,\ b,\ c,\ d;\ x;\ \gamma). \end{aligned}$$

They satisfy the recurrence relation

$$(6.4)\qquad -xR_n = b_n R_{n+1} + d_n R_{n-1} - (\lambda_n + \mu_n) R_n,$$

with λ_n and μ_n are as in (2.27) and

$$(6.5)\qquad b_n = \frac{(n+\gamma+1)(n+\gamma+s-1)(n+\gamma+a+b)(n+\gamma+c+d)}{(2n+2\gamma+s-1)(2n+2\gamma+s)},$$

$$(6.6)\qquad d_n = \frac{[(n+\gamma+a+c-1)(n+\gamma+a+d-1)(n+\gamma+b+c-1)(n+\gamma+b+d-1)]}{(2n+2\gamma+s-2)(2n+2\gamma+s-1)}.$$

We then study the symmetries of the recurrence relation (6.4) under changes of the parameters. Consider the transformations

$$(6.7)\qquad \{\mathcal{T} : \gamma \to \gamma+a+b-1,\ a \to 1-a,\ b \to 1-b,\ c \to c,\ d \to d\},$$

$$(6.8)\qquad \{\mathcal{T}' : \gamma \to \gamma+c+d-1,\ a \to a,\ b \to b,\ c \to 1-c,\ d \to 1-d\}.$$

Note that the variable t does not change under the transformations (6.7) and (6.8) but x, being $(a^2-t^2)^{1/2}$ will change if a changes.
It is clear that

$$\mathcal{T}^2 = \mathcal{T}'^2 = I,\ \mathcal{T}\,\mathcal{T}' = \mathcal{T}'\mathcal{T}.$$

The transformations $\mathcal{T}$ and $\mathcal{T}'$ generate a group $\mathcal{L}$ of order 4,

$$\mathcal{L} = \{I,\ \mathcal{T},\ \mathcal{T}',\ \mathcal{T}\,\mathcal{T}'\}.$$

One can easily see that b_n, d_n and $x - \lambda_n - \mu_n$ are invariant under $\mathcal{L}$. Hence the polynomials R_n are also invariant under $\mathcal{L}$.

Let us define

$$(6.9) \qquad \mathcal{G}_\gamma(w) = \sum_{n=0}^{\infty} R_n(a,\ b,\ c,\ d;\ x;\ \gamma) w^{n+\gamma}.$$

The differential equation for the generating function $\mathcal{G}_\gamma(w)$ takes the form

$$(6.10) \qquad [w^2 \mathcal{F}_\delta + w(x\mathcal{C}_\delta - \mathcal{A}_\delta - \mathcal{B}_\delta) + \mathcal{D}_\delta]\mathcal{G}_\gamma(w) = w^\gamma \mathcal{D}_\gamma,$$

with

$$\begin{aligned}
\mathcal{D}_\gamma &= \gamma(\gamma + s - 2)(\gamma + a + b - 1)(\gamma + c + d - 1)(2\gamma + s - 4),\\
\mathcal{A}_\delta &= (2\delta + s - 2)(\delta + s - 1)(\delta + a + b)(\delta + a + c)(\delta + a + d),\\
\mathcal{B}_\delta &= \delta(2\delta + s)(\delta + b + c - 1)(\delta + b + d - 1)(\delta + c + d - 1),\\
\mathcal{C}_\delta &= (2\delta + s - 2)(2\delta + s - 1)(2\delta + s),\\
\mathcal{F}_\delta &= (2\delta + s + 2)(\delta + a + c)(\delta + a + d)(\delta + b + c)(\delta + b + d),\\
\delta &:= wD_w = w\frac{d}{dw}.
\end{aligned}$$

Note that $\mathcal{D}_\gamma$ is invariant under $\mathcal{L}$. The differential equation (6.10) can be expressed in the form

$$(6.11) \qquad \mathcal{P}\mathcal{G}_\gamma(w) = \sum_{j=0}^{5} \mathcal{P}_{5-j}\delta^j \mathcal{G}_\gamma(w) = \frac{1}{2}\mathcal{D}_\gamma w^\gamma.$$

with

$$(6.12) \qquad \mathcal{P}_0 = (1 - w)^2,\ \mathcal{P}_1 = (w - 1)[(1 + 5s/2)w + 6 - 5s/2].$$

The remaining $\mathcal{P}_j$'s will not be used in the sequel. Applying the factorization method developed in Section 5 we see that $\mathcal{P}$ has a factorization

$$(6.13) \qquad \mathcal{P} = \mathcal{R}\,\mathcal{S},\ \text{where}\ \mathcal{R} = \delta + r(w).$$

Furthermore $\mathcal{S}$ is a differential operator of order 4 which annihilates the following four linearly independent functions:

$$\begin{aligned}
(6.14) \qquad & G_1 = Y_1Y_2,\ G_2 = \mathcal{T}G_1 = Y_3Y_2,\\
& G_3 = \mathcal{T}'G_1 = Y_1Y_4,\ G_4 = \mathcal{T}\mathcal{T}'G_1 = Y_3Y_4,
\end{aligned}$$

and Y_1, Y_2, Y_3 and Y_4 are

$$(6.15)\quad Y_1 = {}_2F_1\left(\begin{matrix} a+t, b+t \\ a+b \end{matrix} ; w\right),$$

$$Y_2 = {}_2F_1\left(\begin{matrix} c-t, d-t \\ c+d \end{matrix} ; w\right),$$

$$Y_3 = {}_2F_1\left(\begin{matrix} 1-a+t, 1-b+t \\ 2-a-b \end{matrix} ; w\right),$$

$$Y_4 = {}_2F_1\left(\begin{matrix} 1-c-t, 1-d-t \\ 2-c-d \end{matrix} ; w\right),$$

Following Orr [**29**] we define

$$(6.16)\quad y_1 = w^{(a+b)/2}(1-w)^{t+1/2}Y_1,$$

and find that it is a solution of the second order differential equation

$$(6.17)\quad \frac{d^2y_1}{dw^2} + I_1 y_1 = 0,$$

provided that

$$(6.18)\quad I_1 = I_1(a,\ b;\ t) := \frac{1-(1-a-b)^2}{4w^2} + \frac{1-4t^2}{4(1-w)^2} + \frac{1-(1-a-b)^2+(b-a)^2-4t^2}{4w(1-w)}.$$

Similarly

$$(6.19)\quad y_2 = w^{(c+d)/2}(1-w)^{-t+1/2}Y_2,$$

satisfies

$$(6.20)\quad \frac{d^2y_2}{dw^2} + I_2 y_2 = 0,\ \text{with}\ I_2 = I_2(c,\ d;\ t) := I_1(c,\ d;\ -t).$$

We set

$$(6.21)\quad Q(w) := I_1 - I_2 = \frac{L}{4w^2} + \frac{M}{4w(1-w)},$$
$$L = (a+b)(2-a-b) - (c+d)(2-c-d),$$
$$M = L + (a-b+c-d)(a-b-c+d).$$

Observe that L and M are also invariant under $\mathcal{L}$.

Orr [**29**] has shown that the product y_1y_2 satisfies a fourth order differential equation

$$\{D_w^4 - (D_wQ)D_w^3 + \cdots\}y_1y_2 = 0. \tag{6.22}$$

Thus we have shown that the product Y_1Y_2 satisfies the ordinary differential equation

$$\begin{aligned} \mathcal{S}Y_1Y_2 &:= \mathcal{P}_0(\delta^4 + \mathcal{Q}\delta^3 + \cdots)Y_1Y_2 = 0, \\ \mathcal{Q} &:= 2s - \sigma - 4w/(1-w) - (wD_wQ)/Q. \end{aligned} \tag{6.23}$$

Taking into account the factorization of $\mathcal{P}$ in (6.13) we find $r(w)$ from (6.11), (6.12) and (6.23). The result is

$$r(w) = \frac{s}{2} - 2 + \frac{(M-L)w}{L+(M-L)w}. \tag{6.24}$$

Now the differential equation (6.11) becomes

$$(\delta + r(w))\mathcal{S}\mathcal{G}_\gamma(w) = (1/2)\mathcal{D}_\gamma w^\gamma, \tag{6.25}$$

which can be integrated once to give

$$\mathcal{S}\mathcal{G}_\gamma(w) = \frac{\mathcal{D}_\gamma}{2[L+(M-L)w]}\left\{L\frac{w^\gamma}{\gamma-2+s/2} + (M-L)\frac{w^{\gamma+1}}{\gamma-1+s/2}\right\}. \tag{6.26}$$

The integration constant in (6.26) vanishes since $\mathcal{G}_\gamma(w) \sim w^\gamma$ as $w \longrightarrow 0$. We may now solve (6.26) using a Green's function technique since we know four solutions of the corresponding homogeneous differential equation, Ince [**13**, p. 122]. A tedious calculation and simplification give the following integral representation for the generating function:

$$\begin{aligned} \mathcal{G}_\gamma(w) = {} & -w^\gamma\{I + \mathcal{T} + \mathcal{T}' + \mathcal{T}\,\mathcal{T}'\} \\ & \times \left\{w^{-\gamma}G_1(w)\int_0^w \frac{\tau^{\gamma-2+s/2}\mathcal{D}_\gamma}{\rho[L+(M-L)\tau]^2}\right. \\ & \left.\times\, \phi(\tau)\int_0^\tau y^{2-s/2}(1-y)Q(y)G_4(y)\,dy\,d\tau\right\}, \end{aligned} \tag{6.27}$$

with $\rho = (1-a-b)(1-c-d)$ and $\phi(\tau) = L/(\gamma-2+s/2) + \tau(M-L)/(\gamma-1+s/2)$. After expanding $G_1(w)$ and $G_4(w)$ in terms of the Wilson polynomials $P_n(a, b, c, d; x)$ then performing the integrations in (6.27) we establish the following explicit representation of the associated Wilson polynomials in terms of the Wilson polynomials

$$(6.28)\quad P_n(a,\ b,\ c,\ d;\ x;\ \gamma) = -\frac{\gamma(\gamma+s-2)(\gamma+a+b-1)(\gamma+c+d-1)(\gamma+1)_n(\gamma+c+d)_n}{(1-a-b)(1-c-d)(\gamma+a+c)_n(\gamma+a+d)_n}$$
$$\times (I-\mathcal{T}-\mathcal{T}'+\mathcal{T}\,\mathcal{T}')\left\{\sum_{j=0}^{n} V_{n-j}\left[\frac{\mathcal{T}\,\mathcal{T}'V_j}{(\gamma+j)(2j-s+2)} + \frac{4}{2\gamma+s-2}\sum_{i=0}^{j-1}\left(1-\frac{M}{L}\right)^{j-i}\frac{\mathcal{T}\,\mathcal{T}'V_i}{(2i-s+2)(2i-s+4)}\right]\right\}.$$

In (6.28) we used the notation

$$V_n = \frac{(a+c)_n(a+d)_n}{n!(c+d)_n}P_n(a,\ b,\ c,\ d;\ x).$$

Note that if $\gamma \to 0,\ 2-s,\ 1-a-b$, or $1-c-d$ then (6.28) reduces to an obvious equality.

We conclude this section by proving a generalization of (6.2) to the case of associated Wilson polynomials. The method of proof is very similar to what we used to prove (6.27). To do so we again renormalize the polynomials $P_n(a,\ b,\ c,\ d;\ x;\ \gamma)$. Set

$$(6.29)\quad S_n = S_n(a,\ b,\ c,\ d;\ x;\ \gamma) = \frac{(s+\gamma-1)_n}{(\gamma+1)_n}P_n(a,\ b,\ c,\ d;\ x;\ \gamma).$$

The S_n's satisfy the recurrence relation

$$(6.30)\quad -xS_n = b_nS_{n+1} + d_nS_{n-1} - (\lambda_n+\mu_u)S_n,$$

where λ_n and μ_n are the same as in (2.27) but b_n and d_n are now defined as

$$b_n = \frac{(n+\gamma+1)(n+\gamma+a+b)(n+\gamma+a+c)(n+\gamma+a+d)}{(2n+2\gamma+s-1)(2n+2\gamma+s)},$$
$$d_n = \frac{[(n+\gamma+s-2)(n+\gamma+b+c-1)(n+\gamma+b+d-1)(n+\gamma+c+d-1)]}{(2n+2\gamma+s-2)(2n+2\gamma+s-1)}.$$

The analogue of the group of transformations $\mathcal{L}$ is the four element group $\mathcal{L}'$,

$$\mathcal{L}' = \{I,\ \mathcal{H},\ \mathcal{H}',\ \mathcal{H}\mathcal{H}'\},$$

and

$$\{\mathcal{H}:\gamma\to\gamma+a+b-1,\ a\to 1-b,\ b\to 1-a,\ c\to d,\ d\to c\},$$
$$\{\mathcal{H}':\gamma\to\gamma+a+c-1,\ a\to 1-c,\ b\to d,\ c\to 1-a,\ d\to b\},$$

Recall that the variable t is invariant under the above transformations but $x = (a^2 - t^2)^{1/2}$ will change if a changes. Here we also have

$$\mathcal{H}^2 = \mathcal{H}'\mathcal{P}^2 = I, \mathcal{H}\mathcal{H}' = \mathcal{H}'\mathcal{H}.$$

In the present case b_n and d_n as well as $\lambda_n + \mu_n - x$ are invariant under $\mathcal{L}'$.

The generating function

$$\mathcal{G}_\gamma^*(w) = \sum_{n=0}^{\infty} S_n(a, b, c, d; x; \gamma) w^{n+\gamma}, \tag{6.31}$$

satisfies the differential equation

$$[w^2 \mathcal{F}_\delta^* + w(x\mathcal{C}_\delta - \mathcal{A}_\delta - \mathcal{B}_\delta) + \mathcal{D}_\delta^*]\mathcal{G}_\gamma^*(w) = w^\gamma \mathcal{D}_\gamma^*, \tag{6.32}$$

with

$$\mathcal{D}_\gamma^* = \gamma(2\gamma + s - 4)(\gamma + a + b - 1)(\gamma + a + c - 1)(\gamma + a + d - 1),$$
$$\mathcal{F}_\delta^* = (2\delta + s + 2)(\delta + s - 1)(\delta + b + c)(\delta + b + d)(\delta + c + d),$$

and $\mathcal{A}_\delta$, $\mathcal{B}_\delta$ and $\mathcal{C}_\delta$ are as in (6.10). The differential equation (6.32) can be put in the form

$$\mathcal{P}^* \mathcal{G}_\gamma^*(w) = \sum_{j=0}^{5} \mathcal{P}_{5-j}^* \delta^j \mathcal{G}_\gamma^*(w) = \frac{1}{2}\mathcal{D}_\gamma^* w^\gamma \tag{6.33}$$

with

$$\mathcal{P}_0^* = (1-w)^2, \ \mathcal{P}_1^* = (1-w)[(2a - 7s/2)w + 2a - 5 + 3s/2]. \tag{6.34}$$

As we pointed out earlier, $\mathcal{P}_j^*$, $j = 2, 3, 4, 5$ will not be used so we will not record their values. Applying the factorization method we obtain

$$\mathcal{P}^* = \mathcal{R}^* \mathcal{S}^*, \text{ where } \mathcal{R}^* = \delta + r^*(w).$$

Performing a change of variable in the differential equation satisfied by a generalized hypergeometric function of the type ${}_4F_3$, [**11**] we can prove that the function

$$G(w) := (1-w)^{1-s} {}_4F_3\left(\begin{matrix} a+t, \ a-t, \ s/2, \ (s-1)/2 \\ a+b, \ a+c, \ a+d \end{matrix} ; \frac{-4w}{(1-w)^2}\right) \tag{6.35}$$

is a solution of the differential equation

$$\mathcal{S}^* G = \{\mathcal{P}_0^* \delta^4 + \mathcal{Q}^* \delta^3 + \cdots\} G = 0, \tag{6.36}$$

with

$$\mathcal{Q}^* := 4w(1-s)(1-w) + \frac{1-w}{1+w}$$
$$\times \left[6\delta(1-w^2) - \frac{5}{2}(1-w)^2 + (2a+s-1/2)(1+w)^2\right].$$

We use the factorization $\mathcal{P}^* = \mathcal{R}^* \mathcal{S}^*$ then solve an inhomogenous linear first order differential equation to find that $r^*(w)$ must be

$$r^*(w) = -1 + s/2 - 1/(1+w),$$

and that $\mathcal{G}_\gamma(w)$ satisfies

$$\text{(6.37)} \quad \mathcal{S}^* \mathcal{G}_\gamma^*(w) = \mathcal{D}_\gamma^* \frac{1}{1+w}\left[\frac{w^\gamma}{2\gamma+s-4} + \frac{w^{\gamma+1}}{2\gamma+s-2}\right].$$

Finally this leads to the generating function

$$\mathcal{G}_\gamma^*(w) = w^\gamma \mathcal{D}_\gamma^*[I + \mathcal{H} + \mathcal{H}' + \mathcal{H}\mathcal{H}']w^{-\gamma} G(w) \int_0^w \frac{(1-y)^{s-2} H^*(y)}{\xi(1+a+b-c-d)^2}$$
$$\times \left[\frac{y^{\gamma-1}}{2\gamma+s-4} + \frac{y^\gamma}{2\gamma+s-2}\right] dy,$$

where

$$\xi = (a+b-1)(a+c-1)(a+d-1),$$

and

$$H^*(y) := {}_4F_3\left(\begin{matrix} 1-a+t,\ 1-a-t,\ 1-s/2,\ (3-s)/2 \\ 2-a-b,\ 2-a-c,\ 2-a-d \end{matrix};\ \frac{-4y}{(1-y)^2}\right).$$

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