Canad. J. Math. Vol. 69 (3), 2017 pp. 532–547
http://dx.doi.org/10.4153/CJM-2016-024-2

Dirichlet's Theorem in Function Fields

Arijit Ganguly and Anish Ghosh

Abstract. We study metric Diophantine approximation for function fields, specifically, the problem of improving Dirichlet's theorem in Diophantine approximation.

1 Introduction

1.1 The Set Up

Let p be a prime and let $q := p^r$, where $r \in \mathbb{N}$, and consider the function field $\mathbb{F}_q(T)$. We define a function $|\cdot| : \mathbb{F}_q(T) \to \mathbb{R}_{\geq 0}$ as follows:

$$|0| := 0 \quad \text{and} \quad \Big|\frac{P}{Q}\Big| := e^{\deg P - \deg Q} \text{ for all nonzero } P, Q \in \mathbb{F}_q[T].$$

Clearly, $|\cdot|$ is a nontrivial, non-archimedian, and discrete absolute value in $\mathbb{F}_q(T)$. This absolute value gives rise to a metric on $\mathbb{F}_q(T)$.

The completion field of $\mathbb{F}_q(T)$ is $\mathbb{F}_q((T^{-1}))$, *i.e.*, the field of Laurent series over $\mathbb{F}_q$. The absolute value of $\mathbb{F}_q((T^{-1}))$, which we again denote by $|\cdot|$, is given as follows. Let $a \in \mathbb{F}_q((T^{-1}))$. For $a = 0$, define $|a| = 0$. If $a \neq 0$, then we can write

$$a = \sum_{k \leq k_0} a_k T^k, \text{ where } k_0 \in \mathbb{Z}, a_k \in \mathbb{F}_q, \text{ and } a_{k_0} \neq 0.$$

We define k_0 as the *degree* of a, which will be denoted by $\deg a$, and $|a| := e^{\deg a}$. This clearly extends the absolute value $|\cdot|$ of $\mathbb{F}_q(T)$ to $\mathbb{F}_q((T^{-1}))$, and, moreover, the extension remains non-archimedian and discrete. Let Λ and F denote $\mathbb{F}_q[T]$ and $\mathbb{F}_q((T^{-1}))$ respectively from now on. It is obvious that Λ is discrete in F. For any $n \in \mathbb{N}$, F^n is throughout assumed to be equipped with the supremum norm, which is defined by

$$\|\mathbf{x}\| := \max_{1\leq i\leq n} |x_i| \text{ for all } \mathbf{x} = (x_1, x_2, \ldots, x_n) \in F^n,$$

and with the topology induced by this norm. Clearly, Λ^n is discrete in F^n. Since the topology on F^n considered here is the usual product topology on F^n, it follows that F^n is locally compact as F is locally compact. We shall also fix a Haar measure λ on F.

In this paper, we study analogues of Dirichlet's theorem in Diophantine approximation and its improvability for vectors in F^n. An analogue of Dirichlet's theorem for local fields of positive characteristic can be formulated as follows.

Received by the editors December 9, 2015; revised April 27, 2016.
Published electronically July 29, 2016.
Ghosh is supported by an ISF-UGC grant.
AMS subject classification: **11J83**, 11K60, 37D40, 37A17, 22E40.
Keywords: Dirichlet's theorem, Diophantine approximation, positive characteristic.

Theorem 1.1 *Let t be a nonnegative integer. For $\mathbf{y} := (y_1, y_2, \dots, y_n) \in F^n$, there exist $q \in \Lambda \smallsetminus \{0\}$ and $p \in \Lambda$ such that*

$$|y_1 q_1 + y_2 q_2 + \cdots + y_n q_n - p| < \frac{1}{e^{nt}} \quad \textit{and} \quad \max_{1 \le j \le n} |q_j| \le e^t.$$

The theorem above is clearly well known (see [21, Theorem 1.1] and [2, Appendix 1] for the case of a single linear form), and Diophantine approximation in the context of local fields of positive characteristic has been extensively studied of late. We refer the reader to [9] for a survey and to [3, 20, 22, 23] for more recent results. Indeed, the geometry of numbers, which can be used to prove Dirichlet's theorem, was developed in the context of function fields by Mahler [24] as early as the 1940's. In Section 2, we prove a stronger, multiplicative result. There are many interesting parallels and contrasts between the theory of Diophantine approximation over the real numbers and in positive characteristic. Many results hold in both settings; the main result of this paper being one such, while there are some striking exceptions. For instance, the theory of badly approximable numbers and vectors in positive characteristic offers several surprises. For instance, there is no analogue of Roth's theorem, provided that the base field is finite, which we assume throughout this paper. We refer the reader to [1] for other results in this vein.

1.2 Improving Dirichlet's Theorem

Following Kleinbock–Weiss [18], the notion of Dirichlet improvability can now be introduced as follows. Let $0 < \varepsilon \le \frac{1}{e}$. A vector $\mathbf{y} := (y_1, y_2, \dots, y_n) \in F^n$ is said to be *Dirichlet ε-improvable* if there is some $t_0 > 0$ such that for any choice of n and nonnegative integers $t_1, t_2, \dots, t_n$ with $\max\{t, t_1, \dots, t_n\} > t_0$, where $t = t_1 + t_2 + \cdots + t_n$, one can always find nonzero $(p, q_1, q_2, \dots, q_n) \in \Lambda \times \Lambda^n$ satisfying

$$|y_1 q_1 + y_2 q_2 + \cdots + y_n q_n - p| < \frac{\varepsilon}{e^t} \quad \text{and} \quad |q_j| < \varepsilon e^{t_j} \text{ for } j = 1, 2, \dots, n.$$

Let $\mathrm{DI}_\varepsilon(n)$ denote the set of Dirichlet improvable vectors in F^n or in $\mathbb{R}^n$; the context will make the field clear.

(a) In the definition above, we have invoked the more general, multiplicative analogue of Dirichlet's theorem, for which we provide a proof in Theorem 2.1. The results of this paper are valid in this stronger setting.

(b) This notion can be considered in greater generality for systems of linear forms as was done by Kleinbock and Weiss. We refer the reader to Definition 3.1.

(c) Dirichlet's theorem can be formulated for global fields; *e.g.,* one could consider number fields or finite extensions of positive characteristic fields. However, it is in general an open problem to determine the optimal constant in Dirichlet's theorem in this setting, without which of course, the question of improvement does not arise. There are some cases where the constant can be determined. For example in [14], the theory of metric Diophantine approximation for certain imaginary quadratic extensions of function fields was developed. In these fields, an analogue of Dirichlet's theorem with the same constant, *i.e.,* 1, holds, and it is plausible that the results of this paper will work in that setting as well.

We review briefly the state of the art on the question of improving Dirichlet's theorem in the context of real numbers. Davenport and Schmidt [7, 8] showed that the Lebesgue measure of $\mathrm{DI}_\varepsilon(n)$ is zero for every $\varepsilon < 1$. Starting with work of Mahler, the question of Diophantine approximation on manifolds has received considerable attention. In this subject, one asks if Diophantine properties that are typical with respect to Lebesgue measure are also typical with respect to the push forward of Lebesgue measures via smooth maps. The starting point of this theory was a conjecture due to Mahler, which asked if almost every point on the curve

$$(x, x^2, \dots, x^n) \tag{1.1}$$

is not very well approximable by rationals. Such maps (or measures) are referred to as *extremal*. This conjecture was resolved by V. G. Sprindzhuk who in turn stated two generalisations of Mahler's conjecture that involved a nondegenerate collection of functions replacing the map above. We refer the reader to the work of Kleinbock–Margulis [15], where Sprindzhuk's conjectures are resolved, for the definitions. In a subsequent striking work, Kleinbock, Lindenstrauss, and Weiss [16] extended the results of [15] to a much wider class of measures, the so-called *friendly* measures. This class includes push-forwards of Lebesgue measure as well as many other self similar measures including the uniform measure on the middle-third Cantor set. As regards improving Dirichlet's theorem for manifolds, in [8], the authors showed that for any $\varepsilon < 4^{-1/3}$ the set of $x \in \mathbb{R}$ for which $(x, x^2) \in \mathrm{DI}_\varepsilon(2)$ has zero Lebesgue measure. Further results in this vein were obtained by Baker and by Bugeaud in [4–6]. In [18], Kleinbock and Weiss proved several results in this direction and, in particular, showed the existence of $\varepsilon > 0$ such that for continuous, good and nonplanar maps $\mathbf{f}$ and Radon, Federer measures ν, $\mathbf{f}_*(\nu)(\mathrm{DI}_\varepsilon(n)) = 0$. We will define these terms later in the paper. In particular, this generalises the work of Baker and Bugeaud. The result that is obtained in [18] holds for ε that are quite a bit smaller than 1, and to prove an analogous result for every $\varepsilon < 1$ remains an outstanding open problem. In the case of curves, N. Shah has resolved this problem. See [28] and also [29, 30] for related results.

In another direction, Kleinbock and Tomanov [17] established S-arithmetic analogues of Sprindzhuk's conjectures. In positive characteristic, Sprindzhuk [31] established the analogues of Mahler's conjecture, namely the extremality of the curve (1.1) over F^n and also proved other interesting results, including a transference principle interpolating between simultaneous Diophantine approximation and systems of linear forms. The analogues of Sprindzhuk's conjectures in positive characteristic were established by the second author in [13]. However, the question of improving Dirichlet's theorem in positive characteristic has been completely open as far as we are aware. In this paper, we study the question of Dirichlet improvability of vectors, maps, and measures in positive characteristic.

Here is a special case of our main result, Theorem 3.7.

Theorem 1.2 *Let $f_1, f_2, \dots, f_n$ be polynomials so that $1, f_1, f_2, \dots, f_n$ are linearly independent over F. Fix some open set U of F and consider the map*

$$\mathbf{f}(x) = \big(f_1(x), f_2(x), \dots, f_n(x)\big)$$

https://doi.org/10.4153/CJM-2016-024-2 Published online by Cambridge University Press

defined for all $x \in U$. Then there exists $\varepsilon_0 > 0$ such that whenever $\varepsilon < \varepsilon_0$, $\mathbf{f}(x)$ is not Dirichlet ε-improvable for λ almost all $x \in U$.

Theorem 3.7, the main result of this paper, is far more general and holds for good, non-planar maps and Radon, Federer measures. It may be regarded as a positive characteristic version of [18, Theorem 1.5]. Since the statement of the general form of the theorem is fairly technical, we have chosen to postpone it to later in the paper. The constant ε can be estimated, so the proof is "effective" in that sense. However, it is likely to be far from optimal. We compute ε_0 in the special case $n = 2$ and $f_i(x) = x^i$ for $i = 1, 2$ (see Section 7) as an example. Our proof proceeds along the lines of [18], and the main tool is a quantitative non-divergence result for certain maps in the space of unimodular lattices, which can be identified with the non-compact quotient $\mathrm{SL}(n+1, F)/\mathrm{SL}(n+1, \Lambda)$.

2 Review of the Classical Theory

In this section, we provide a proof of Dirichlet's theorem in positive characteristic for completeness and to aid the reader. In what follows, for $k \in \mathbb{N}$, $\mathbb{Z}_+^k$ denotes the set of all k tuples $(t_1, t_2, \dots, t_k)$ where each t_i is a nonnegative integer . We prove the following theorem.

Theorem 2.1 *Let $m, n \in \mathbb{N}$, $k = m + n$, and*

$$\mathfrak{a}^+ := \Big\{ \mathbf{t} := (t_1, t_2, \dots, t_k) \in \mathbb{Z}_+^k : \sum_{i=1}^m t_i = \sum_{j=1}^n t_{m+j} \Big\}.$$

Consider m linear forms $Y_1, Y_2, \dots, Y_m$ over F in n variables. Then for any $\mathbf{t} \in \mathfrak{a}^+$, there exist solutions $\mathbf{q} = (q_1, q_2, \dots, q_n) \in \Lambda^n \setminus \{\mathbf{0}\}$ and $\mathbf{p} = (p_1, p_2, \dots, p_m) \in \Lambda^m$ of the following system of inequalities:

$$\begin{aligned} &|Y_i \mathbf{q} - p_i| < e^{-t_i} && \text{for } i = 1, 2, \dots, m, \\ &|q_j| \le e^{t_{m+j}} && \text{for } j = 1, 2, \dots, n. \end{aligned} \tag{2.1}$$

To prove this theorem, we first introduce the "polynomial part" and the "fractional part" of a Laurent series. For any Laurent series

$$a = \cdots + \frac{a_2}{T^2} + \frac{a_1}{T} + (a_0 + a_1 T + a_2 T^2 + \cdots + a_k T^k)$$

in F, where $k \in \mathbb{Z}$, $a_i \in \mathbb{F}_q$, and $a_k \neq 0$, let us define the *polynomial part* of a as

$$a_0 + a_1 T + a_2 T^2 + \cdots + a_k T^k$$

if $k \ge 0$; otherwise, it is defined to be 0. The *fractional part* of a, denoted by $\langle a \rangle$, is defined as

$$\alpha - \text{polynomial part} = \frac{a_1}{T} + \frac{a_2}{T^2} + \cdots.$$

Now, let $a := \frac{a_1}{T} + \frac{a_2}{T^2} + \cdots \in F$ and $\alpha := \alpha_0 + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k \in \Gamma \setminus \{0\}$ with degrees $\le k$, where $k \ge 0$ is an integer. Let us observe that, for any $s \in \mathbb{N}$, the coefficient of $\frac{1}{T^s}$ in αa is

$$a_s \alpha_0 + \cdots + a_{s+k} \alpha_k.$$

It follows that, for any $m \in \mathbb{N}$, $|\langle \alpha a \rangle| < \frac{1}{e^m}$ if and only if the system $A\boldsymbol{x} = \mathbf{0}$ of linear equations over $\mathbb{F}_q$, where the coefficient matrix

$$A := \begin{bmatrix} a_1 & a_2 & \cdots & a_{k+1} \\ a_2 & a_3 & \cdots & a_{k+2} \\ \vdots & \vdots & & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+k} \end{bmatrix}$$

has $(\alpha_0, \alpha_1, \dots, \alpha_k)$ as a nontrivial solution.

Continuing along the same line, let us now take two Laurent series $a = \frac{a_1}{T} + \frac{a_2}{T^2} + \cdots$, $b = \frac{b_1}{T} + \frac{b_2}{T^2} + \cdots$, and two nonzero polynomials $\alpha := \alpha_0 + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k$, $\beta := \beta_0 + \beta_1 T + \beta_2 T^2 + \cdots + \beta_l T^l$, with degree $\leq k, l$, respectively, where $k, l = 0, 1, 2, \dots$. For any $s \in \mathbb{N}$, the coefficient of $\frac{1}{T^s}$ in $\alpha a + \beta b$ is easily seen to be

$$a_s \alpha_0 + \cdots + a_{s+k}\alpha_k + b_s \beta_0 + \cdots + b_{s+l}\beta_l.$$

Therefore, for any $m \in \mathbb{N}$, $|\langle \alpha a + \beta b \rangle| < \frac{1}{e^m}$ if and only if $(\alpha_0, \alpha_1, \dots, \alpha_k, \beta_0, \beta_1, \dots, \beta_l)$ is a nontrivial solution of the system

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \mathbf{0},$$

where

$$A := \begin{bmatrix} a_1 & a_2 & \cdots & a_{k+1} \\ a_2 & a_3 & \cdots & a_{k+2} \\ \vdots & \vdots & & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+k} \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} b_1 & b_2 & \cdots & b_{l+1} \\ b_2 & b_3 & \cdots & b_{l+2} \\ \vdots & \vdots & & \vdots \\ b_m & b_{m+1} & \cdots & b_{m+l} \end{bmatrix}.$$

It is obvious that we can generalize this observation for any such n Laurent series and nonzero polynomials.

Now we are ready to start the proof of Theorem 2.1. Each Y_i, being a linear form over F in n variables, must be of the form

$$y_{i1}x_1 + y_{i2}x_2 + \cdots + y_{in}x_n,$$

for some $y_{ij} \in \mathbb{F}_q$, $j = 1, 2, \dots, n$. It suffices to consider the case where $|y_{ij}| < 1$; *i.e.,* the polynomial part of y_{ij} is zero, for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

From the observations we made earlier, we see that each y_{ij} gives rise to a matrix M_{ij} having t_i rows and $t_{m+j} + 1$ columns, and more importantly, the existence of solution of the system (2.1) is equivalent to the existence of nontrivial solutions of the following system of linear equations over $\mathbb{F}_q$:

$$\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_n \end{bmatrix} = \mathbf{0}.$$

Clearly the above coefficient matrix has $\sum_{i=1}^m t_i$ rows and $\sum_{j=1}^n (t_{m+j} + 1)$ columns. As $\sum_{i=1}^m t_i = \sum_{j=1}^n t_{m+j}$, we see that the matrix has more columns than rows, and hence nontrivial solution exists. This completes our proof. ∎

3 The Main Theorem

We shall now introduce the notion of "Dirichlet improvability" in a greater generality. Let $\mathfrak{a}^+$ be as given in Theorem 2.1, $\mathcal{T}$ be an unbounded subset of $\mathfrak{a}^+$ and let $0 < \varepsilon \leq \frac{1}{e}$.

Definition 3.1 For a system of linear forms $Y_1, Y_2, \ldots, Y_m$ over F in n variables, we say that *DT can be ε-improved along* $\mathcal{T}$, or we use the notation $Y \in DI_\varepsilon(\mathcal{T})$, where Y is the $m \times n$ matrix having Y_i as the i-th row for each i, if there exists $t_0 > 0$ such that for every $\mathbf{t} := (t_1, t_2, \ldots, t_k) \in \mathcal{T}$ with $\|\mathbf{t}\| > t_0$ the following system admits nontrivial solutions $(\mathbf{p}, \mathbf{q}) \in \Lambda^m \times \Lambda^n$:

$$
\begin{aligned}
&|Y_i\mathbf{q} - p_i| < \frac{\varepsilon}{e^{t_i}} && \text{for } i = 1, 2, \ldots, m, \\
&|q_j| < \varepsilon e^{t_{m+j}} && \text{for } j = 1, 2, \ldots, n.
\end{aligned} \tag{3.1}
$$

In particular, a vector $\mathbf{y} := (y_1, y_2, \ldots, y_n) \in F^n$ is said to be *Dirichlet ε-improvable along* $\mathcal{T}$ if the corresponding row matrix $[y_1 y_2 \cdots y_n] \in DI_\varepsilon(\mathcal{T})$.

By analogy to what was shown in [18], here also we want to prove that if $\varepsilon > 0$ is sufficiently small and an unbounded subset $\mathcal{T}$ of $\mathfrak{a}^+$ is chosen, the set of all Dirichlet ε-improvable vectors along $\mathcal{T}$ is negligible. The setup here is *multiplicative*; *i.e.*, one studies Diophantine inequalities where the Euclidean or supremum norm is replaced with the product of coordinates. The changed "norm" introduces several complications, and the subject of multiplicative Diophantine approximation is generally considered more difficult than its euclidean counterpart.

Before proceeding to our main theorem, we will recall some terminology introduced in the papers of Kleinbock and Margulis, and Kleinbock, Lindenstrauss, and Weiss and used in several subsequent works by many authors. The following is taken from [17, §1 and 2].

Definition 3.2 Let X be a metric space. We will say X is *Besicovitch* if there exists a constant N_X such that for any bounded subset A of X and for any collection $\mathcal{C}$ of nonempty open balls in X such that every $a \in A$ is the center of some ball in $\mathcal{C}$, there exists a countable subcollection $\{B_i\}$ of $\mathcal{C}$ with

$$1_A \leq \sum_i 1_{B_i} \leq N_X.$$

Standard examples of Besicovitch spaces are the Euclidean spaces $\mathbb{R}^n$ (see [25, Theorem 2.7]) and F^n with metric induced by supremum norm as defined in Section 1. In fact, the constant N_{F^n} is equal to 1, as this is an ultrametric space. In other words, any covering by balls of any bounded subset of F^n admits a countable subcover consisting of pairwise mutually disjoint balls.

Suppose X is a Besicovitch metric space, $U \subseteq X$ is open, ν is a Radon measure on X, $(\mathcal{F}, |\cdot|)$ is a valued field and $f\colon X \to \mathcal{F}$ is a given function such that $|f|$ is measurable. For any $B \subseteq X$, we set

$$\|f\|_{\nu,B} := \sup_{x \in B \cap \operatorname{supp}(\nu)} |f(x)|.$$

Definition 3.3 For $C, \alpha > 0$, f is said to be (C,α)-good on U with respect to ν if for every ball $B \subseteq U$ with center in $\operatorname{supp}(\nu)$, one has

$$\nu(\{x \in B : |f(x)| < \varepsilon\}) \leq C\Big(\frac{\varepsilon}{\|f\|_{\nu,B}}\Big)^{\alpha}\nu(B).$$

The following properties are immediate from Definition 3.3.

Lemma 3.4 *Let $X, U, \nu, \mathcal{F}, f, C, \alpha$, be as given above.*

(i) *f is (C,α)-good on U with respect to ν if and only if so is $|f|$.*
(ii) *If f is (C,α)-good on U with respect to ν, then so is cf for all $c \in \mathcal{F}$.*
(iii) *For all $i \in I$, f_i are (C,α)-good on U with respect to ν, and $\sup_{i\in I}|f_i|$ is measurable $\Rightarrow$ so is $\sup_{i\in I}|f_i|$.*
(iv) *If f is (C,α)-good on U with respect to ν and $g\colon V \to \mathbb{R}$ is a continuous function such that $c_1 \leq |\frac{f}{g}| \leq c_2$ for some $c_1, c_2 > 0$, then g is $(C(\frac{c_2}{c_1})^{\alpha}, \alpha)$-good on U with respect to ν.*
(v) *Let $C_2 > 1$ and $\alpha_2 > 0$. If f is (C_1,α_1)-good on U with respect to ν and $C_1 \leq C_2$, $\alpha_2 \leq \alpha_1$, then f is (C_2,α_2)-good on V with respect to ν.*

We say a map $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ from U to $\mathcal{F}^n$, where $n \in \mathbb{N}$, is (C,α)-good on U with respect to ν, or simply $(\mathbf{f},\nu)$ is (C,α)-good on U, if every $\mathcal{F}$-linear combination of $1, f_1, \ldots, f_n$ is (C,α)-good on U with respect to ν.

Definition 3.5 Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ be a map from U to $\mathcal{F}^n$, where $n \in \mathbb{N}$. We say that $(\mathbf{f},\nu)$ is *nonplanar* if for any ball $B \subseteq U$ with center in $\operatorname{supp}(\nu)$, the restrictions of the functions $1, f_1, \ldots, f_n$ on $B \cap \operatorname{supp}(\nu)$ are linearly independent. In other words, if $\mathbf{f}(B \cap \operatorname{supp}(\nu))$ is not contained in any affine subspace of $\mathcal{F}^n$ for any ball $B \subseteq U$ with center in $\operatorname{supp}(\nu)$.

For $m \in \mathbb{N}$ and a ball $B = B(x;r) \subseteq X$, where $x \in X$ and $r > 0$, we will use the notation $3^m B$ to denote the ball $B(x;3^m r)$.

Definition 3.6 Let $D > 0$. The measure ν is said to be D-Federer on U if for every ball B with center in $\operatorname{supp}(\nu)$ such that $3B \subseteq U$, one has

$$\frac{\nu(3B)}{\nu(B)} \leq D.$$

We are now ready to state our main theorem, which addresses improvements of Dirichlet's theorem in the multiplicative setting for good, nonplanar maps and Federer measures over local fields of positive characteristic.

Theorem 3.7 *For any $d, n \in \mathbb{N}$ and $C, \alpha, D > 0$ there exists $\varepsilon_0 = \varepsilon_0(n, C, \alpha, D)$ satisfying the following: if ν is a Radon measure on F^d, U is an open set of F^d such that $\nu(U) > 0$ and ν is D-Federer on U, and $\mathbf{f}\colon U \to F^n$ is a continuous map such that $(\mathbf{f},\nu)$ is (C,α)-good and nonplanar is given then for any $\varepsilon < \varepsilon_0$,*

$$\mathbf{f}_*\nu(DI_{\varepsilon}(\mathcal{T})) = 0 \textit{ for any unbounded } \mathcal{T} \subseteq \mathfrak{a}^+.$$

We use the so-called "quantitative nondivergence", a generalization of non-divergence of unipotent flows on homogeneous spaces, to prove our main theorem. Similarly to the approach adopted in [18], in the following section we will first translate the property of a system of linear forms over F being Dirichlet improvable into certain recurrent properties of flows on some homogeneous space.

4 The Correspondence

Let $G := \mathrm{SL}(k, F)$, $\Gamma := \mathrm{SL}(k, \Lambda)$, and let π be the quotient map $G \to G/\Gamma$. Then G acts on G/Γ by left translations via the rule $g\pi(h) = \pi(gh)$ for $g, h \in G$. For $Y \in M_{m\times n}(F)$, define

$$\tau(Y) := \begin{bmatrix} I_m & Y \\ 0 & I_n \end{bmatrix} \quad \text{and} \quad \overline{\tau} := \pi \circ \tau,$$

where I_l stands for the $l \times l$ identity matrix, $l \in \mathbb{N}$.

Since Γ is the stabilizer of Λ^k under the transitive action of G on the set $\mathcal{L}_k(F)$ of unimodular lattices in F^k we can identify $G/\Gamma \simeq \mathcal{L}_k(F)$. Thus, $\overline{\tau}(Y)$ becomes identified with

$$\left\{ (Y\mathbf{q} - \mathbf{p}, \mathbf{q}) : \mathbf{p} \in \Lambda^m, \mathbf{q} \in \Lambda^n \right\}.$$

Now for $\varepsilon > 0$, let K_ε denote the collection of all unimodular lattices in F^k that contain no nonzero vector of norm smaller than ε, that is,

$$K_\varepsilon := \pi\Big(\left\{ g \in G : \|g\mathbf{v}\| \geq \varepsilon \forall \mathbf{v} \in \Lambda^k \smallsetminus \{\mathbf{0}\} \right\} \Big). \tag{4.1}$$

Next, for $\mathbf{t} := (t_1, t_2, \dots, t_k) \in \mathfrak{a}^+$, we associate the diagonal matrix

$$g_{\mathbf{t}} := \mathrm{diag}(T^{t_1}, \dots, T^{t_m}, T^{-t_{m+1}}, \dots, T^{-t_k}) \in G.$$

Let us come to the relevance of defining the above objects. An immediate observation shows that, for given $\mathbf{t} \in \mathfrak{a}^+$, the system (3.1) has nonzero polynomial solutions if and only if $g_{\mathbf{t}}\overline{\tau}(Y) \notin K_\varepsilon$. Thus, we have the following proposition.

Proposition 4.1 *Let $0 < \varepsilon \leq \frac{1}{e}$ and unbounded $\mathcal{T} \subseteq \mathfrak{a}^+$ be given. Then for any $Y \in M_{m\times n}(F)$,*

$$Y \in DI_\varepsilon(\mathcal{T}) \Longleftrightarrow g_{\mathbf{t}}\overline{\tau}(Y) \notin K_\varepsilon \forall \mathbf{t} \in \mathcal{T} \textit{ with } \|\mathbf{t}\| \gg 1,$$

or equivalently one has,

$$DI_\varepsilon(\mathcal{T}) = \bigcup_{n=1}^{\infty} \bigcap_{\substack{\mathbf{t}\in\mathcal{T}, \\ \|\mathbf{t}\|>n}} \left\{ Y \in M_{m\times n}(F) : g_{\mathbf{t}}\overline{\tau}(Y) \notin K_\varepsilon \right\}.$$

Hence, in view of the above proposition, it is clear that if in addition a Radon measure ν on F^d, an open set U of F^d, and a map $F\colon U \to M_{m\times n}(F)$ are given, then to prove that $F_*\nu(DI_\varepsilon(\mathcal{T})) = \nu(F^{-1}(DI_\varepsilon(\mathcal{T})) = 0$, it is enough to show that

$$\nu\Big(F^{-1}\Big(\bigcap_{\substack{\mathbf{t}\in\mathcal{T}, \\ \|\mathbf{t}\|>n}} \{ Y \in M_{m\times n}(F) : g_{\mathbf{t}}\overline{\tau}(Y) \notin K_\varepsilon \} \Big) \Big) = 0 \tag{4.2}$$

for all $n \in \mathbb{N}$. Suppose now that we have some $c \in (0,1)$ with the property that for any ball $B \subseteq U$ centered in $\operatorname{supp}(\nu)$, there exists $s > 0$ such that

$$\nu\Big(B \cap F^{-1}\big(\big\{Y : g_{\mathbf{t}}\overline{\tau}(Y) \notin K_\varepsilon\big\}\big)\Big) = \nu\big(\big\{\mathbf{x} \in B : g_{\mathbf{t}}\overline{\tau}(F(\mathbf{x})) \notin K_\varepsilon\big\}\big) \leq c\nu(B) \tag{4.3}$$

holds for any $\mathbf{t} \in \mathfrak{a}^+$ with $\|\mathbf{t}\| \geq s$. Then it is easy to see that for any $n \in \mathbb{N}$ and any ball $B \subseteq U$ centered in $\operatorname{supp}(\nu)$,

$$\begin{aligned} &\frac{\nu(B \cap F^{-1}(\bigcap_{\mathbf{t}\in\mathcal{T},\|\mathbf{t}\|>n}\{Y \in M_{m\times n}(F) : g_{\mathbf{t}}\overline{\tau}(Y) \notin K_\varepsilon\}))}{\nu(B)} \\ &\quad = \frac{\nu(\bigcap_{\mathbf{t}\in\mathcal{T},\|\mathbf{t}\|>n} B \cap F^{-1}(\{Y \in M_{m\times n}(F) : g_{\mathbf{t}}\overline{\tau}(Y) \notin K_\varepsilon\}))}{\nu(B)} \\ &\quad \leq \frac{c\nu(B)}{\nu(B)} = c < 1. \end{aligned} \tag{4.4}$$

It follows that for any given $n \in \mathbb{N}$, no $\mathbf{x} \in U \cap \operatorname{supp}(\nu)$ is a point of density of the set

$$F^{-1}\Big(\bigcap_{\substack{\mathbf{t}\in\mathcal{T},\\ \|\mathbf{t}\|>n}} \big\{Y \in M_{m\times n}(F) : g_{\mathbf{t}}\overline{\tau}(Y) \notin K_\varepsilon\big\}\Big),$$

as (4.4) holds true for any ball B with $\mathbf{x} \in B \subseteq U$. Thus, (4.2) will be achieved in view of the following theorem, an analogue of Lebesgue's density Theorem for ultrametric spaces. This result is well known, and, for probability measures, a proof can be found in [26]. Indeed, it can be proved by making appropriate modifications to the standard argument for Euclidean space [12].

Theorem 4.2 *For any measurable $\Omega \subseteq F^d$, almost every $\mathbf{x} \in \Omega \cap \operatorname{supp}(\nu)$ is a point of density of Ω,* i.e.,

$$\lim_{\substack{\nu(B)\to 0\\ \mathbf{x}\in B}} \frac{\nu(B\cap\Omega)}{\nu(B)} = 1.$$

5 The Proof of Theorem 3.7

As F^d is locally compact, Hausdorff, and second countable, every open set is the union of some countable collection of compact subsets. Hence, to prove the Theorem 3.7, once the correct $\varepsilon_0 = \varepsilon_0(n, C, \alpha, D)$ is found, it suffices to show that for all $\mathbf{y} \in U \cap \operatorname{supp}(\nu)$, there exists a ball $\mathfrak{B} \subseteq U$ containing $\mathbf{y}$ such that

$$\nu\Big(\mathfrak{B} \cap \mathbf{f}^{-1}\big(DI_\varepsilon(\mathfrak{T})\big)\Big) = \nu\Big(\big\{\mathbf{x} \in \mathfrak{B} : \mathbf{f}(\mathbf{x}) \in DI_\varepsilon(\mathfrak{T})\big\}\Big) = 0 \tag{5.1}$$

for all $\varepsilon < \varepsilon_0$. From our discussion in Section 4, we see that (5.1) is guaranteed as soon as we can show the existence of some $c \in (0,1)$ that satisfies the following: if a ball B with center in $\operatorname{supp}(\nu)$ is contained in $\mathfrak{B}$, then

(5.2) there exists $s > 0$ such that for all $\mathbf{t} \in \mathfrak{a}^+$ with $\|\mathbf{t}\| \geq s$, (4.3) holds.

Now we will need the following proposition.

https://doi.org/10.4153/CJM-2016-024-2 Published online by Cambridge University Press

Proposition 5.1 *For any $d, n \in \mathbb{N}$ and any $C, \alpha, D > 0$ there exists $\widetilde{C} = \widetilde{C}(n, C, D)$ with the following property: let B be a ball centered in $\operatorname{supp}(\nu)$, let ν be a Radon measure on F^d that is D-Federer on $\widetilde{B} := 3^{n+1}B$ and let $\mathbf{f}: \widetilde{B} \to F^n$ be a continuous map so that*

(i) *any F-linear combination of $1, f_1, \dots, f_n$ is (C, α)-good on $\widetilde{B}$ with respect to ν and*
(ii) *the restrictions of $1, f_1, \dots, f_n$ to $B \cap \operatorname{supp}(\nu)$ are linearly independent over F.*

Then we can find some $s > 0$ such that for all $\mathbf{t} \in \mathfrak{a}^+$ with $\|\mathbf{t}\| \geq s$ and any $\varepsilon \leq \frac{1}{e}$, one has

$$\nu\Big(\big\{ \mathbf{x} \in B : g_{\mathbf{t}} \overline{\tau}(\mathbf{f}(\mathbf{x})) \notin K_\varepsilon \big\} \Big) \leq \widetilde{C} \varepsilon^\alpha \nu(B). \tag{5.3}$$

Theorem 3.7 now follows easily from Proposition 5.1. In fact, we first choose $0 < \varepsilon_0 \leq \frac{1}{e}$ so that $\widetilde{C}\varepsilon_0^\alpha < 1$. Clearly, this ε_0 depends only on (n, C, α, D). Let $\mathbf{y} \in U \cap \operatorname{supp}(\nu)$. Choose a ball $\mathfrak{B}$ such that $\mathbf{y} \in \mathfrak{B} \subseteq \widetilde{\mathfrak{B}} := 3^{n+1}\mathfrak{B} \subseteq U$. Now pick any ball $B \subseteq \mathfrak{B}$ having center in $\operatorname{supp}(\nu)$ and consider the corresponding $\widetilde{B}$. Since $(\mathbf{f}, \nu)$ is (C, α)-good and nonplanar, conditions (i) and (ii) of Proposition 5.1 hold here immediately. Hence, if we set $c = \widetilde{C}\varepsilon_0^\alpha$, the assertion (5.2) is immediate from Proposition 5.1 whenever $0 < \varepsilon < \varepsilon_0$. Thus, the proof of Theorem 3.7 is complete. ■

We now need to prove Proposition 5.1. We shall show this as a consequence of a more general result, namely the "Quantitative Nondivergence Theorem".

6 Quantitative Nondivergence and the Proof of Proposition 5.1

We first recall the "Quantitative Nondivergence Theorem" in its most general form, as it is developed in [17, §7]. From this we prove Proposition 5.1.

6.1 Quantitative Nondivergence

We assume in this subsection that $\mathcal{D}$ is an integral domain, K is the field of quotients of $\mathcal{D}$, and $\mathcal{R}$ is a commutative ring containing K as a subring.

Let $m \in \mathbb{N}$. If Δ is a $\mathcal{D}$-submodule of $\mathcal{R}^m$, let us denote by $K\Delta$ (resp. $\mathcal{R}\Delta$) its K- (resp. $\mathcal{R}$) linear span inside $\mathcal{R}^m$. We use the notation $\operatorname{rank}(\Delta)$ to denote the rank of Δ, which is defined as

$$\operatorname{rank}(\Delta) := \dim_K(K\Delta).$$

For example, $\operatorname{rank}(\mathcal{D}^m) = m$. If Θ is a $\mathcal{D}$-submodule of $\mathcal{R}^m$ and Δ is a submodule of Θ, we say that Δ is *primitive in* Θ if any submodule of Θ containing Δ and having rank equal to $\operatorname{rank}(\Delta)$ is equal to Δ. We see that the set of all nonzero primitive submodules of a fixed $\mathcal{D}$-submodule Θ of $\mathcal{R}^m$ is a partially ordered set with respect to set inclusion and its length is equal to $\operatorname{rank}(\Theta)$. When $\Theta = \mathcal{D}^m$, we can even characterize the primitive submodules of $\mathcal{D}^m$ from the following observation:

$$\Delta \text{ is primitive} \iff \Delta = K\Delta \cap \mathcal{D}^m \iff \Delta = \mathcal{R}\Delta \cap \mathcal{D}^m.$$

This also shows that for any submodule Δ' of $\mathcal{D}^m$ there exists a unique primitive submodule $\Delta \supseteq \Delta'$ such that $\operatorname{rank}(\Delta) = \operatorname{rank}(\Delta')$, namely $\Delta := K\Delta' \cap \mathcal{D}^m$.

In addition, let $\mathcal{R}$ have a topological ring structure. We consider the topological group $\operatorname{GL}(m, \mathcal{R})$ of $m \times m$ invertible matrices with entires in $\mathcal{R}$. It is obvious that any

$g \in \mathrm{GL}(m, \mathcal{R})$ maps $\mathcal{D}$-submodules of $\mathcal{R}^m$ to $\mathcal{D}$-submodules of $\mathcal{R}^m$ preserving their rank and inclusion relation. Let

$$\mathfrak{M}(\mathcal{R}, \mathcal{D}, m) := \{ g\Delta : g \in \mathrm{GL}(m, \mathcal{R}), \Delta \text{ is a submodule of } \mathcal{D}^m \}.$$

We also denote the set of all nonzero primitive submodules of $\mathcal{D}^m$, which is a poset of length m with respect to inclusion relation, as we have already seen by $\mathfrak{P}(\mathcal{D}, m)$.

For a given function $\|\cdot\|: \mathfrak{M}(\mathcal{R}, \mathcal{D}, m) \to \mathbb{R}_{\geq 0}$, one says that $\|\cdot\|$ is *norm-like* if the following three conditions hold:

(N1) For any $\Delta, \Delta' \in \mathfrak{M}(\mathcal{R}, \mathcal{D}, m)$ with $\Delta' \subseteq \Delta$ and $\mathrm{rank}(\Delta') = \mathrm{rank}(\Delta)$, we always have $\|\Delta'\| \geq \|\Delta\|$.

(N2) there exists $C_{\|\cdot\|} > 0$ such that $\|\Delta + \mathcal{D}\gamma\| \leq C_{\|\cdot\|}\|\Delta\|\|\mathcal{D}\gamma\|$ holds for any $\Delta \in \mathfrak{M}(\mathcal{R}, \mathcal{D}, m)$ and any $\gamma \notin \mathcal{R}\Delta$.

(N3) the function $\mathrm{GL}(m, \mathcal{R}) \to \mathbb{R}_{\geq 0}, g \mapsto \|g\Delta\|$ is continuous for every submodule Δ of $\mathcal{D}^m$.

With the notation and terminology defined above, it is now time to state the "Quantitative Nondivergence Theorem".

Theorem 6.1 ([17, Theorem 7.3]) *Let $B \subseteq X$ be a ball in a Besicovitch metric space X and let $h: \widetilde{B} \to \mathrm{GL}(m, \mathcal{R})$, where $\widetilde{B} := 3^m B$, be a continuous map. Suppose ν is a Radon measure on X that is D-Federer on $\widetilde{B}$. Assume that a norm-like function $\|\cdot\|$ is given on $\mathfrak{M}(\mathcal{R}, \mathcal{D}, m)$. Assume further that for some $C, \alpha > 0$ and $\rho \in (0, 1/C_{\|\cdot\|}]$, the following conditions hold.*

(C1) *For every $\Delta \in \mathfrak{P}(\mathcal{D}, m)$, the function $x \mapsto \|h(x)\Delta\|$ is (C, α)-good on $\widetilde{B}$ with respect to ν.*

(C2) *For every $\Delta \in \mathfrak{P}(\mathcal{D}, m)$, $\sup_{x \in B \cap \mathrm{supp}(\nu)} \|h(x)\Delta\| \geq \rho$.*

(C3) *For all $x \in \widetilde{B} \cap \mathrm{supp}(\nu)$, $\#\{\Delta \in \mathfrak{P}(\mathcal{D}, m) : \|h(x)\Delta\| < \rho\} < \infty$.*

Then for any positive $\varepsilon \leq \rho$, one has

$$\nu\Big(\{ x \in B : \|h(x)\gamma\| < \varepsilon \textit{ for some } \gamma \in \mathcal{D}^m \smallsetminus \{\mathbf{0}\} \} \Big) \leq mC(N_X D^2)^m \Big(\frac{\varepsilon}{\rho}\Big)^\alpha \nu(B),$$

where N_X is the "Besicovitch constant".

For the proof, see [17, Theorem 7.3].

6.2 The Proof of Proposition 5.1

From the definition of K_ε, as in (4.1), it is obvious that for $\mathbf{t} \in \mathfrak{a}^+$ and $\mathbf{x} \in B$,

$$g_{\mathbf{t}}\overline{\tau}(\mathbf{f}(\mathbf{x})) \notin K_\varepsilon \iff \|(g_{\mathbf{t}}\tau(\mathbf{f}(\mathbf{x})))\mathbf{v}\| < \varepsilon \text{ for some } \mathbf{v} \in \Lambda^{n+1} \smallsetminus \{\mathbf{0}\}.$$

This inspires us to use Theorem 6.1 with the following assumptions:

- $\mathcal{D} = \Lambda$, $\mathcal{R} = F$, $X = F^d$, $m = n+1$;
- ν, B, C, α and D as in Proposition 5.1;
- $h(\mathbf{x}) = g_{\mathbf{t}}\tau(\mathbf{f}(\mathbf{x})) \forall \mathbf{x} \in \widetilde{B}$.

We define $\|\cdot\|$ as follows: since Λ is a PID, any submodule of the Λ module Λ^{n+1}, being a submodule of a free module of rank $n+1$, is free of rank $\leq n+1$. Thus, any nonzero

$\Delta \in \mathfrak{M}(F, \Lambda, n+1)$ has a Λ basis, say $\{\mathbf{v}_1, \dots, \mathbf{v}_j\}$, where $1 \le j \le n+1$. We consider the j-vector $\mathbf{w} := \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j \in \bigwedge^j(F^{n+1})$. Recall that the j-vectors $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}$ with integers $1 \le i_1 < i_2 < \cdots < i_j \le n+1$ form a basis of $\bigwedge^j(F^{n+1})$, and thus $\bigwedge^j(F^{n+1})$ can be identified with $F^{\binom{n+1}{j}}$. Therefore one can naturally talk about the *supremum norm* on $\bigwedge^j(F^{n+1})$ using this identification. We define $\|\Delta\|$ to be the supremum norm of $\mathbf{w}$.

It is a routine verification that this definition does not depend on the choice of the ordered basis of Δ. If $\Delta = \{\mathbf{0}\}$, we define $\|\Delta\| = 1$. In order to prove that the just defined $\|\cdot\|$ is indeed norm-like, we need to verify conditions (N1)–(N3). (N1) and (N3) follow easily from the basic properties of exterior product, while (N2) can be proved by a verbatim repetition of the proof of [15, Lemma 5.1] as follows.

We claim that $C_{\|\cdot\|}$ can be taken as 1. If $\Delta = \{\mathbf{0}\}$, then it is immediate. Otherwise, let $\{\mathbf{v}_1, \dots, \mathbf{v}_j\}$ be a basis of Δ. Clearly, $\{\mathbf{v}_1, \dots, \mathbf{v}_j, \gamma\}$ is a basis of $\Delta + \Lambda\gamma$. Now writing

$$\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j = \sum_{\substack{I \subseteq \{1,2,\dots,n+1\},\\ \#I = j}} w_I e_I \quad \text{and} \quad \gamma = \sum_{i=1}^{n+1} w_i e_i$$

(in the usual notations) and using the ultrametric property, we see that

$$\begin{aligned}\|\Delta + \Lambda\gamma\| &= \Big\| \sum_{\substack{I \subseteq \{1,2,\dots,n+1\},\\ \#I = j}} w_I e_I \wedge \sum_{i=1}^{n+1} w_i e_i \Big\| \le \max_{1 \le i \le n+1} \Big\| \sum_{\substack{I \subseteq \{1,2,\dots,n+1\},\\ \#I = j}} w_I w_i (e_I \wedge e_i) \Big\| \\ &\le \max_{1 \le i \le n+1} \max_{\substack{I \subseteq \{1,2,\dots,n+1\},\\ \#I = j}} |w_I w_i| \le \max_{\substack{I \subseteq \{1,2,\dots,n+1\},\\ \#I = j}} |w_I| \max_{1 \le i \le n+1} |w_i| = \|\Delta\| \|\Lambda\gamma\|.\end{aligned}$$

Now we have to check conditions (C1), (C2), and (C3) of Theorem 6.1. From the discreteness of $\bigwedge^j(\Lambda^{n+1})$ in $\bigwedge^j(F^{n+1})$ for all $j = 1, 2, \dots, n+1$, (C3) is immediate. To investigate the validity of the others, we have to do the explicit computation exactly as in [18, §3.3].

• **Checking (C1):** Here, for the sake of convenience in computation, it is customary to bring a few minor changes in some of the notation we have been using so far. For the rest of this section, we write $\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n\}$, the standard basis of F^{n+1}, and for

$$I = \{i_1, \dots, i_j\} \subseteq \{0, \dots, n\}, \text{ where } i_i < i_2 < \cdots < i_j, \tag{6.1}$$

we let $\mathbf{e}_I$ denote $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_j}$. Similarly, it will be convenient to put any $\mathbf{t} \in \mathfrak{a}^+$ as

$$\mathbf{t} = (t_0, t_1, \dots, t_n), \text{ where } t_0 = \sum_{i=1}^{n} t_i.$$

Let us observe that for any $\mathbf{y} \in F^n$, $\tau(\mathbf{y})$ fixes $\mathbf{e}_0$ and sends any other $\mathbf{e}_i$ to $\mathbf{e}_i + y_i \mathbf{e}_0$. Thus, for any I as in (6.1), we have

$$\tau(\mathbf{y})\mathbf{e}_I = \begin{cases} \mathbf{e}_I & \text{if } 0 \in I, \\ \mathbf{e}_I + \sum_{i \in I} \pm y_i \mathbf{e}_{I \cup \{0\} \smallsetminus \{i\}} & \text{otherwise.} \end{cases} \tag{6.2}$$

Likewise, we can also see that for any I as in (6.1),

$$g_{\mathbf{t}} \mathbf{e}_I = \begin{cases} T^{t_0 - \sum_{i \in I \smallsetminus \{0\}} t_i} \mathbf{e}_I & \text{if } 0 \in I, \\ T^{-\sum_{i \in I} t_i} \mathbf{e}_I & \text{otherwise.} \end{cases} \tag{6.3}$$

Suppose $\Delta \in \mathfrak{P}(\Lambda, n+1)$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ is a basis of Δ and let

$$\mathbf{w} := \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j = \sum_{\substack{I \subseteq \{0,\ldots,n\},\\ \#I=j}} w_I \mathbf{e}_I; w_I \in \Lambda.$$

From (6.2) and (6.3), it follows that for any $\mathbf{x} \in \widetilde{B}$, one has

$$h(\mathbf{x})\mathbf{w} = \sum_{\substack{I \subseteq \{0,\ldots,n\},\\ \#I=j}} h_I(\mathbf{x}) \mathbf{e}_I,$$

where

$$h_I(\mathbf{x}) := \begin{cases} T^{-\sum_{i\in I} t_i} w_I & \text{if } 0 \notin I, \\ T^{\sum_{i\notin I} t_i}\big(w_I + \sum_{i\notin I} \pm w_{I\cup\{i\}\smallsetminus\{0\}} f_i(\mathbf{x})\big) & \text{otherwise.} \end{cases} \tag{6.4}$$

In particular, the coordinate maps h_I of the map $\mathbf{x} \mapsto h(\mathbf{x})\mathbf{w}, \mathbf{x} \in \widetilde{B}$ are F-linear combinations of $1, f_1, \ldots, f_n$ and hence, by Proposition 5.1(i), all of them are (C, α)-good on $\widetilde{B}$ with respect to ν. Therefore, from Lemma 3.4(iii), it follows that the function

$$\mathbf{x} \mapsto \|h(\mathbf{x})\Delta\| = \|h(\mathbf{x})\mathbf{w}\| = \max_I |h_I(\mathbf{x})|$$

is (C, α)-good on $\widetilde{B}$ with respect to ν. Thus, (C1) is established.

- **Checking (C2):** Let $\Delta \in \mathfrak{P}(\Lambda, n+1)$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ be a basis of Δ and let

$$\mathbf{w} := \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j = \sum_{\substack{I \subseteq \{0,\ldots,n\},\\ \#I=j}} w_I \mathbf{e}_I; w_I \in \Lambda.$$

Case 1: Assume $w_I = 0$ whenever $0 \notin I$. Then there must be some $J \subseteq \{0, \ldots, n\}$ containing 0 such that $w_J \neq 0$, as all w_I cannot be zero. Pick any $\mathbf{t} \in \mathfrak{a}^+$. Now from (6.4), we see that

$$|h_J(\mathbf{x})| = |T^{\sum_{i\notin J} t_i} w_J| \geq 1 \text{ for any } \mathbf{x} \in \widetilde{B}.$$

Therefore in this case, for all $\mathbf{t} \in \mathfrak{a}^+$, we have

$$\begin{aligned} \sup_{\mathbf{x}\in B\cap\operatorname{supp}(\nu)} \|h(\mathbf{x})\Delta\| &= \sup_{\mathbf{x}\in B\cap\operatorname{supp}(\nu)} \|h(\mathbf{x})\mathbf{w}\| = \sup_{\mathbf{x}\in B\cap\operatorname{supp}(\nu)} \max_I |h_I(\mathbf{x})| \\ &\geq \sup_{\mathbf{x}\in B\cap\operatorname{supp}(\nu)} |h_J(\mathbf{x})| \geq 1. \end{aligned} \tag{6.5}$$

Case 2: Suppose $w_I \neq 0$ for some $I \subseteq \{1, \ldots, n\}$. Choose $l \in \{1, \ldots, n\}$ such that $t_l = \max_{1\leq i\leq n} t_i$. If $l \in I$, set $J = I \cup \{0\} \smallsetminus \{l\}$. Clearly, J contains 0 but does not contain l. In view of (6.4), the coefficient of f_l in the expression of h_J is easily seen to be $\pm T^{\sum_{i\notin J} t_i} w_I$, and its absolute value is

$$|T^{\sum_{i\notin J} t_i} w_I| \geq e^{\sum_{i\notin J} t_i} \geq e^{t_l} \geq e^{t_0/n} = e^{\|\mathbf{t}\|/n}. \tag{6.6}$$

If $l \notin I$, choose any $i \in I$ and let $J = I \cup \{0\} \smallsetminus \{i\}$. As before, $\pm T^{\sum_{i\notin J} t_i} w_I$ turns out as the coefficient of f_i in h_J so that we obviously get the analogue of (6.6). Thus, in this case, there always exists J such that

(6.7) at least one of the coefficients of $f_1, f_2, \ldots, f_n$ in h_J has absolute value $\geq e^{\|\mathbf{t}\|/n}$.

https://doi.org/10.4153/CJM-2016-024-2 Published online by Cambridge University Press

Now, from Proposition 5.1(ii), it follows that there exists $\delta > 0$ such that

$$\sup_{\mathbf{x}\in B\cap \operatorname{supp}(\nu)} |c_0 + c_1 f_1(\mathbf{x}) + \cdots + c_n f_n(\mathbf{x})| \geq \delta$$

for any $c_0, c_1, \ldots, c_n \in F$ with $\max_{0\leq i\leq n} |c_i| \geq 1$. We choose $M \in \mathbb{N}$ such that $\delta e^M \geq 1$.

Let $\|\mathbf{t}\| \geq nM$. Then, because of (6.7), surely at least one of the coefficients of $f_1, f_2, \ldots, f_n$ in $\frac{1}{T^M} h_J$ has absolute value at least 1, and thus

$$\sup_{\mathbf{x}\cap \operatorname{supp}(\nu)} \Big| \frac{1}{T^M} h_J(\mathbf{x}) \Big| \geq \delta.$$

This gives

$$\sup_{\mathbf{x}\cap \operatorname{supp}(\nu)} \Big| h_J(\mathbf{x}) \Big| \geq \delta e^M \geq 1.$$

So, even here, we can see that for all $\|\mathbf{t}\| \geq nM$,

$$\begin{aligned} \sup_{\mathbf{x}\in B\cap \operatorname{supp}(\nu)} \|h(\mathbf{x})\Delta\| &= \sup_{\mathbf{x}\in B\cap \operatorname{supp}(\nu)} \|h(\mathbf{x})\mathbf{w}\| = \sup_{\mathbf{x}\in B\cap \operatorname{supp}(\nu)} \max_I |h_I(\mathbf{x})| \\ &\geq \sup_{\mathbf{x}\in B\cap \operatorname{supp}(\nu)} |h_J(\mathbf{x})| \geq 1. \end{aligned} \tag{6.8}$$

Letting $\rho = 1$, $(C2)$ is thus immediate from (6.5) and (6.8) whenever $\|\mathbf{t}\| \geq nM$.

Finally, $\widetilde{C}$ and s are taken as $(n+1)CD^{2(n+1)}$ and nM respectively, and one applies Theorem 6.1 to show (5.3). ∎

7 Explicit Constants: an Example

In this section, we talk about a simple application of Theorem 3.7 to a concrete example, with special attention to the explicit constant ε_0. Here $d = 1$, $n = 2$, and ν is the unique Haar measure on F that satisfies $\nu(B[0;1]) = 1$. It is not difficult to show that ν is e^2-Federer. Let

$$f\colon B(0,1) \longrightarrow F^2, \quad x \longmapsto (x, x^2).$$

We claim that f is $(2, 1/2)$-good; in fact, so is any $\phi \in F[x]$ having degree ≤ 2. To see this, we shall apply the same technique that used in the proof of [15, proposition 3.2].

Let $\varepsilon > 0$ and $\mathfrak{B} \subseteq B(0;1)$. We have to show

$$\nu\big(\{x \in \mathfrak{B} : |\phi(x)| < \varepsilon\}\big) \leq 2\Big(\frac{\varepsilon}{\|\phi\|_{\mathfrak{B}}}\Big)^{1/2} \nu(\mathfrak{B}). \tag{7.1}$$

For convenience, put $S := \{x \in \mathfrak{B} : |\phi(x)| < \varepsilon\}$. If $\nu(S)$ (*i.e.*, the LHS of (7.1)) is 0, then there is nothing to prove. Otherwise, we will show that

$$m \leq 2\Big(\frac{\varepsilon}{\|\phi\|_{\mathfrak{B}}}\Big)^{1/2} \nu(\mathfrak{B}),$$

or equivalently,

$$|\phi(x)| \leq \varepsilon\Big(\frac{\nu(\mathfrak{B})}{m/2}\Big)^2 \text{ for all } x \in \mathfrak{B} \tag{7.2}$$

whenever $0 < m < \nu(S)$.

From the continuity of ϕ, we see that for each $x \in S$, there is a ball B_x with center at x and radius $< \frac{m}{2}$ such that $B_x \subseteq S$. Now from the Besicovitch nature of F, one can

https://doi.org/10.4153/CJM-2016-024-2 Published online by Cambridge University Press

extract a countable subcover $\{B_1, B_2, \dots\}$ consisting of mutually disjoint open balls from the cover $\{B_x : x \in S\}$ of S. Clearly, $\nu(B_i) \le \frac{m}{2}$ for each i. Thus, in view of their size, it follows that the subcover has at least three balls. Let us denote their centers as x_1, x_2, and x_3. Then the centers x_i are in S, and they must satisfy

$$|x_i - x_j| \ge \frac{m}{2} \text{ for all } i, j = 1, 2, 3;\ i \ne j. \tag{7.3}$$

We now employ Lagrange's interpolation formula to complete the proof. Using this formula, we can write ϕ as

$$\begin{aligned}\phi(x) = \phi(x_1)\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} &+ \phi(x_2)\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \\ &+ \phi(x_3)\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}.\end{aligned} \tag{7.4}$$

As $\mathfrak{B}$ is a ball, certainly there exist a and $m \in \mathbb{N}$ such that $\mathfrak{B} = B[a; \frac{1}{e^m}]$. Therefore, diameter$(\mathfrak{B}) = \frac{1}{e^m} = \nu(\mathfrak{B})$. In view of this, (7.3), and (7.4), it follows at once that

$$|\phi(x)| \le \varepsilon \frac{(\text{diameter}(\mathfrak{B}))^2}{m^2/4} = \varepsilon\Big(\frac{\nu(\mathfrak{B})}{m/2}\Big)^2 \text{ for all } x \in \mathfrak{B},$$

and that shows (7.2).

Thus, all the conditions of the hypothesis of Theorem 3.7 hold here, and the existence of the desired $\varepsilon_0 > 0$ is confirmed. We are interested in computing it. In the proof of Theorem 3.7, we have also observed that our ε_0 can be taken as any positive quantity that is $< \frac{1}{\widetilde{C}^{1/\alpha}}$, where $\widetilde{C}$ was set, as we did in the proof of Proposition 5.1, as $(n+1)CD^{2(n+1)}$. Therefore in our example, we obtain that

$$\varepsilon_0 < \frac{1}{\widetilde{C}^2} = \frac{1}{(3 \times 2 \times (e^2)^{2\times 3})^2} = \frac{1}{36e^{24}}.$$

Acknowledgements We thank the referee for several helpful suggestions and Yann Bugeaud and Sanju Velani for helpful comments.

References

[1] B. Adamczewski and Y. Bugeaud, *On the Littlewood conjecture in fields of power series*. In: Probability and number theory-Kanazawa 2005, Adv. Stud. Pure Math., 49, Math. Soc. Japan, Tokyo, 2007, pp. 1–20.

[2] M. Amou, *A metrical result on transcendence measures in certain fields*. J. Number Theory **59**(1996), no. 2, 389–397. http://dx.doi.org/10.1006/jnth.1996.0104

[3] J. S. Athreya, A. Ghosh, and A.Prasad, *Ultrametric logarithm laws, II*. Monatsh. Math. **167**(2012), no. 3–4, 333–356. http://dx.doi.org/10.1007/s00605-012-0376-y

[4] R. C. Baker, *Metric diophantine approximation on manifolds*. J. London Math. Soc. (2) **14**(1976), 43–48. http://dx.doi.org/10.1112/jlms/s2-14.1.43

[5] ———, *Dirichlet's theorem on Diophantine approximation*. Math. Proc. Cambridge Phil. Soc. **83**(1978), no. 1, 37–59. http://dx.doi.org/10.1017/S030500410005427X

[6] Y. Bugeaud, *Approximation by algebraic integers and Hausdorff dimension*. J. London Math. Soc. (2) **65**(2002), no. 3, 547–559. http://dx.doi.org/10.1112/S0024610702003137

[7] H. Davenport and W. M. Schmidt, *Dirichlet's theorem on diophantine approximation*. In: Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69), Academic Press, London, 1970, pp. 113–132.

[8] ———, *Dirichlet's theorem on diophantine approximation. II*. Acta Arith. **16**(1969/1970), 413–424.

[9] B. de Mathan, *Approximations diophantiennes dans un corps local*. Bull. Soc. Math. France Suppl. Mé m. **21**(1970), 1–93.
[10] M. Dodson, B. Rynne, and J. Vickers, *Dirichlet's theorem and Diophantine approximation on manifolds*. J. Number Theory **36**(1990), no. 1, 85–88. http://dx.doi.org/10.1016/0022-314X(90)90006-D
[11] M. M. Dodson, S. Kristensen, and J. Levesley, *A quantitative Khintchine- Groshev type theorem over a field of formal series*. Indag. Math. (N.S.) **16**(2005), no. 2, 171–177. http://dx.doi.org/10.1016/S0019-3577(05)80020-5
[12] A. Ganguly, *Problems in Diophantine approximation and dynamical systems*. PhD thesis, Tata Institute of Fundamental Research, forthcoming.
[13] A. Ghosh, *Metric Diophantine approximation over a local field of positive characteristic*. J. Number Theory **124**(2007), no. 2, 454–469. http://dx.doi.org/10.1016/j.jnt.2006.10.009
[14] A. Ghosh and R. Royals, *An extension of Khintchine's theorem*. Acta Arith. **167**(2015), no. 1, 1–17. http://dx.doi.org/10.4064/aa167-1-1
[15] D. Y. Kleinbock and G. A. Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds*. Ann. of Math. **148**(1998), no. 1, 339–360. http://dx.doi.org/10.2307/120997
[16] D. Kleinbock, E. Lindenstrauss, and B. Weiss, *On fractal measures and Diophantine approximation*. Selecta Math. **10**(2004), no. 4, 479–523. http://dx.doi.org/10.1007/s00029-004-0378-2
[17] D. Kleinbock and G. Tomanov, *Flows on S-arithmetic homogeneous spaces and applications to metric Diophantine approximation*. Comment. Math. Helv. **82**(2007), no. 3, 519–581. http://dx.doi.org/10.4171/CMH/102
[18] D. Kleinbock and B. Weiss, *Dirichlet's theorem on diophantine approximation and homogeneous flows*. J. Mod. Dyn. 2(2008), no. 1, 43–62.
[19] ______, *Friendly measures, homogeneous flows and singular vectors*. In: Algebraic and topological dynamics, Contemp. Math., 385, Amer. Math. Soc., Providence, RI, 2005, pp. 281–292. http://dx.doi.org/10.1090/conm/385/07201
[20] S. Kristensen, *On well approximable matrices over a field of formal series*. Math. Proc. Cambridge Philos. Soc. **135**(2003), no. 2, 255–268. http://dx.doi.org/10.1017/S0305004103006911
[21] ______, *Badly approximable systems of linear forms over a field of formal series*. J. Théor. Nombres Bordeaux **18**(2006), no. 2, 421–444. http://dx.doi.org/10.5802/jtnb.552
[22] A. Lasjaunias, *A survey of Diophantine approximation in fields of power series*. Monatsh. Math. **130**(2000), no. 3, 211–229. http://dx.doi.org/10.1007/s006050070036
[23] ______, *Diophantine approximation and continued fractions in power series fields*. In: Analytic number theory, Cambridge Univ. Press, Cambridge, 2009, pp. 297–305.
[24] K. Mahler, *An analogue to Minkowski's geometry of numbers in a field of series*. Ann. of Math. (2) **42**(1941), no. 2, 488–522. http://dx.doi.org/10.2307/1968914
[25] P. Mattila, *Geometry of sets and measures in Euclidean space. Fractals and rectifiability*. Cambridge Studies in Advanced Mathematics, 44, Cambridge University Press, Cambridge, 1995. http://dx.doi.org/10.1017/CBO9780511623813
[26] B. Miller, *The existence of measures of a given cocycle, I: atomless, ergodic σ-finite measures*. Ergodic Theory Dynam. System **28**(2008), no. 5, 1599–1613. http://dx.doi.org/10.1017/S0143385707001113
[27] W. H. Schikhof, *Ultrametric calculus. An introduction to p-adic analysis*. Cambridge Studies in Advanced Mathematics, 4, Cambridge University Press, Cambridge, 1984.
[28] N. A. Shah, *Equidistribution of expanding translates of curves and Dirichlet's theorem on Diophantine approximation*. Invent. Math. **177**(2009), no. 3, 509–532. http://dx.doi.org/10.1007/s00222-009-0186-6
[29] ______, *Expanding translates of curves and Dirichlet-Minkowski theorem on linear forms*. J. Amer. Math. Soc. **23**(2010), no. 2, 563–589. http://dx.doi.org/10.1090/S0894-0347-09-00657-2
[30] ______, *Equidistribution of translates of curves on homogeneous spaces and Dirichlet's approximation*. Proceedings of the International Congress of Mathematicians, Volume III, Hindustan Book Agency, New Delhi, 2010, pp. 1332–1343.
[31] V. G. Sprindzuk, *Mahler's problem in metric number theory.* Translations of Mathematical Monographs, 25, American Mathematical Society, Providence, RI, 1969.
[32] ______, *Achievements and problems in Diophantine approximations.* (Russian) Uspekhi Mat. Nauk **35**(1980), no. 4(214), 3–68, 248.

School of Mathematics, Tata Institute of Fundamental Research, Mumbai, 400005, India
e-mail: arimath@math.tifr.res.in ghosh.anish@gmail.com