Maps on Quantum States in C^*-algebras Preserving von Neumann Entropy or Schatten p-norm of Convex Combinations

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Abstract. Very recently, Karder and Petek completely described maps on density matrices (positive semidefinite matrices with unit trace) preserving certain entropy-like convex functionals of any convex combination. As a result, maps could be characterized that preserve von Neumann entropy or Schatten p-norm of any convex combination of quantum states (whose mathematical representatives are the density matrices). In this note we consider these latter two problems on the set of invertible density operators, in a much more general setting, on the set of positive invertible elements with unit trace in a C^*-algebra.

1 Introduction and Statements of the Results

In [8], Karder and Petek completely described maps on density matrices preserving certain entropy-like convex functionals of any convex combination. More precisely, for a given strictly convex function $f:[0,1] \to \mathbb{R}$ and density matrix A, they introduced the numerical quantity

$$F(A) := \sum_{k=1}^{n} f(\lambda_k)$$

where the numbers λ_k-s are the eigenvalues of A counted with multiplicity, and described the structure of those transformations on density matrices satisfying

$$F\big(tA + (1-t)B\big) = F\big(t\phi(A) + (1-t)\phi(B)\big)$$

for all $t \in [0,1]$ and density matrices A, B. By particular choices of the function f, maps could be characterized that preserve von Neumann entropy or Schatten p-norm of any convex combination of quantum states. Driven by the aforementioned result, in this note we consider these latter two problems in the setting of C^*-algebras carrying faithful normalized traces, on the set of positive invertible elements with unit trace.

To formulate our results, we need a short summary of some notation, basic concepts and facts, which is given in the next paragraphs.

Throughout the sequel, $\mathcal{A}$ denotes a unital complex C^*-algebra with unit e. The symbols $\mathcal{A}_{sa}, \mathcal{A}^+, \mathcal{A}_{-1}^+$ stand for the space of self-adjoint elements, the cone of positive elements, and the cone of positive invertible elements in $\mathcal{A}$, respectively. Furthermore,

we shall assume that $\mathcal{A}$ admits at least one faithful normalized trace τ, by which we mean a linear functional $\tau:\mathcal{A} \to \mathbb{C}$ satisfying

(a) $\tau(ab) = \tau(ba)$,
(b) $\tau(a^*a) \geq 0$,
(c) $\tau(a^*a) = 0$ if and only if $a = 0$,
(d) $\tau(e) = 1$

for all $a, b \in \mathcal{A}$. Fundamental examples for such algebras are the irrational rotational algebras (or, in another words, noncommutative tori) and finite von Neumann factors.

In the operator algebraic framework of quantum mechanics, quantum states are often identified with positive elements with unit trace [4]. As this definition depends on the choice of τ, the set

$$\mathcal{D}_\tau := \{ a \in \mathcal{A}^+ : \tau(a) = 1 \}$$

is called the τ-density space of $\mathcal{A}$. The set of invertible elements in $\mathcal{D}_\tau$ will be denoted by $\mathcal{M}_\tau$, that is,

$$\mathcal{M}_\tau := \{ a \in \mathcal{A}_{-1}^+ : \tau(a) = 1 \}.$$

In certain applications in quantum theory, especially when differential geometric considerations are made and corresponding analytic tools are applied, it is very natural to deal with $\mathcal{M}_\tau$ instead the whole $\mathcal{D}_\tau$ (see *e.g.,* [6] and the references therein).

Let $\eta(x) := x \log x$ be the so-called standard entropy function. Note that if a faithful normalized trace τ exists on $\mathcal{A}$, then the von Neumann entropy and the Schatten p-norm of an element $a \in \mathcal{A}$ could be defined as

$$S(a) := -\tau(\eta(a)) \quad \text{and} \quad \|a\|_p := (\tau(|a|^p))^{1/p},$$

respectively, without any difficulty. We remark that if $\mathcal{A}$ is such an algebra, then $\|\cdot\|_p$ is indeed a norm on $\mathcal{A}$ for any number $p \geq 1$; see *e.g.,* [3]. Moreover, we recall that a linear bijection J of $\mathcal{A}$ is called a *Jordan $*$-isomorphism* if it satisfies $J(a^2) = J(a)^2$ and $J(a^*) = J(a)^*$ for any $a \in \mathcal{A}$.

Our main result reads as follows.

Theorem 1.1   *For a C^*-algebra $\mathcal{A}$ with faithful normalized trace τ and a bijective map $\phi:\mathcal{M}_\tau \to \mathcal{M}_\tau$, the following statements are equivalent:*

(i)   *ϕ preserves the von Neumann entropy of any convex combination, that is, it satisfies*

$$S\big(ta + (1-t)b \big) = S\big(t\phi(a) + (1-t)\phi(b) \big) \quad \big(a, b \in \mathcal{M}_\tau, \ t \in [0,1] \big).$$

(ii)   *For a fixed number $p > 1$, ϕ preserves the Schatten p-norm of any convex combination; that is, we have*

$$\|ta + (1-t)b\|_p = \|t\phi(a) + (1-t)\phi(b)\|_p \quad (a, b \in \mathcal{M}_\tau, \ t \in [0,1]).$$

(iii) *There exists a trace preserving Jordan $*$-isomorphism J of $\mathcal{A}$ such that*

$$\phi(a) = J(a) \quad \big(a \in \mathcal{M}_\tau \big).$$

Jordan $*$-isomorphisms are quite general, but somewhat more information can be elicited under certain restrictions on the underlying C^*-algebra. Recall that a von Neumann algebra with center $\{\lambda e : \lambda \in \mathbb{C}\}$ is called a *factor*.

Corollary 1.2 *Let $\mathcal{A} \neq \{\lambda e : \lambda \in \mathbb{C}\}$ be a finite von Neumann factor.*

(i) *A unique faithful normalized trace τ exists on $\mathcal{A}$.*

(ii) *The bijective transformation $\phi \colon \mathcal{M}_\tau \to \mathcal{M}_\tau$ satisfies*

$$S\big(ta + (1-t)b\big) = S\big(t\phi(a) + (1-t)\phi(b)\big) \quad (a, b \in \mathcal{M}_\tau)$$

or

$$\|ta + (1-t)b\|_p = \|t\phi(a) + (1-t)\phi(b)\|_p \quad (a, b \in \mathcal{M}_\tau)$$

for all $t \in [0,1]$ if and only if ϕ extends to a trace preserving map $\tilde{\phi}$ on $\mathcal{A}$ that is either an algebra $*$-isomorphism or an algebra $*$-anti-isomorphism.

In the classical setting of matrix algebras, the following known result (*cf.* [10, Theorem 1]) is recovered.

Corollary 1.3 *Let $\mathcal{A}$ be the matrix algebra of all n by n complex matrices and let $\mathcal{M} \subseteq \mathcal{A}$ be the set of invertible density matrices. The bijective map $\phi \colon \mathcal{M} \to \mathcal{M}$ satisfies*

$$S(tA + (1-t)B) = S(t\phi(A) + (1-t)\phi(B)) \quad (A, B \in \mathcal{M})$$

for all $t \in [0,1]$ if and only if there exists a unitary matrix $U \in \mathcal{A}$ such that ϕ is of one of the following forms:

(i) $\phi(A) = UAU^* \quad (A \in \mathcal{M})$;

(ii) $\phi(A) = UA^{tr}U^* \quad (A \in \mathcal{M})$.

Here, the symbol ^{tr} refers for the transposition. The following result is new in the matrix algebra setting as well.

Corollary 1.4 *Keeping the notation in Corollary 1.3, the bijection $\phi \colon \mathcal{M} \to \mathcal{M}$ satisfies*

$$\|tA + (1-t)B\|_p = \|t\phi(A) + (1-t)\phi(B)\|_p \quad (A, B \in \mathcal{M})$$

for all $t \in [0,1]$ if and only if there exists a unitary $U \in \mathcal{A}$ such that ϕ is of either of form (i) *or form* (ii).

2 Proofs

In the proofs, our major tool is the following formula, which is known as the Dixmier–Kadison–Fuglede differential rule [2, p. 119, Lemma 3].

Lemma 2.1 *Let $\mathcal{A}$ be a C^*-algebra with a normalized trace τ and assume that $f \colon \Omega \to \mathbb{C}$ is a holomorphic function on an open set $\Omega \subseteq \mathbb{C}$. Let $t \mapsto h_t$ $(t \in {]}0,1{[})$ be a continuously differentiable path of elements in $\mathcal{A}_{sa}$ such that $\bigcup_{t \in {]}0,1{[}} \sigma(h_t) \subseteq \Omega$. Then the map $t \mapsto f(h_t)$ $(t \in {]}0,1{[})$ is differentiable and*

$$\frac{d}{dt}\tau\big(f(h_t)\big) = \tau\Big(f'(h_t)\frac{dh_t}{dt}\Big) \quad (t \in {]}0,1{[}).$$

We remark that Lemma 2.1 was originally formulated for von Neumann algebras but a closer look at its proof shows that the statement remains true for C^*-algebras equipped with a normalized trace.

We also need the following observation, which is a part of [9, Lemma 6].

Lemma 2.2 *Assume that $\mathcal{A}$ is a C^*-algebra that possesses a faithful normalized trace τ. For any given $a \in \mathcal{A}$, we have $\tau(ax) = 0$ for all $x \in \mathcal{A}_{-1}^+$ if and only if $a = 0$.*

Now we are in a position to present the proof of our first result.

Proof of Theorem 1.1 The condition in the theorem implies that

$$(2.1) \qquad \tau\big(\eta(ta + (1-t)b)\big) = \tau\big(\eta(t\phi(a) + (1-t)\phi(b))\big) \quad (a, b \in \mathcal{M}_\tau)$$

holds for all $t \in [0, 1]$. Differentiating both sides of (2.1) at $t = 0+$ and an application of Lemma 2.1 yield that

$$\tau\big((\log b + e)(a - b)\big) = \tau\big((\log \phi(b) + e)(\phi(a) - \phi(b))\big) \quad (a, b \in \mathcal{M}_\tau).$$

Rearranging this equality and taking into account that $\tau(a) = \tau(b) = \tau(\phi(a)) = \tau(\phi(b)) = 1$, we get

$$\tau\big((\log b)a\big) - \tau\big((\log b)b\big) = \tau\big((\log \phi(b))\phi(a)\big) - \tau\big((\log \phi(b))\phi(b)\big) \ (a, b \in \mathcal{M}_\tau).$$

We also obtain from (2.1) and by taking $t = 0$ that

$$\tau\big(\eta(b)\big) = \tau\big((\log b)b\big) = \tau\big((\log \phi(b))\phi(b)\big) \quad (a, b \in \mathcal{M}_\tau),$$

and thus

$$\tau\big((\log b)a\big) = \tau\big((\log \phi(b))\phi(a)\big) \quad (a, b \in \mathcal{M}_\tau).$$

Define $\psi \colon \mathcal{A}_{-1}^+ \to \mathcal{A}_{-1}^+$ by the formula

$$\psi(a) := \tau(a)\phi\Big(\frac{a}{\tau(a)}\Big).$$

We assert that the transformation ψ is a bijection. To verify this, assume first that $\psi(a) = \psi(b)$ holds. As ϕ maps into $\mathcal{M}_\tau$, ψ preserves the trace. Hence, $\tau(a) = \tau(\psi(a)) = \tau(\psi(b)) = \tau(b)$ implying that $\phi(a/\tau(a)) = \phi(b/\tau(b))$. Now the injectivity of ψ follows from that of ϕ. As for the surjectivity, let $y := \tau(a)\phi^{-1}(a/(\tau(a)))$ with some $a \in \mathcal{A}_{-1}^+$. Then we have $\psi(y) = a$, which verifies our claim.

We proceed as follows. For any $a, b \in \mathcal{A}_{-1}^+$, we have

$$(2.2) \qquad \tau\Bigg(\Big[\log\Big(\frac{b}{\tau(b)}\Big)\Big]\frac{a}{\tau(a)}\Bigg) = \tau\Bigg(\Big[\log\Big(\frac{\psi(b)}{\tau(b)}\Big)\Big]\frac{\psi(a)}{\tau(a)}\Bigg).$$

Multiplying both sides of (2.2) by $\tau(a)$ and taking into account that $\log(c/\tau(c)) = \log c - (\log \tau(c))e$, one finds that

$$\tau\big((\log b)a\big) - \big(\log \tau(b)\big)\tau(a) = \tau\big((\log \psi(b))\psi(a)\big)$$
$$- \big(\log \tau(b)\big)\tau\big(\psi(a)\big) \quad (a, b \in \mathcal{A}_{-1}^+).$$

The aforementioned trace preserving property of ψ yields that

$$(2.3) \qquad \tau\big((\log b)a\big) = \tau\big((\log \psi(b))\psi(a)\big) \quad (a, b \in \mathcal{A}_{-1}^+)$$

is also satisfied.

As $\log b$ runs through the whole $\mathcal{A}_{sa} \supseteq \mathcal{A}_{-1}^+$, an application of Lemma 2.2 shows that the transformation ψ is additive. The structure of additive bijections on $\mathcal{A}_{-1}^+$

is described in [1]. According to [1, Lemma 8] there exist a Jordan $*$-isomorphism $J: \mathcal{A} \to \mathcal{A}$ and an invertible element $c \in \mathcal{A}_{-1}^{+}$ such that

$$\psi(a) = cJ(a)c \quad (a \in \mathcal{A}_{-1}^{+}).$$

From (2.3), we infer that

(2.4)
$$\begin{aligned}\tau\big((\log b)a\big) &= \tau\big((\log cJ(b)c)cJ(a)c\big) \\ &= \tau\big(c(\log cJ(b)c)cJ(a)\big) \quad (a, b \in \mathcal{A}_{-1}^{+}).\end{aligned}$$

As any Jordan $*$-isomorphism sends the unit to itself, substituting $b = e$ into (2.4) yields that

$$0 = \tau\big(c(\log c^2)cJ(a)\big) = 2\tau\big(c(\log c)cJ(a)\big) \quad (a \in \mathcal{A}_{-1}^{+}).$$

By Lemma 2.2, again, we obtain that $c(\log c)c = 0$. Hence, $\log c = 0$ or, equivalently, $c = e$. This gives us the necessity part of the statement in the theorem.

As for the sufficiency, it can be checked straightforwardly by referring to the fact that Jordan $*$-isomorphisms are compatible with the continuous function calculus in the sense that $J(f(a)) = f(J(a))$ holds true for any self-adjoint element $a \in \mathcal{A}_{sa}$, and complex-valued continuous function f defined on the spectrum of a. (This assertion follows from the well-known fact that a Jordan $*$-homomorphism J respects the Jordan triple product $(a, b) \mapsto aba$ (see *e.g.*, [5, Lemma 2]) from which it can be seen that J is compatible with the power operations whenever J is unital, and from polynomial approximation of continuous functions.) The proof is complete. ∎

Proof of Corollary 1.2 (i) is well known (*cf.* [7, 8.2.8. Theorem]). As for (ii), we recall that an algebra is called *prime* whenever, for any $a, b \in \mathcal{A}$, the equality $a\mathcal{A}b = \{0\}$ implies that $a = 0$ or $b = 0$. By an old result of Herstein [11, 6.3.7 Theorem], any Jordan $*$-homomorphism onto a prime ring is either multiplicative or anti-multiplicative. Thus Jordan $*$-isomorphisms onto finite von Neumann algebras are either algebra $*$-isomorphisms or algebra $*$-anti-isomorphisms, because any von Neumann factor is a prime algebra. The result now follows from Theorem 1.1. ∎

Proof of Corollaries 1.3 and 1.4 It is apparent that the matrix algebra $\mathcal{M}$ is a finite von Neumann factor. Every algebra $*$-isomorphism on $\mathcal{M}$ is implemented by a unitary similarity transformation, and every algebra $*$-antiisomorphism can be obtained as a unitary similarity transformation composed by transposition, whence the result. ∎

References

[1] R. Beneduci and L. Molnár, *On the standard K-loop structure of positive invertible elements in a C*-algebra*. J. Math. Anal. Appl. **420**(2014), 551–562. http://dx.doi.org/10.1016/j.jmaa.2014.05.009

[2] J. Dixmier, *Von Neumann algebras*. North-Holland Mathematical Library, 27, North-Holland Publishing Company, Amsterdam-New York, 1981.

[3] T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators*. Pacific J. Math. **123**(1986), 269–300. http://dx.doi.org/10.2140/pjm.1986.123.269

[4] D. Farenick, S. Jaques, and M. Rahaman, *The fidelity of density operators in an operator-algebraic framework*. J. Math. Phys. **57**(2016), 102202.   http://dx.doi.org/10.1063/1.4965876

[5] I. N. Herstein, *Jordan homomorphisms*. Trans. Amer. Math. Soc. **81**(1956), 331–341.
http://dx.doi.org/10.1090/S0002-9947-1956-0076751-6

[6] A. Jenčová, *Geodesic distances on density matrices*, J. Math. Phys. **45**(2004), 1787–1794.
http://dx.doi.org/10.1063/1.1689000

[7] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*. Vol II., Pure and Applied Mathematics, 100, Academic Press, Orlando, FL, 1986.

[8] M. Karder and T. Petek, *Maps on quaternion states preserving generalized entropy of convex combination*. Linear Algebra Appl. **532**(2017), 86–98.   http://dx.doi.org/10.1016/j.laa.2017.06.003

[9] L. Molnár, *Maps on the positive definite cone of a C*-algebra preserving certain quasi-entropies*. J. Math. Anal. Appl. **447**(2017), 206–221.   http://dx.doi.org/10.1016/j.jmaa.2016.09.067

[10] L. Molnár and G. Nagy, *Transformations on density operators that leave the Holevo bound invariant*. Int. J. Theor. Phys. **53**(2014), 3273–3278.   http://dx.doi.org/10.1007/s10773-013-1638-8

[11] T. W. Palmer, *Banach algebras and the general theory of ∗-algebras. Vol. I*. Encyclopedia of Mathematics and its Applications, 49, Cambridge University Press, Cambridge, 1994.
http://dx.doi.org/10.1017/CBO9781107325777

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