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ON THE PARTIALLY ORDERED SET OF PRIME IDEALS OF A DISTRIBUTIVE LATTICE

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1. Introduction. For a distributive lattice L, let $\mathscr{P}(L)$ denote the poset of all prime ideals of L together with $\emptyset$ and L. This paper is concerned with the following type of problem. Given a class $\mathscr{C}$ of distributive lattices, characterize all posets P for which $P \cong \mathscr{P}(L)$ for some $L \in \mathscr{C}$. Such a poset P will be called *representable over* $\mathscr{C}$. For example, if $\mathscr{C}$ is the class of all relatively complemented distributive lattices, then P is representable over $\mathscr{C}$ if and only if P is a totally unordered poset with 0, 1 adjoined. One of our main results is a complete characterization of those posets P which are representable over the class of distributive lattices which are generated by their meet irreducible elements. The problem of determining which posets P are representable over the class of all distributive lattices appears to be very difficult. (See [**2**].) It will be shown that this problem is equivalent to the embeddability of P as the set of meet irreducible elements of a certain distributive algebraic lattice.

Results concerning the degree to which $\mathscr{P}(L)$ determines L are presented in §§ 4 and 5. It is shown that if $\mathscr{P}(L)$ is isomorphic with the power set of a non-empty set X, then L is a free distributive lattice.

2. Preliminaries. Let P be a poset and S a non-empty subset of P. Denote by $(S]_P$, or simply $(S]$, the set $\{x \in P | x \leqq s \text{ for some } s \in S\}$; abbreviate $(\{s\}]$ by $(s]$; $[S)$ is defined dually. S is *hereditary* if $x \leqq y$ and $y \in S$ imply $x \in S$. For a non-empty set X, 2^X will denote the poset of all subsets of X.

The class of all distributive lattices will be denoted by $\mathscr{L}$. As stated above, $\mathscr{P}(L)$ is the poset of prime ideals of L together with $\emptyset$ and L (we avoid unnecessary technical complications by *not* excluding $\emptyset$ and L from $\mathscr{P}(L)$). For each $x \in L$, let $x^* = \{I \in \mathscr{P}(L) | x \notin I\}$. Note that $\emptyset \in x^*$ and that $L \notin x^*$. It is well known (see, e.g., [**4**]) that the prime ideal theorem implies (i) $L \cong \{x^* | x \in L\}$, and (ii) if T_1, T_2 are non-empty subsets of L and

$$\bigcap \{x^* | x \in T_1\} \subseteq \bigcup \{y^* | y \in T_2\},$$

then there exist finite subsets $\emptyset \neq T_1' \subseteq T_1$, $\emptyset \neq T_2' \subseteq T_2$ such that

$$\bigcap \{x^* | x \in T_1'\} \subseteq \bigcup \{y^* | y \in T_2'\}.$$

Finally, recall that an element $x \in L$ is meet irreducible ($M.I.$) if $yz \leqq x$ implies that $y \leqq x$ or $z \leqq x$. So $(x] \in \mathscr{P}(L)$ if and only if x is $M.I.$ The class

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of all distributive lattices which are generated by $M.I.$ elements will be denoted by $\mathscr{A}$.

3. Posets representable over $\mathscr{A}$. We begin by giving a sufficient condition for a poset P to be representable over $\mathscr{L}$.

Definition 1. Let P be a poset with 0, 1. A non-zero element $k \in P$ is *weakly compact* provided that if $\emptyset \neq D \subseteq P$, $\sum_P D$ exists, and $k \leqq \sum_P D$, then there exists $\{d_1, \ldots, d_n\} \subseteq D$ such that $[d_1) \cap \ldots \cap [d_n) \subseteq [k)$.

Definition 2. A non-empty subset D of a poset P will be called *prime* provided that if $\{s, \ldots, s_n, t_1, \ldots, t_m\}$ are weakly compact in P, $\{t_1, \ldots, t_m\} \subseteq D$ and $[t_1) \cap \ldots \cap [t_m) \subseteq [s_1) \cup \ldots \cup [s_n)$, then $s_i \in D$ for some $i \in \{1, \ldots, n\}$.

Let P be a poset with $0 < 1$ and K the weakly compact members of P. Consider the following two conditions on P:

($C1$) If $p \nleqq q$ then there exists $k \in K$ such that $k \leqq p$ and $k \nleqq q$.

($C2$) If D is a prime subset of P then $\sum_P D$ exists.

THEOREM 3. *If P is a poset with $0 < 1$ that satisfies ($C1$) and ($C2$) then P is representable over $\mathscr{L}$.*

Proof. Let R be the ring of sets generated by $\{[k)' | k \in K\}$ where $[k)' = P \sim [k)$. For each $p \in P$, let $\psi(p) = \{A \in R | p \notin A\}$. Then $\psi(p) \in \mathscr{P}(R)$, so $p \rightarrow \psi(p)$ defines a function from P into $\mathscr{P}(R)$. Now $[k)'$ is a hereditary subset of P for each $k \in K$, so R is a ring of hereditary sets. It follows that ψ preserves order. Condition ($C1$) implies that $[k)' \in \psi(p) \sim \psi(q)$, so $p \leqq q$ if and only if $\psi(p) \leqq \psi(q)$.

Next, observe that $\psi(0) = \emptyset$ and $\psi(1) = R$. Now let I be a prime ideal in R and set $D = \{k \in K | [k)' \in I\}$. Then $D \neq \emptyset$ since $I \neq \emptyset$. Also, if

$$\{s_1, \ldots, s_n, t_1, \ldots, t_m\} \subseteq K, \quad \{t_1, \ldots, t_m\} \subseteq D,$$

and

$$[t_1) \cap \ldots \cap [t_m) \subseteq [s_1) \cup \ldots \cup [s_n),$$

then

$$[s_1)' \cap \ldots \cap [s_n)' \subseteq [t_1)' \cup \ldots \cup [t_m)';$$

but I is a prime and $\{[t_1)', \ldots, \{t_m)'\} \subseteq I$, so $s_i \in D$ for some $i \in \{1, \ldots, n\}$. That is, D is a prime subset of P.

By ($C2$), $p = \sum_P D$ exists and the proof will be completed by showing that $\psi(p) = I$. Let

$$A = \bigcap_{i=1}^{n} \left(\bigcup_{j=1}^{n_i} [k_{ij})' \right).$$

First, suppose that $A \in \psi(p)$; then $p \notin A$, so

$$p \in \bigcap_{j=1}^{n_{i_0}} [k_{i_0 j})$$

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for some $i_0 \in \{1, \ldots, n\}$. Let $j \in \{1, \ldots, n_{i_0}\}$. Then $k_{i_0 j} \leqq p = \sum_P D$, and since $k_{i_0 j}$ is weakly compact there exists $\{q_1, \ldots q_n\} \subseteq D$ such that $[q_1) \cap \ldots \cap [q_n) \subseteq [k_{i_0 j})$. But D is prime, so $k_{i_0 j} \in D$; hence $[k_{i_0 j})' \in I$. This means that $[k_{i_0 j})' \in I$ for $i = 1, \ldots, n_{i_0}$, and so $A \in I$. To show the reverse inclusion, let $A \in I$. Say

$$\bigcup_{j=1}^{n_1} [k_{1j})' \in I.$$

Then $\{k_{1j}, \ldots, k_{1n_1}\} \subseteq D$ so $k_{1j} \leqq \sum D = p$ for each j. Therefore,

$$p \notin \bigcup_{j=1}^{n_1} [k_{1j})'$$

and $p \notin A$ which means that $A \in \psi(p)$.

Next, we determine the weakly compact elements in $\mathscr{P}(L)$ for $L \in \mathscr{A}$.

LEMMA 4. *If $L \in \mathscr{A}$ then the following are equivalent:*
(i) *I is weakly compact in $\mathscr{P}(L)$;*
(ii) *$I = (x]_L$ for some M.I. element $x \in L$.*

Proof. Let M be the $M.I.$ elements of I and set $P = \mathscr{P}(L)$.

(i) $\Rightarrow$ (ii). Let $D = \{(u]_L | u \in M \cap I\}$. $D \neq \emptyset$ since $I \neq \emptyset$. Now I is the ideal generated by $\bigcup D$ so, in fact, $I = \sum_P D$. Since I is weakly compact, there exist $\{(u_1]_L, \ldots, (u_n]_L\} \subseteq D$ such that

$$[(u_1]_L)_P \cap \ldots \cap [(u_n]_L)_P \subseteq [I)_P.$$

Now let $x = u_1 + \ldots + u_n$. We will show $(x] = I$. Indeed, $u_i \in I$ for each $i \in \{1, \ldots, n\}$ so $(x] \subseteq I$. If $I \not\subseteq (x]$ then there exists $y \in I$ such that $y \nleqq x$ and therefore a prime ideal J such that $x \in J$, $y \notin J$. But $x \in J$ implies $J \in [(u_1]_L)_P \cap \ldots \cap [(u_n]_L)_P \subseteq [I)_P$ so $y \in I \subseteq J$, which is a contradiction. Thus, $I = (x]$ and since I is prime, x is $M.I.$

(ii) $\Rightarrow$ (i) Firstly, $x \in M$ implies that $(x] \in P$. Suppose that $\emptyset \neq D \subseteq P$, $\sum_P D$ exists, and that $(x] \subseteq \sum_P D$. Let J be the ideal generated by $\bigcup D$ and suppose that $x \notin J$. Then there is a prime ideal J' such that $x \notin J'$, $J \subseteq J'$. So $K \subseteq J \subseteq J'$ for each $K \in D$ and hence $(x] \subseteq \sum_P D \subseteq J'$, which is a contradiction. Thus, since $x \in J$, there exist $I_1, \ldots, I_n$ in D and $x_i \in I_i$ such that $x \leqq x_1 + \ldots + x_n$. Finally,

$$[I_1) \cap \ldots \cap [I_n) \subseteq [(x]_L)_P;$$

for if $K \in P$ and $K \in [I_1) \cap \ldots \cap [I_n)$, then for each $i \in \{1, \ldots, n\}$, $x_i \in I_i \subseteq K$, so $x \in K$. Hence $(x]_L \subseteq K$.

THEOREM 5. *A poset P is representable over $\mathscr{A}$ if and only if P satisfies $(C1)$ and $(C2)$.*

Proof. Suppose first that P satisfies $(C1)$ and $(C2)$. Now the ring R constructed in Theorem 3 was generated by $\{[k)' | k \in K\}$ for some non-empty subset $K \subseteq P$. From this, it is easily verified that $[k)'$ is $M.I.$ for each $k \in K$.

Conversely, suppose that $P = \mathscr{P}(L)$, where L is generated by the set M of $M.I.$ elements of L. For $(C1)$, suppose that $\{I, J\} \subseteq P$ and that $I \nsubseteq J$. Then there is an element $x \in I \sim J$. But since x is a sum of products of members of M, there is an element $u \in M \cap (I \sim J)$. By Lemma 4, $(u]$ is weakly compact in P, and also $(u] \subseteq I$, $(u] \nsubseteq J$. For $(C2)$, suppose that D is prime in P. Let J be the ideal in L generated by $\bigcup D$. Clearly, if $J \in P$, then $\sum D$ will exist and equal J. Now to prove that J is a prime ideal, it is sufficient to show that if $\{u_1, \ldots, u_n\} \subseteq M$ and $u_1 \cdot \ldots \cdot u_n \in J$, then $u_i \in J$ for some $i \in \{1, \ldots, n\}$. But if $u_1 \cdot \ldots \cdot u_n \in J$, then there are members $I_1, \ldots, I_m$ of D and elements $x_i \in I_i$ such that $u_1 \cdot \ldots \cdot u_n \leqq x_1 + \ldots + x_m$. Now I_i is weakly compact since D is prime, so $I_i = (y_i]$ where $y_i \in M$ for each $i \in \{1, \ldots, m\}$. Hence $u_1 \ldots u_n \leqq y_1 + \ldots + y_m$ and so

$$[(y_1]_L)_P \cap \ldots \cap [(y_m]_L)_P \subseteq [(u_1]_L)_P \cup \ldots \cup [(u_n]_L)_P.$$

Invoking the primeness of D again, we find that $(u_i]_L \in D$ for some i, so $u_i \in J$.

To show how conditions $(C1)$ and $(C2)$ can be applied in specific cases we present the following corollary:

COROLLARY 6. *If P is a poset with $0 < 1$ and $[p)$ is finite for each $p \neq 0$, then it is representable over $\mathscr{A}$.*

Proof. Let D be a non-empty subset of $P \sim \{0\}$. For each finite, non-empty subset $T \subseteq D$, $\bigcap_{t \in T}[t)$ is finite and contains 1. Let n be the least number of elements in $\bigcap_{t \in T}[t)$ for any such $T \subseteq D$ and let T_0 be a finite non-empty subset of D such that $\bigcap_{t \in T_0}[t) = \{x_1, \ldots, x_n\}$. Then the elements $\{x_1, \ldots, x_n\}$ are all upper bounds of D, for clearly $t \leqq x_i$ for all $t \in T_0$ and if $d \in D \sim T_0$, then

$$(\bigcap_{t \in T_0}[t)) \cap [d) \subseteq \bigcap_{t \in T_0}[t) = \{x_1, \ldots, x_n\},$$

so by the minimality of n,

$$(\bigcap_{t \in T_0}[t)) \cap [d) = \{x_1, \ldots, x_n\}$$

and hence $d \leqq x_i$.

We now proceed to verify $(C1)$. Suppose that $p \neq 0$, $\sum_P D$ exists, and that $p \leqq \sum_P D$. But then $u \in \bigcap_{t \in T_0}[t)$ implies that $x_i \leqq u$ for some $i \in \{1, \ldots, n\}$ and as x_i is an upper bound for D, $p \leqq \sum_P D \leqq x_i \leqq u$. Thus, $\bigcap_{t \in T_0}[t) \subseteq [p)$. For $(C2)$, suppose that D is prime in P. Let $\{u_1, \ldots, u_m\}$ be the minimal elements of $\{x_1, \ldots, x_n\}$ so that $\bigcap_{t \in T_0}[t) = [u_1) \cup \ldots \cup [u_m)$. By the definition of prime, $u_{i_0} \in D$ for some $i_0 \in \{1, \ldots, m\}$. But then

$$(\bigcap_{t \in T_0}[t)) \cap [u_{i_0}) \subseteq \{x_1, \ldots, x_n\}$$

and again by the minimality of n,

$$[u_{i_0}) = (\bigcap_{t \in t_0}[t)) \cap [u_{i_0}) = \{x_1, \ldots, x_n\}.$$

It follows that $\sum_P D$ exists and equals u_{i_0}.

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COROLLARY 7. *Every finite poset and every totally unordered poset, with* 0 *and* 1 *adjoined, is representable over* $\mathscr{A}$.

4. Uniqueness of posets representable over $\mathscr{A}$. Corollary 7 implies that posets representable over $\mathscr{A}$ may also be representable by distributive lattices outside of $\mathscr{A}$. Indeed, choose a non-atomic Boolean algebra B. Then there is a lattice $L \in \mathscr{A}$ such that $\mathscr{P}(B) = \mathscr{P}(L)$. However, within the class $\mathscr{A}$, we do have uniqueness.

THEOREM 8. *If L and L' are members of $\mathscr{A}$ and $\mathscr{P}(L) \cong \mathscr{P}(L')$, then $L \cong L'$.*

Proof. Let M and M' be the sets of $M.I.$ elements of L and L', respectively, and let f: $\mathscr{P}(L) \to \mathscr{P}(L')$ be an isomorphism. Since f induces an isomorphism between the set of weakly compact elements of $\mathscr{P}(L)$ and the set of weakly compact elements of $\mathscr{P}(L')$, Lemma 4 implies the existence of an isomorphism $g:M \to M'$ such that $f((x]) = (g(x)]$. To show that g can be extended to a homomorphism $G:L \to L'$, it is sufficient to prove that if S and T are finite non-empty subsets of M, and $\Pi S \leqq \sum T$, then $\Pi g(S) \leqq \sum g(T)$. Indeed, this condition implies that the function $G:L \to L'$ defined by

$$G(\Pi S_1 + \ldots + \Pi S_n) = \Pi g(S_n) + \ldots + \Pi g(S_n)$$

is well defined. It is easy then to verify that G is a homomorphism; the details can be found, for example, in [**1**, Lemma 1.7].

Now suppose that $\Pi g(S) \nleqq \sum g(T)$. Then there exists $I \in P(L')$ such that $\sum g(T) \in I$ and $\Pi g(S) \notin I$. For each $t \in T$, $g(t) \in I$, so $f((t]) = (g(t)] \subseteq I$. Hence, $t \in (t] \subseteq f^{-1}(I)$. But then $\sum T \in f^{-1}(I)$, so $s \in f^{-1}(I)$ for some $s \in S$ and, therefore, $(s] \subseteq f^{-1}(I)$. Finally, $g(s) \in (g(s)] = f((s]) \subseteq I$, which is a contradiction. Thus, there is a homomorphism $G:L \to L'$ such that $G|M = g$. Similarly, there is a homomorphism $G':L' \to L$ such that $G'|:M' = g^{-1}$. It follows that G is an isomorphism.

The existence and uniqueness of representable chains can now be described completely.

THEOREM 9. *If C is a chain which is representable over $\mathscr{L}$, then C is complete and each interval $(a, b]$ contains an element with an immediate predecessor. Moreover, the representation of C over $\mathscr{L}$ is unique. Conversely, if C is a complete chain in which each interval $(a, b]$ contains an element with an immediate predecessor then $C \cong \mathscr{P}(C_1)$ for some chain C_1.*

Proof. If $C = \mathscr{P}(L)$ for some $L \in \mathscr{L}$, then C is closed under arbitrary unions, so C is complete. For $\{I, J\} \subseteq \mathscr{P}(L)$, if $I \subset J$, then there is an element $x \in J \sim I$, so $I \subset (x] \subseteq J$. Since C is a chain, so is L, and hence $(x] \in \mathscr{P}(L)$. The immediate predecessor of $(x]$ is $\{u \in L | u < x\}$. Next, if $\mathscr{P}(L) \cong C \cong \mathscr{P}(L')$, then L and L' are chains and hence in $\mathscr{M}$. By Theorem 8, $L \cong L'$. For the converse, it is sufficient to prove that if $c \in C$ has an immediate predecessor c', then c is weakly compact. Thus, suppose that $c \leqq \sum_P D$. If $d < c$ for all $d \in D$, then $d \leqq c'$, so $c \leqq \sum_P D \leqq c' < c$. Hence, $c \leqq d_0$ for some $d_0 \in D$.

5. Free distributive lattices. In this section we show that $P = 2^X$ is representable only as the free distributive lattice on $|X|$ free generators. The fact that $\mathscr{P}(L) \cong 2^X$ when L is free is well known.

LEMMA 10. *Let L be a distributive lattice and suppose that $P = \mathscr{P}(L)$ is complete. If T is a finite non-empty set of M.I. elements of L, then*

$$\sum_P \{(t] | t \in T\} = (\sum T], \tag{3}$$

and $\sum T$ is M.I. in L.

Proof. For each $t \in T$, $t \in (t] \subseteq \sum_P \{(t] | t \in T\}$, so

$$(\sum T] \subseteq \sum_P \{(t] | t \in T\}.$$

Conversely, if $u \notin (\sum T]$, then there is a prime ideal I such that $u \notin I$, $\sum T \in I$. But then $T \subseteq I$, so $(t] \subseteq I$ for each $t \in I$. Hence $\sum_P \{(t] | t \in I\} \subseteq I$, and so $u \notin \sum_P \{(t] | t \in I\}$. Since $(\sum T] \in P$, $\sum T$ is *M.I.*

LEMMA 11. *Let $L \in \mathscr{L}$ and let $P = \mathscr{P}(L) \cong 2^X$ for some $X \neq \emptyset$. Then I is an atom in P if and only if $I = (m]$, where m is M.I. in L and is minimal in the set of all M.I. elements in L.*

Proof. Sufficiency. Suppose that I has no greatest element. Then for each $u \in I$, there exists $v_u \in I$ and $I(u, v_u) \in P$ such that $v_u \nleqq u$, $u \in I(u, v_u)$, and $v_u \notin I(u, v_u)$. Let $S = \{I(u, v_u) | u \in I\}$. Since $v_u \in I \sim I(u, v_u)$, we have $I \nsubseteq J$ for all $J \in S$. But I is an atom in P which implies that $I \cdot J = 0_P$ for all $J \in S$. Now $I \subseteq \bigcup S \subseteq \sum_P S$, and since the Boolean algebra P is $(2, \infty)$-distributive,

$$\begin{aligned} I &= I \cdot \textstyle\sum_P S \\ &= \textstyle\sum_P \{I \cdot J | J \in S\} \\ &= 0_P, \end{aligned}$$

contradicting the definition of an atom.

So I has a greatest element m. It follows that $I = (m]$, m is *M.I.*, and is, in fact, minimal in the set of all *M.I.* elements in L.

Necessity. Under the conditions of the hypothesis, $(m] \in P$ and $(m] \neq \emptyset = 0_P$. So there is an atom $J \in P$ such that $J \subseteq (m]$. But from the converse, $J = (n]$ where n is *M.I.* Since m is minimal in the set of *M.I.* elements and $n \leqq m$, we have $n = m$, so $(m] = J$ is an atom in P.

THEOREM 12. *If L is a distributive lattice and $\mathscr{P}(L) \cong 2^X$ for some non-empty set X, then L is the free distributive lattice on $|X|$ free generators.*

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Proof. Let $P = \mathscr{P}(L)$ and let S be the minimal elements in the set of $M.I.$ elements of L. $|S| = |X| > 0$ by Lemma 11. We prove first that S is an independent set.

Let T_1 and T_2 be finite non-empty subsets of S such that $\Pi T_1 \leqq \sum T_2$. By Lemma 10, $\sum T_2$ is $M.I.$, so there exists $t_1 \in T_1$ such that $t_1 \leqq \sum T_2$. Now

$$(t_1] \subseteq (\textstyle\sum T_2] = \sum_P \{(t] | t \in T_2\},$$

and since $t_1 \in S$, $(t_1]$ is an atom in P; so there exists $t_2 \in T_2$ such that $(t_1] \subseteq (t_2]$. The minimality of t_2 and $t_1 \leqq t_2$ imply $t_1 = t_2$.

Since independent sets generate free distributive lattices, it suffices to prove that S generates L. For this purpose, let $S^\Sigma = \{\sum T | T$ is a finite non-empty subset of $S\}$. Recall from Lemma 10 that the members of S^Σ are $M.I.$ in L. We prove that if $I \in P$ and $I \neq \emptyset$, then $I \cap S^\Sigma \neq \emptyset$ and

$$\textstyle\sum_P \{(t] | t \in I \cap S^\Sigma\} = \bigcup \{(t] | t \in I \cap S^\Sigma\}.$$

Since $I \neq \emptyset$, it is a sum of atoms in P. By Lemma 11, there is a member $y \in S$ such that $(y] \subseteq I$, so $y \in I \cap S \subseteq I \cap S^\Sigma$. For the second part of the assertion it is easily verified that

$$\bigcup \{(t] | t \in I \cap S^\Sigma\} \in P.$$

We will now show that each $x \in L$ is a finite product of members of S^Σ. The work is divided into two cases.

Firstly, assume that $S^\Sigma \cap [x) \neq \emptyset$. Then

$$x^* = \bigcap \{y^* | y \in S^\Sigma \cap [x)\}. \tag{4}$$

To see this, let $I \in x^*$. Then $x \notin I$, so $y \in S^\Sigma \cap [x)$ implies that $y \notin I$ and, therefore, that $I \in y^*$. Conversely, suppose that

$$I \in \bigcap \{y^* | y \in S^\Sigma \cap [x)\} \sim x^*.$$

Now

$$I = \textstyle\sum_P \{(t] | t \in S^\Sigma \cap I\}$$
$$= \bigcup \{(t] | t \in S^\Sigma \cap I\},$$

and as $x \in I$, $x \in (t]$ for some $t \in S^\Sigma \cap I$. So $t \in S^\Sigma \cap [x)$ and, therefore, $I \in t^*$, which is a contradiction. But by (2), (4) implies that $x^* = y_1^* \cap \ldots \cap y_n^*$ for some $y_i \in S^\Sigma$ and hence that $x = y_1 \cdot \ldots \cdot y_n$, which completes the proof for this case.

Finally, suppose that $S^\Sigma \cap [x) = \emptyset$. Thus, $x \nleqq t$ for all $t \in S^\Sigma$. Then

$$x^* = \bigcup \{s^* | s \in S\}. \tag{5}$$

Indeed, if $I \notin x^*$ then $x \in I = \bigcup \{(t] | t \in I \cap S^\Sigma\}$, so $x \leqq t$ for some $t \in S^\Sigma$, which is a contradiction. If $I \in x^*$, then $x \notin I$, so either $I = \emptyset$ or there is an atom $(s]$, $s \in S$, such that $(s] \nsubseteq I$ and therefore, in either case, $I \in \bigcup \{s^* | s \in S\}$.

From (5), it follows that $x = s_1 + \ldots + s_n \in S^\Sigma$.

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6. Posets representable over $\mathscr{L}$. A characterization of those posets representable over L can be obtained immediately from (1) and (2).

THEOREM 13. *A poset P is representable over $\mathscr{L}$ if and only if P has $0 < 1$ and there is a ring R of non-empty, proper, hereditary subsets of P satisfying:*
(i) *If $p \nleqq q$ then there exists $A \in R$ such that $p \notin A$ and $q \in A$.*
(ii) *If $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ are non-empty families in R and*

$$\bigcap \{A_i | i \in I\} \subseteq \{B_j | j \in J\},$$

then there exist finite non-empty subsets $I' \subseteq I$ and $J' \subseteq J$ such that

$$\bigcap \{A_i | i \in I'\} \subseteq \bigcup \{B_j | j \in J'\}.$$

Another characterization can be obtained by distinguishing the prime ideals in the class of all ideals of a distributive lattice.

THEOREM 14. *A poset P with $0 < 1$ is representable over $\mathscr{L}$ if and only if P is the set of M.I. elements of a distributive algebraic lattice L in which the non-zero compact elements K form a sublattice of L.*

Proof. For the sufficiency of the condition, we show that $P \cong \mathscr{P}(K)$. For each $p \in P$, let $\psi(p) = \{q \in K | q \leqq p\}$. The relation $p \mapsto \psi(p)$ establishes a function from P into $\mathscr{P}(K)$ which is order preserving in both directions. To show that ψ is onto, first note that $\psi(0) = \emptyset$ and that $\psi(1) = K$. Now let I be a prime ideal in K. Set $p = \sum_L I$. To show that p is *M.I.*, suppose that $xy \leqq p$ but $x \nleqq p$ and $y \nleqq p$ for some $\{x, y\} \subseteq P$. Since L is algebraic, there exists $\{s, t\} \subseteq K$ such that $s \leqq x$, $s \nleqq p$ and $t \leqq y$, $t \nleqq p$. But $st \leqq xy \leqq p = \sum_L I$, and since K is a sublattice of L, there exists $\{x_1, \ldots, x_n\} \subseteq I$ such that $st \leqq x_1 + \ldots + x_n$. But I is prime, so $s \in I$ or $t \in I$. Thus, either $s \leqq \sum I = p$ or $t \leqq \sum I = p$, which is a contradiction. Hence, $p \in P$ and it follows that $\psi(p) = I$.

Conversely, suppose that L is a distributive lattice and that $P = \mathscr{P}(L)$. Let $\mathscr{I}(L)$ be the poset of all ideals in L together with $\emptyset$. $\mathscr{I}(L)$ is a complete lattice where $\Pi S = \bigcap S$ and $\sum S$ is the ideal generated by $\bigcup S$. Since L is distributive, it follows that $\mathscr{I}(L)$ is also distributive. Since $I \in \mathscr{I}(L)$ can be represented by $I = \sum \{(x] | x \in I\}$, it is easily verified that $\mathscr{I}(L)$ is an algebraic lattice. It remains to show that $I \in \mathscr{P}(L)$ if and only if I is *M.I.* in $\mathscr{I}(L)$. Let $I \in \mathscr{P}(L)$ and let $J \cdot J_1 \subseteq I$. If $J \nsubseteq I$, then there is an element $u \in J \sim I$. But then $J_1 \subseteq I$; for, if $x \in J_1$, then $xu \in J \cap J_1 = J \cdot J_1 \subseteq I$, and so $x \in I$. On the other hand, if I is *M.I.* in $\mathscr{I}(L)$ and $xy \in I$, then $(x] \cdot (y] = (xy] \subseteq I$, so $(x] \subseteq I$ or $(y] \subseteq I$. Hence, $x \in I$ or $y \in I$, and $I \in \mathscr{P}(L)$.

Neither Theorem 13 nor Theorem 14 is an optimal solution, since neither really tells us much about P itself. We therefore ask for a characterization of posets representable over $\mathscr{L}$, which is analogous the solution for $\mathscr{A}$ given in Theorem 5.

References

1. R. Balbes, *Projective and injective distributive lattices*, Pacific J. Math. *21* (1967), 405–420.

2. G. Grätzer, *Lattice theory: First concepts and distributive lattices* (Freeman, San Francisco, 1971).

3. L. Nachbin, *Une proprieté characteristique des algébres Booléiennes*, Portugal. Math. *6* (1947), 115–118.

4. M. H. Stone, *Topological representations of distributive lattices and Browerian logics*, Časopis Pěst. Mat. *67* (1937), 1–25.

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