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OSCILLATIONS IN HIGHER-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT Consider the n-th order ($n \geq 1$) neutral differential equation

$$\text{(E)} \qquad \frac{d^n}{dt^n}\left[x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s)\right] + \zeta \int_{\sigma_1}^{\sigma_2} x(t+s)\, d\eta(s) = 0,$$

where $\delta \in \{0, +1, -1\}$, $\zeta \in \{+1, -1\}$, $-\infty < \tau_1 < \tau_2 < \infty$ with $\tau_1\tau_2 \neq 0$, $-\infty < \sigma_1 < \sigma_2 < \infty$ and μ and η are increasing real-valued functions on $[\tau_1, \tau_2]$ and $[\sigma_1, \sigma_2]$ respectively The function μ is assumed to be not constant on $[\tau_1, \tau]$ and $[\tau, \tau_2]$ for every $\tau \in (\tau_1, \tau_2)$, similarly, for each $\sigma \in (\sigma_1, \sigma_2)$, it is supposed that η is not constant on $[\sigma_1, \sigma]$ and $[\sigma, \sigma_2]$ Under some mild restrictions on τ_i and σ_i ($i = 1, 2$), it is proved that all solutions of (E) are oscillatory if and only if the characteristic equation

$$\lambda^n\left[1 + \delta \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\right] + \zeta \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) = 0$$

of (E) has no real roots

1. **Introduction and statement of the main result.** A neutral differential equation is a differential equation in which the highest order derivative of the unknown function is evaluated both at the present state and at one or more past or future states. Neutral differential equations arise naturally in the theory of transmission lines where the hyperbolic partial differential equations are linear and the boundary conditions are nonlinear. Other problems of nonlinear vibrations can also be formulated in terms of these equations.

Consider the n-th order ($n \geq 1$) neutral differential equation

$$\text{(E)} \qquad \frac{d^n}{dt^n}\left[x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s)\right] + \zeta \int_{\sigma_1}^{\sigma_2} x(t+s)\, d\eta(s) = 0,$$

where: $\delta \in \{0, +1, -1\}$ and $\zeta \in \{+1, -1\}$; τ_1 and τ_2 are nonzero real numbers with $\tau_1 < \tau_2$; σ_1 and σ_2 are real constants with $\sigma_1 < \sigma_2$; μ is an increasing real-valued function on $[\tau_1, \tau_2]$, which is not constant on any interval of the form $[\tau_1, \tau]$ or of the form $[\tau, \tau_2]$ where $\tau_1 < \tau < \tau_2$; η is an increasing real-valued function on $[\sigma_1, \sigma_2]$, which is not constant on the interval $[\sigma_1, \sigma]$ and on $[\sigma, \sigma_2]$ for every σ with $\sigma_1 < \sigma < \sigma_2$.

Throughout the paper we will use the notation

$$\gamma = \min\{0, \tau_1, \sigma_1\}.$$

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Clearly, γ is a nonpositive number.

By a *solution* of (E) we mean a continuous real-valued function x on the interval $[\gamma, \infty)$ such that the function $x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s)$ is n-times continuously differentiable for $t \geq 0$ and x satisfies (E) for all $t \geq 0$. Concerning existence, uniqueness and continuous dependence of solutions of neutral differential equations the reader is referred to [6], [10], [11], and [18].

As usual, a solution of (E) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*.

The *characteristic equation* of (E) is

$$(*) \qquad F(\lambda) \equiv \lambda^n \Big[1 + \delta \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big] + \zeta \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) = 0.$$

It is known that the behavior of solutions of neutral differential equations exhibit features which are not true for nonneutral equations. There are examples (see [8], [9], [18], [26], and [27]) of neutral differential equations with all the characteristic roots in the negative half-plane or with all the characteristic roots simple and on the imaginary axis and yet the equations have unbounded solutions; such a behavior is not possible in the case of nonneutral equations.

The oscillatory behavior of solutions of neutral differential equations has been the subject of intensive investigations during the past few years. In the oscillation theory of neutral differential equations one of the most important problems is to obtain necessary and sufficient conditions for the oscillation via the characteristic equation. Among numerous papers dealing with this problem we refer in particular to [1], [2], [5], [7], [12]–[15], [20]–[22], [24], [25], [28], and [30]. Sufficient conditions for the oscillation of the solutions of neutral equations have been obtained in many recent papers; we choose to refer in particular to [3], [4], [19], and [23]. Most of the papers mentioned above concern the case of first order neutral equations and two of these papers (see [21] and [22]) are referred to second order equations; the papers [1], [7], [19], [23], [28], and [30] are concerned with the oscillation of neutral differential equations of higher order.

Our aim in this paper is to show that, under some mild restrictions on τ_i and σ_i ($i = 1,2$), all solutions of the differential equation (E) are oscillatory if and only if its characteristic equation $(*)$ has no real roots. Such a result has been established by Philos and Sficas [24] for the first order case (*i.e.* when $n = 1$) with $\zeta = +1$ and under the additional assumption that $\tau_1\tau_2 > 0$.

Some restrictions on τ_i and σ_i ($i = 1,2$) will be imposed. More precisely, for $\zeta = +1$ it will be assumed that

$$(\mathrm{H}_+) \qquad \begin{cases} n \text{ is odd}, \delta = +1, \text{ and } \tau_1 < 0 \Rightarrow \tau_1 \neq \sigma_1 \\ n \text{ is even}, \delta = -1, \text{ and } \tau_1 < 0 \Rightarrow \tau_1 \neq \sigma_1 \\ \delta = -1 \text{ and } \tau_2 > 0 \Rightarrow \tau_2 \neq \sigma_2 \end{cases}$$

and for $\zeta = -1$ it will be supposed that

$$(\mathrm{H}_-) \qquad \begin{cases} n \text{ is odd}, \delta = -1, \text{ and } \tau_1 < 0 \Rightarrow \tau_1 \neq \sigma_1 \\ n \text{ is even}, \delta = +1, \text{ and } \tau_1 < 0 \Rightarrow \tau_1 \neq \sigma_1 \\ \delta = +1 \text{ and } \tau_2 > 0 \Rightarrow \tau_2 \neq \sigma_2. \end{cases}$$

Our main result is the following:

THEOREM. *Assume that (H_+) holds for $\zeta = +1$ and (H_-) holds for $\zeta = -1$. Then a necessary and sufficient condition for the oscillation of all solutions of (E) is that its characteristic equation (*) has no real roots.*

It is remarkable that the assumption (H_+) for $\zeta = +1$ or the assumption (H_-) for $\zeta = -1$ is needed only for the proof of the fact that, if the characteristic equation ($*$) has no real roots, then all solutions of the differential equation(E) are oscillatory. In Section 4, we will show that in the case of neutral difference-differential equations the condition (H_+) for $\zeta = +1$ or (H_-) for $\zeta = -1$ is a consequence of the assumption that the characteristic equation has no real roots (and so in this case the hypothesis (H_+) for $\zeta = +1$ or (H_-) for $\zeta = -1$ is not needed in our theorem). The same is also true in several other cases of neutral differential equations (see Section 4).

The method used in proving our theorem is based on the theory of Laplace transforms. The arguments rely on a known result (Lemma 1 in Section 2) about the abscissa of convergence of the Laplace transform of a nonnegative function. These arguments were presented for the first time in the excellent paper by Győri, Ladas and Pakula [17]; this technique is an improved version of a similar one used by Arino and Győri [1] (see also Győri, Ladas and Pakula [16]).

It must be noted that the use of Laplace transforms in equations with mixed (or advanced) arguments (which are included in equation (E)) exhibits some particular difficulties which are faced mainly by Lemmas 5 and 6 of this paper. More precisely, in the case of delay differential equations it is well known (see Hale [18]) that any solution is of exponential order and it does not tend to zero faster than any exponential. (This result was used in [1].) However, it is not known that such a result is true for the case of equations with mixed (or advanced) arguments. This difficulty is faced by Lemmas 5 and 6, which establish that, if (E) has a nonoscillatory solution, then the differential equation (E) admits also a nonoscillatory solution which is of exponential order and does not tend to zero faster than any exponential.

In the special case where $n = 1$ and $\zeta = +1$, our theorem leads to an improved version of the main result in the recent paper by Philos and Sficas [24] where the restriction $\tau_1\tau_2 > 0$ was imposed. The method used here patterns after that of [24]. However, the arguments have considerably been simplified. Moreover, the technique has been improved so that the restriction $\tau_1\tau_2 > 0$ has been removed. Finally, let us notice that in Lemma 6 it is established that any n-times continuously differentiable, positive and strictly monotone solution of (E) is of exponential order and it does not tend to zero faster than any exponential.

The proof of the theorem will be given in Section 3. Some lemmas which are needed for the proof of the main result are presented in Section 2. The last section (Section 4) contains some applications of the theorem and, in addition, a discussion.

2. **Some useful lemmas.** This section is devoted to some lemmas which will be used in proving our main result.

We first recall some facts about Laplace transforms. Let φ be a continuous real-valued function on the interval $[0,\infty)$. For the improper integral

$$\int_0^\infty e^{-\lambda t}\varphi(t)\,dt$$

three possibilities arise:

(a) the integral converges for no point λ in the complex plane;

(b) it converges for all points λ;

(c) it converges for every point λ with $\operatorname{Re}\lambda > \alpha_0$ and diverges for all λ with $\operatorname{Re}\lambda < \alpha_0$, where α_0 is a real number.

Assume that (b) or (c) is true and define

$$\Phi(\lambda) = \int_0^\infty e^{-\lambda t}\varphi(t)\,dt \quad \text{for} \quad \operatorname{Re}\lambda > \alpha,$$

where $\alpha = -\infty$ in case (b), and $\alpha = \alpha_0$ in case (c). Then the function Φ is called the *Laplace transform* of φ. Moreover, we refer to α as the *abscissa of convergence* of the Laplace transform of φ. Note that Φ is an analytic function in the half-plane $\operatorname{Re}\lambda > \alpha$, when $\alpha > -\infty$.

We will use the following known result from Widder [29].

LEMMA 1. *Let φ be a nonnegative continuous real-valued function on the interval $[0,\infty)$. If Φ is the Laplace transform of φ and has abscissa of convergence $\alpha > -\infty$, then the real point $\lambda = \alpha$ is a singularity of Φ.*

The function φ is *of exponential order* c, for some real number c, if there exist $M > 0$ and $t_0 \geq 0$ such that $|\varphi(t)| \leq Me^{ct}$ for all $t \geq t_0$. It is easy to see that, if φ is of exponential order c, then the abscissa of convergence of the Laplace transform of φ is less than or equal to c.

The next lemma gives conditions under which the Laplace transform (if it is defined) of a function has abscissa of convergence $\alpha > -\infty$; this lemma gives also sufficient conditions for a function to be of exponential order c for some real number c.

LEMMA 2. *(a) Let φ be a positive function on the interval $[0,\infty)$. Assume that, for some $T \geq 0$, the function φ is strictly decreasing on $[T,\infty)$, continuous on $[0,T]$ and such that*

$$\varphi(t) \leq M\varphi(t+\xi) \ \textit{for} \ t \geq T,$$

where $M > 1$ and $\xi > 0$ are real constants. Then there exist $C > 0$ and $k > 0$ so that

$$\varphi(t) \geq Ce^{-kt} \ \textit{for all} \ t \geq 0. \tag{1}$$

(b) Let φ be a positive function on the interval $[0,\infty)$. Assume that, for some $T \geq 0$, the function φ is strictly increasing on $[T,\infty)$, continuous on $[0,T]$ and such that

$$\varphi(t) \geq M\varphi(t+\xi) \ \textit{for} \ t \geq T,$$

where $M \in (0,1)$ and $\xi > 0$ are real constants Then there exist $C > 0$ and $k > 0$ so that

$$\varphi(t) \leq Ce^{kt} \text{ for all } t \geq 0 \tag{2}$$

PROOF (a) Consider an arbitrary point $t \geq T$ and set

$$\nu = \left[\frac{t-T}{\xi}\right] + 1$$

Then we obtain

$$\varphi(t) > \varphi(T+\nu\xi) \geq \frac{1}{M}\varphi\big(T+(\nu-1)\xi\big) \geq \qquad \geq \frac{1}{M^{\nu}}\varphi(T)$$
$$\geq \frac{1}{M^{1+\frac{t\ T}{\xi}}}\varphi(T) = \frac{1}{M^{1\ \ T/\xi}}\varphi(T)e^{-(\ln M/\xi)t}$$

So, if we put

$$k = \frac{\ln M}{\xi} > 0$$

and

$$C = \min\Big\{\min_{t\in[0\ T]}[\varphi(t)e^{kt}], \frac{1}{M^{1\ \ T/\xi}}\varphi(T)\Big\} > 0,$$

then we see that (1) is true

(b) The conclusion can be obtained by applying (a) for the function $1/\varphi$

Let the function φ in Lemma 2 be continuous on $[0,\infty)$ If (1) holds, then we obtain

$$\int_0^\infty e^{-(-k)t}\varphi(t)\,dt \geq C\int_0^\infty dt = \infty$$

Thus, (1) means that the Laplace transform (if it is defined) of φ has abscissa of convergence $\alpha \geq -k > -\infty$ Moreover, note that (2) implies that φ is of exponential order k

Lemma 3 below provides necessary conditions for the characteristic equation $(*)$ to have no real roots

LEMMA 3 *Assume that the characteristic equation $(*)$ has no real roots If $\zeta = +1$, then we have*

(i) *n is odd and $\delta \in \{0,+1\} \Rightarrow \sigma_1 < 0$,*
(ii) *n is odd, $\delta = -1$, and $\tau_1 > 0 \Rightarrow \sigma_1 < 0$,*
(iii) *n is odd and $\delta = +1 \Rightarrow \tau_1 \geq \sigma_1$,*
(iv) *n is even, $\delta = -1$, and $\tau_1 < 0 \Rightarrow \tau_1 \geq \sigma_1$,*
(v) *$\delta = -1$ and $\tau_2 > 0 \Rightarrow \tau_2 \leq \sigma_2$*

Moreover, if $\zeta = -1$, then we have

(vi) *n is even and $\delta \in \{0,+1\} \Rightarrow \sigma_1 < 0$,*
(vii) *n is even, $\delta = -1$, and $\tau_1 > 0 \Rightarrow \sigma_1 < 0$,*
(viii) *n is odd, $\delta = -1$, and $\tau_1 < 0 \Rightarrow \tau_1 \geq \sigma_1$,*

(ix) *n is even and* $\delta = +1 \Rightarrow \tau_1 \geq \sigma_1$,
(x) $\delta \in \{0, +1\} \Rightarrow \sigma_2 > 0$,
(xi) $\delta = -1$ *and* $\tau_2 < 0 \Rightarrow \sigma_2 > 0$,
(xii) $\delta = +1 \Rightarrow \tau_2 \leq \sigma_2$.

PROOF. Assume that $\zeta = +1$. Then $F(0) = \int_{\sigma_1}^{\sigma_2} d\eta(s) > 0$. So, as $F(\lambda) = 0$ has no real roots, we have

(3) $$F(\lambda) > 0 \text{ for all } \lambda \in (-\infty, \infty).$$

Assume that n is odd, $\delta \in \{0, +1\}$, and $\sigma_1 \geq 0$. For each $\lambda < 0$, we get

$$F(\lambda) = \lambda^n \Big[1 + \delta \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big] + \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) < \lambda^n + \int_{\sigma_1}^{\sigma_2} d\eta(s)$$

and so $F(-\infty) = -\infty$. This contradicts (3) and hence (i) is proved.

Let n be odd, $\delta = -1$, $\tau_1 > 0$, and $\sigma_1 \geq 0$. We have for $\lambda < 0$

$$\begin{aligned} F(\lambda) &= \lambda^n \Big[1 - \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big] + \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) \\ &< \lambda^n - \lambda^n e^{\lambda \tau_1} \int_{\tau_1}^{\tau_2} d\mu(s) + \int_{\sigma_1}^{\sigma_2} d\eta(s). \end{aligned}$$

This guarantees that $F(-\infty) = -\infty$, which contradicts (3) and so (ii) is established.

Now, we introduce the function R defined by

(4) $$R(\lambda) = \frac{\int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)}{\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s)}, \quad -\infty < \lambda < \infty.$$

For this function we have

(5) $$\tau_1 < \sigma_1 \Rightarrow R(-\infty) = \infty.$$

Indeed, in the case where $\tau_1 < \sigma_1$ we can consider a positive number ε with $\sigma_1 - \tau_2 < \varepsilon < \sigma_1 - \tau_1$. Then $\tau_1 < \sigma_1 - \varepsilon < \tau_2$ and so, for every $\lambda < 0$, we obtain

$$R(\lambda) > \frac{\int_{\tau_1}^{\sigma_1 - \varepsilon} e^{\lambda s}\, d\mu(s)}{\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s)} > \frac{e^{\lambda(\sigma_1 - \varepsilon)} \int_{\tau_1}^{\sigma_1 - \varepsilon} d\mu(s)}{e^{\lambda \sigma_1} \int_{\sigma_1}^{\sigma_2} d\eta(s)} = e^{-\lambda \varepsilon} \frac{\int_{\tau_1}^{\sigma_1 - \varepsilon} d\mu(s)}{\int_{\sigma_1}^{\sigma_2} d\eta(s)}.$$

This ensures that $R(-\infty) = \infty$, *i.e.* (5) is true. Furthermore, if $\tau_2 > \sigma_2$, then we can choose an $\varepsilon > 0$ so that $\tau_1 - \sigma_2 < \varepsilon < \tau_2 - \sigma_2$ and hence we derive for $\lambda > 0$

$$R(\lambda) > \frac{\int_{\sigma_2 + \varepsilon}^{\tau_2} e^{\lambda s}\, d\mu(s)}{\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s)} > \frac{e^{\lambda(\sigma_2 + \varepsilon)} \int_{\sigma_2 + \varepsilon}^{\tau_2} d\mu(s)}{e^{\lambda \sigma_2} \int_{\sigma_1}^{\sigma_2} d\eta(s)} = e^{\lambda \varepsilon} \frac{\int_{\sigma_2 + \varepsilon}^{\tau_2} d\mu(s)}{\int_{\sigma_1}^{\sigma_2} d\eta(s)}.$$

Therefore,

(6) $$\tau_2 > \sigma_2 \Rightarrow R(\infty) = \infty.$$

Next, we proceed to prove (iii). We suppose that n is odd, $\delta = +1$, and $\tau_1 < \sigma_1$. Then we have

$$F(\lambda) = \lambda^n + [\lambda^n R(\lambda) + 1] \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s), \quad \lambda \in \mathbb{R}.$$

But (5) implies the existence of a $\lambda_0 < 0$ such that $\lambda^n R(\lambda) < -1$ for all $\lambda \leq \lambda_0$. Hence, $F(\lambda) < 0$ for $\lambda \leq \lambda_0$, which contradicts (3). So, (iii) holds.

Assume that n is even, $\delta = -1$, $\tau_1 < 0$, and $\tau_1 < \sigma_1$. First of all, we will prove that $\sigma_1 < 0$. To this end, suppose that $\sigma_1 \geq 0$. We choose a $\tau < 0$ with $\tau_1 < \tau < \tau_2$ and we obtain for $\lambda < 0$

$$\begin{aligned} F(\lambda) &= \lambda^n\Big[1 - \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big] + \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) \\ &< \lambda^n\Big[1 - \int_{\tau_1}^{\tau} e^{\lambda s}\, d\mu(s)\Big] + e^{\lambda\sigma_1}\int_{\sigma_1}^{\sigma_2} d\eta(s) \\ &< \lambda^n\Big[1 - e^{\lambda\tau}\int_{\tau_1}^{\tau} d\mu(s)\Big] + \int_{\sigma_1}^{\sigma_2} d\eta(s). \end{aligned}$$

This gives $F(-\infty) = -\infty$, which contradicts (3) and so $\sigma_1 < 0$. Now, we can consider a $\sigma < 0$ such that $\sigma_1 < \sigma < \sigma_2$ and hence

$$\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) > \int_{\sigma_1}^{\sigma} e^{\lambda s}\, d\eta(s) > e^{\lambda\sigma}\int_{\sigma_1}^{\sigma} d\eta(s)$$

for all $\lambda < 0$. This guarantees that

$$\lim_{\lambda\to-\infty} \frac{\lambda^n}{\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s)} = 0. \tag{7}$$

We have

$$F(\lambda) = \left[\frac{\lambda^n}{\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s)} - \lambda^n R(\lambda) + 1\right]\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) \text{ for } \lambda \in \mathbb{R}. \tag{8}$$

Thus, by (5) and (7), we can conclude that there exists a $\lambda_0 < 0$ so that $F(\lambda) < 0$ for every $\lambda \leq \lambda_0$. This contradicts (3) and hence (iv) is true.

Next, let us assume that $\delta = -1$, $\tau_2 > 0$, and $\tau_2 > \sigma_2$. If we suppose that $\sigma_2 \leq 0$ and we consider a $\tau > 0$ with $\tau_1 < \tau < \tau_2$, then we get for $\lambda > 0$

$$\begin{aligned} F(\lambda) &< \lambda^n\Big[1 - \int_{\tau}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big] + e^{\lambda\sigma_2}\int_{\sigma_1}^{\sigma_2} d\eta(s) \\ &< \lambda^n\Big[1 - e^{\lambda\tau}\int_{\tau}^{\tau_2} d\mu(s)\Big] + \int_{\sigma_1}^{\sigma_2} d\eta(s). \end{aligned}$$

This gives $F(\infty) = -\infty$, which contradicts (3). This contradiction establishes that $\sigma_2 > 0$. Thus, we can choose a $\sigma > 0$ with $\sigma_1 < \sigma < \sigma_2$. Then we have

$$\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\mu(s) > e^{\lambda\sigma}\int_{\sigma}^{\sigma_2} d\eta(s) \text{ for } \lambda > 0,$$

which implies that

$$\lim_{\lambda\to\infty} \frac{\lambda^n}{\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s)} = 0. \tag{9}$$

From (6), (8) and (9) it follows that $F(\lambda) < 0$ for all large λ, which contradicts (3) and proves that (v) is valid.

Consider now the case where $\zeta = -1$. In this case we have $F(0) = -\int_{\sigma_1}^{\sigma_2} d\eta(s) < 0$ and consequently

(10) $$F(\lambda) < 0 \text{ for all } \lambda \in \mathbb{R}.$$

Assume that n is even, $\delta \in \{0, +1\}$, and $\sigma_1 \geq 0$. Then we have for $\lambda < 0$

$$F(\lambda) = \lambda^n \Big[1 + \delta \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big] - \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) > \lambda^n - \int_{\sigma_1}^{\sigma_2} d\eta(s)$$

and hence $F(-\infty) = \infty$, which contradicts (10). This contradiction proves (vi).

Let n be even, $\delta = -1$, $\tau_1 > 0$, and $\sigma_1 \geq 0$. Then, for each $\lambda < 0$, we obtain

$$\begin{aligned} F(\lambda) &= \lambda^n \Big[1 - \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big] - \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) \\ &> \lambda^n - \lambda^n e^{\lambda \tau_1} \int_{\tau_1}^{\tau_2} d\mu(s) - \int_{\sigma_1}^{\sigma_2} d\eta(s), \end{aligned}$$

which gives $F(-\infty) = \infty$. This contradicts (10) and so (vii) is true.

Let us suppose that n is odd, $\delta = -1$, $\tau_1 < 0$, and $\tau_1 < \sigma_1$. We have for $\lambda < 0$

$$\begin{aligned} F(\lambda) &= \lambda^n \Big[1 - \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big] - \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) \\ &> \lambda^n \Big[1 - \int_{\tau_1}^{\tau} e^{\lambda s}\, d\mu(s)\Big] - e^{\lambda \sigma_1} \int_{\sigma_1}^{\sigma_2} d\eta(s) \\ &> \lambda^n \Big[1 - e^{\lambda \tau} \int_{\tau_1}^{\tau} d\mu(s)\Big] - e^{\lambda \sigma_1} \int_{\sigma_1}^{\sigma_2} d\eta(s), \end{aligned}$$

where τ is a negative number with $\tau_1 < \tau < \tau_2$. So, if $\sigma_1 \geq 0$, then we get $F(-\infty) = \infty$, which contradicts (10). Hence, we always have $\sigma_1 < 0$. Thus, as previously, we can see that (7) holds. Now, we have

(11) $$F(\lambda) = \Big[\frac{\lambda^n}{\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s)} - \lambda^n R(\lambda) - 1\Big] \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) \text{ for } \lambda \in \mathbb{R},$$

where the function R is defined by (4). From (5), (7) and (11) it follows that there exists a $\lambda_0 < 0$ such that $F(\lambda) > 0$ for all $\lambda \leq \lambda_0$. This contradicts (10) and so (viii) has been proved.

Assume that n is even, $\delta = +1$, and $\tau_1 < \sigma_1$. We have

(12) $$F(\lambda) = \lambda^n [\lambda^n R(\lambda) - 1] \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s), \quad \lambda \in \mathbb{R}.$$

By (5), there exists a $\lambda_0 < 0$ such that $\lambda^n R(\lambda) > 1$ for every $\lambda \leq \lambda_0$. So, $F(\lambda) > 0$ for $\lambda \leq \lambda_0$, which contradicts (10) and proves (ix).

Let us assume that $\delta \in \{0, +1\}$ and $\sigma_2 \leq 0$. We have

$$F(\lambda) = \lambda^n \Big[1 + \delta \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big] - \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) > \lambda^n - \int_{\sigma_1}^{\sigma_2} d\eta(s)$$

for all $\lambda > 0$, and hence $F(\infty) = \infty$. This contradicts (10) and so (x) has been proved.

Assume that $\delta = -1$, $\tau_2 < 0$, and $\sigma_2 \leq 0$. We have for $\lambda > 0$

$$\begin{aligned} F(\lambda) &= \lambda^n\Big[1 - \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big] - \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) \\ &> \lambda^n - \lambda^n e^{\lambda\tau_2} \int_{\tau_1}^{\tau_2} d\mu(s) - \int_{\sigma_1}^{\sigma_2} d\eta(s), \end{aligned}$$

which gives $F(\infty) = \infty$. This contradicts (10) and so (xi) is true.

Finally, let $\delta = +1$ and $\tau_2 > \sigma_2$. From (6) it follows that there exists a $\lambda_0 > 0$ such that $\lambda^n R(\lambda) > 1$ for every $\lambda \geq \lambda_0$. Hence, (12) gives $F(\lambda) > 0$ for all $\lambda \geq \lambda_0$, which contradicts (10). Thus, (xii) has been established.

LEMMA 4. *Let x be a solution of the differential equation (E). Then we have:*
(a) If $t_0 \geq 0$, then the function $z(t) = x(t + t_0)$, $t \geq \gamma$ is also a solution of (E).
(b) Set

$$u_1(t) = \int_{\tau_1}^{\tau_2} x(t + s)\, d\mu(s) \text{ for } t \geq 0.$$

Then the function $u(t) = u_1(t - \gamma)$, $t \geq \gamma$ is also a solution of (E).

PROOF. (a) The conclusion follows easily and it is a consequence of the autonomous nature of (E).

(b) For any $t \geq -\gamma$, we obtain

$$\begin{aligned} &\Big[u_1(t) + \delta \int_{\tau_1}^{\tau_2} u_1(t+s)\, d\mu(s)\Big]^{(n)} + \zeta \int_{\sigma_1}^{\sigma_2} u_1(t+s)\, d\eta(s) \\ &= \Big\{\int_{\tau_1}^{\tau_2}\Big[x(t+s) + \delta \int_{\tau_1}^{\tau_2} x(t+s+r)\, d\mu(r)\Big]\, d\mu(s)\Big\}^{(n)} + \zeta \int_{\sigma_1}^{\sigma_2}\Big[\int_{\tau_1}^{\tau_2} x(t+s+r)\, d\mu(r)\Big]\, d\eta(s) \\ &= \int_{\tau_1}^{\tau_2}\Big[x(t+s) + \delta \int_{\tau_1}^{\tau_2} x(t+s+r)\, d\mu(r)\Big]^{(n)} d\mu(s) + \zeta \int_{\tau_1}^{\tau_2}\Big[\int_{\sigma_1}^{\sigma_2} x(t+s+r)\, d\eta(r)\Big]\, d\mu(s) \\ &= \int_{\tau_1}^{\tau_2}\Big\{\Big[x(t+s) + \delta \int_{\tau_1}^{\tau_2} x(t+s+r)\, d\mu(r)\Big]^{(n)} + \zeta \int_{\sigma_1}^{\sigma_2} x(t+s+r)\, d\eta(r)\Big\}\, d\mu(s) \\ &= 0 \end{aligned}$$

and hence u_1 satisfies (E) for every $t \geq -\gamma$. Since (E) is autonomous, we conclude that u satisfies (E) for all $t \geq 0$. Thus, u is a solution of (E).

LEMMA 5. *Assume that the differential equation (E) admits a nonoscillatory solution. Then (E) also has a solution which is n-times continuously differentiable, positive and strictly monotone on the interval* $[\gamma, \infty)$.

PROOF. Let x be a nonoscillatory solution of (E). As the negative of a solution of (E) is also a solution of the same equation, we may (and do) assume that x is eventually positive. Consider the function y defined by

$$y(t) = x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) \text{ for } t \geq 0.$$

Lemma 4 guarantees that the function $\tilde{y}(t) = y(t - \gamma)$, $t \geq \gamma$ is also a solution of the differential equation (E), which is n-times continuously differentiable on the interval $[\gamma, \infty)$. For every $t \geq 0$, we have

$$y^{(n)}(t) = \Big[x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s)\Big]^{(n)} = -\zeta \int_{\sigma_1}^{\sigma_2} x(t+s)\, d\eta(s)$$

and consequently $y^{(n)}$ is either eventually positive or eventually negative. This implies that y (and hence $\tilde{y}$) is either eventually positive or eventually negative. Set

$$z_1 = \begin{cases} \tilde{y}, & \text{if } \tilde{y}(t) > 0 \text{ for all large } t \\ -\tilde{y}, & \text{if } \tilde{y}(t) < 0 \text{ for all large } t. \end{cases}$$

Then z_1 is a solution of (E) which is n-times continuously differentiable on $[\gamma, \infty)$, eventually positive and eventually strictly monotone. Let $T \geq \gamma$ be such that z_1 is positive and strictly monotone on $[T, \infty)$. Then, by Lemma 4, the function $z(t) = z_1(t + T - \gamma)$, $t \geq \gamma$ is also a solution of (E). This solution is obviously n-times continuously differentiable, positive and strictly monotone on the interval $[\gamma, \infty)$.

LEMMA 6. *For $\zeta = +1$ we suppose that (i) and (ii) hold and:*
(iii)′ n is odd, $\delta = +1$, and $\tau_1 < 0 \Rightarrow \tau_1 > \sigma_1$,
(iv)′ n is even, $\delta = -1$, and $\tau_1 < 0 \Rightarrow \tau_1 > \sigma_1$,
(v)′ $\delta = -1$ and $\tau_2 > 0 \Rightarrow \tau_2 < \sigma_2$.
Also, for $\zeta = -1$ we suppose that (vi), (vii), (x) and (xi) hold and:
(viii)′ n is odd, $\delta = -1$, and $\tau_1 < 0 \Rightarrow \tau_1 > \sigma_1$,
(ix)′ n is even, $\delta = +1$, and $\tau_1 < 0 \Rightarrow \tau_1 > \sigma_1$,
(xii)′ $\delta = +1$ and $\tau_2 > 0 \Rightarrow \tau_2 < \sigma_2$.

Let x be a solution of (E) which is n-times continuously differentiable, positive and strictly monotone and on the interval $[\gamma, \infty)$. Then:
(a) x is of exponential order c for some $c \in (-\infty, \infty)$;
(b) the Laplace transform of x has abscissa of convergence $\alpha > -\infty$.

PROOF. Set $L = \lim_{t\to\infty} x(t)$, $0 \leq L \leq \infty$. If $0 \leq L < \infty$, then the solution x is bounded on the interval $[0, \infty)$ and so x is of exponential order $c = 0$. If $0 < L \leq \infty$, then there exists a $\mu > 0$ so that $x(t) \geq \mu$ for every $t \geq 0$ and consequently

$$\int_0^\infty x(t)\,dt \geq \mu \int_0^\infty dt = \infty.$$

This means that, provided that x has a Laplace transform, the abscissa of convergence α satisfies $\alpha \geq 0 > -\infty$. Hence, it is enough to restrict ourselves to the cases where $L = 0$ or $L = \infty$. Moreover, for $L = \infty$ it suffices to prove that x is of exponential order c for some real number c, while for $L = 0$ it remains to show that the abscissa of convergence α of the Laplace transform of x is such that $\alpha > -\infty$. Furthermore, by Lemma 2, for $L = \infty$ it is enough to prove that there exist $T \geq 0$, $M > 0$ and $\xi > 0$ so that

$$x(t) \geq Mx(t + \xi) \text{ for } t \geq T, \tag{13}$$

while for $L = 0$ it suffices to show that for some constants $T \geq 0$, $M > 0$ and $\xi > 0$ the following inequality is true

$$x(t) \leq Mx(t + \xi) \text{ for } t \geq T. \tag{14}$$

[Note that, if $L = \infty$ and (13) is satisfied, then we always have $0 < M < 1$, while $L = 0$ and (14) imply that $M > 1$.]

Define

$$y(t) = x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) \text{ for } t \geq 0.$$

Then from (E) we obtain

$$y^{(n)}(t) = -\zeta \int_{\sigma_1}^{\sigma_2} x(t+s)\, d\eta(s) \text{ for every } t \geq 0 \tag{15}$$

and so $y^{(n)}$ is either positive on the interval $[0, \infty)$ or negative on $[0, \infty)$. Then it is easy to verify that there exists a $T_0 \geq 0$ such that, for each $i \in \{0, 1, \ldots, n-1\}$, the function $y^{(i)}$ is either positive on the interval $[T_0, \infty)$ or negative on $[T_0, \infty)$. Clearly, for $L = \infty$ the solution x is strictly increasing on the interval $[\gamma, \infty)$, while x is strictly decreasing on $[\gamma, \infty)$ in the case where $L = 0$. Consider now the following four cases:

CASE I. $\zeta = +1$ and $L = \infty$. From (15) we obtain for every $t \geq 0$

$$y^{(n)}(t) = -\int_{\sigma_1}^{\sigma_2} x(t+s)\, d\eta(s) < -\Big[\int_{\sigma_1}^{\sigma_2} d\eta(s)\Big] x(t+\sigma_1),$$

which gives $\lim_{t\to\infty} y^{(n)}(t) = -\infty$, and consequently $y^{(n)}$ is negative on $[0, \infty)$. It is easy to see that $\lim_{t\to\infty} y^{(i)}(t) = -\infty$ $(i = 0, 1, \ldots, n-1)$ and so we have

$$y^{(i)}(t) < 0 \text{ for every } t \geq T_0 \quad (i = 0, 1, \ldots, n-1). \tag{16}$$

If $\delta \in \{0, +1\}$, then the definition of y ensures that $y(t) > 0$ for $t \geq 0$. This contradicts (16). Next, let us suppose that $\delta = -1$ and $\tau_2 < 0$. Then, by (16), we obtain for $t \geq T_0$

$$0 > y(t) = x(t) - \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) > x(t) - \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t+\tau_2)$$

and consequently

$$x(t+\tau_2) > \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big]^{-1} x(t), \quad t \geq T_0.$$

This gives

$$x(t) > \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big]^{-1} x(t-\tau_2) \text{ for every } t \geq T_0$$

and so (13) holds with

$$T = T_0 \geq 0, \quad M = \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big]^{-1} > 0, \text{ and } \xi = -\tau_2 > 0.$$

It remains to consider the case where $\delta = -1$ and $\tau_2 > 0$. By (v)$'$, we have $\tau_2 < \sigma_2$ and hence we can choose a number σ with $0 < \tau_2 < \sigma < \sigma_2$. Set

$$z_1(t) = -y^{(n-1)}(t) - \int_{\sigma}^{\sigma_2} \Big[\int_{t+\sigma}^{t+s} x(r)\, dr\Big] d\eta(s) \text{ for } t \geq 0.$$

Then, by using (15), we obtain for every $t \geq 0$

$$\begin{aligned} z_1'(t) &= -y^{(n)}(t) - \int_{\sigma}^{\sigma_2} [x(t+s) - x(t+\sigma)]\, d\eta(s) \\ &= \int_{\sigma_1}^{\sigma_2} x(t+s)\, d\eta(s) - \int_{\sigma}^{\sigma_2} x(t+s)\, d\eta(s) + \Big[\int_{\sigma}^{\sigma_2} d\eta(s)\Big] x(t+\sigma) \\ &= \int_{\sigma_1}^{\sigma} x(t+s)\, d\eta(s) + \Big[\int_{\sigma}^{\sigma_2} d\eta(s)\Big] x(t+\sigma) \\ &> \Big[\int_{\sigma}^{\sigma_2} d\eta(s)\Big] x(t+\sigma), \end{aligned}$$

which ensures that $\lim_{t\to\infty} z_1'(t) = \infty$. Thus, we have $\lim_{t\to\infty} z_1(t) = \infty$ and so, for some $t_1 \ge T_0$, the function z_1 is positive on $[t_1, \infty)$. Hence, for every $t \ge t_1$

$$-y^{(n-1)}(t) > \int_\sigma^{\sigma_2} \left[\int_{t+\sigma}^{t+s} x(r)\, dr\right] d\eta(s) > \left[\int_\sigma^{\sigma_2} (s-\sigma)\, d\eta(s)\right] x(t+\sigma)$$

and so

$$-y^{(n-1)}(t) > A_1 x(t+\sigma) \text{ for all } t \ge t_1, \tag{17}$$

where $A_1 = \int_\sigma^{\sigma_2}(s-\sigma)\, d\eta(s) > 0$. If $n = 1$, then (17) gives for $t \ge t_1$

$$A_1 x(t+\sigma) < -y(t) = -x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) < \left[\int_{\tau_1}^{\tau_2} d\mu(s)\right] x(t+\tau_2)$$

and consequently

$$x(t+\tau_2) > A_1 \left[\int_{\tau_1}^{\tau_2} d\mu(s)\right]^{-1} x(t+\sigma) \text{ for } t \ge t_1,$$

i.e.

$$x(t) > A_1 \left[\int_{\tau_1}^{\tau_2} d\mu(s)\right]^{-1} x(t-\tau_2+\sigma), \quad t \ge t_1 + \tau_2.$$

Thus, for $n = 1$, (13) is true with

$$T = t_1 + \tau_2 \ge 0, \quad M = A_1 \left[\int_{\tau_1}^{\tau_2} d\mu(s)\right]^{-1} > 0, \text{ and } \xi = -\tau_2 + \sigma > 0.$$

So, we suppose that $n > 1$. Consider a number ϑ with $0 < \vartheta < \sigma - \tau_2$. By using (16) and (17), we obtain for $t \ge t_1$

$$\begin{aligned} -y^{(n-2)}\left(t + \frac{\vartheta}{n-1}\right) &> -y^{(n-2)}\left(t + \frac{\vartheta}{n-1}\right) + y^{(n-2)}(t) = -\int_t^{t+\vartheta/(n-1)} y^{(n-1)}(s)\, ds \\ &> A_1 \int_t^{t+\vartheta/(n-1)} x(s+\sigma)\, ds > A_1 \frac{\vartheta}{n-1} x(t+\sigma) \end{aligned}$$

and hence

$$-y^{(n-2)}(t) > A_1 \frac{\vartheta}{n-1} x\left(t - \frac{\vartheta}{n-1} + \sigma\right) \text{ for } t \ge t_1 + \frac{\vartheta}{n-1}.$$

By this procedure, after $n-1$ steps we find

$$-y(t) > A_1 \left(\frac{\vartheta}{n-1}\right)^{n-1} x(t - \vartheta + \sigma) \text{ for all } t \ge t_1 + \vartheta.$$

Thus, for every $t \ge t_1 + \vartheta$

$$A_1 \left(\frac{\vartheta}{n-1}\right)^{n-1} x(t-\vartheta+\sigma) < -x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) < \left[\int_{\tau_1}^{\tau_2} d\mu(s)\right] x(t+\tau_2),$$

which gives

$$x(t) > A_1 \left(\frac{\vartheta}{n-1}\right)^{n-1} \left[\int_{\tau_1}^{\tau_2} d\mu(s)\right]^{-1} x(t - \tau_2 - \vartheta + \sigma) \text{ for } t \ge t_1 + \vartheta + \tau_2.$$

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So, if we put

$$T = t_1 + \vartheta + \tau_2 \geq 0, \quad M = A_1\Big(\frac{\vartheta}{n-1}\Big)^{n-1}\Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big]^{-1} > 0, \text{ and } \xi = -\tau_2 - \vartheta + \sigma > 0,$$

then (13) is also true.

CASE II. $\zeta = +1$ and $L = 0$. From (15) it follows that $y^{(n)}$ is negative on $[0, \infty)$. Furthermore, we can see that $\lim_{t\to\infty} y^{(i)}(t) = 0\,(i = 0, 1, \ldots, n-1)$. So, we can conclude that

$$(18) \qquad (-1)^{n+1-i} y^{(i)}(t) > 0 \text{ for every } t \geq T_0 \quad (i = 0, 1, \ldots, n-1).$$

In particular, y is negative on $[T_0, \infty)$ for n even, and y is positive on $[T_0, \infty)$ for n odd.

If $\delta \in \{0, +1\}$, then the definition of y ensures that $y(t) > 0$ for $t \geq 0$. We have thus arrived at a contradiction in the case where n is even and $\delta \in \{0, +1\}$. Next, let us assume that n is even, $\delta = -1$, and $\tau_1 > 0$. Then we obtain for $t \geq T_0$

$$0 > y(t) = x(t) - \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) > x(t) - \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t+\tau_1)$$

and consequently

$$x(t) < \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t+\tau_1) \text{ for } t \geq T_0.$$

That is, (14) holds with

$$T = T_0 \geq 0, \quad M = \int_{\tau_1}^{\tau_2} d\mu(s) > 0, \text{ and } \xi = \tau_1 > 0.$$

Consider now the case where n is odd, $\delta = -1$, and $\tau_1 < 0$. Then we choose a $\tau < 0$ with $\tau_1 < \tau < \tau_2$. Then, for every $t \geq T_0$, we have

$$\begin{aligned} 0 < y(t) &= x(t) - \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) \\ &< x(t) - \int_{\tau_1}^{\tau} x(t+s)\, d\mu(s) \\ &< x(t) - \Big[\int_{\tau_1}^{\tau} d\mu(s)\Big] x(t+\tau) \end{aligned}$$

or

$$x(t+\tau) < \Big[\int_{\tau_1}^{\tau} d\mu(s)\Big]^{-1} x(t) \text{ for every } t \geq T_0.$$

This gives

$$x(t) < \Big[\int_{\tau_1}^{\tau} d\mu(s)\Big]^{-1} x(t-\tau) \text{ for all } t \geq T_0,$$

which means that (14) is true if we set

$$T = T_0 \geq 0, \quad M = \Big[\int_{\tau_1}^{\tau} d\mu(s)\Big]^{-1} > 0, \text{ and } \xi = -\tau > 0.$$

It remains to examine the following cases:

$$\begin{cases} n \text{ is even}, \delta = -1, \text{ and } \tau_1 < 0; \\ n \text{ is odd and } \delta \in \{0, +1\}; \\ n \text{ is odd}, \delta = -1, \text{ and } \tau_1 > 0. \end{cases}$$

By the assumptions (i), (ii), (iii)$'$, and (iv)$'$, we can verify that $\sigma_1 < 0$ and $\sigma_1 < \tau_1$. (If $\delta = 0$, then τ_1 can be chosen arbitrarily and so there is no loss of generality to assume that $\sigma_1 < \tau_1$.) Thus, we can choose a $\sigma < 0$ with $\sigma_1 < \sigma < \tau_1$ and $\sigma < \sigma_2$. Set

$$z_2(t) = y^{(n-1)}(t) - \int_{\sigma_1}^{\sigma} \left[\int_{t+s}^{t+\sigma} x(r)\, dr \right] d\eta(s) \text{ for } t \geq 0.$$

Then, by taking into account (15), we obtain for $t \geq 0$

$$\begin{aligned} z_2'(t) &= y^{(n)}(t) - \int_{\sigma_1}^{\sigma} [x(t+\sigma) - x(t+s)]\, d\eta(s) \\ &= -\int_{\sigma_1}^{\sigma_2} x(t+s)\, d\eta(s) - \left[\int_{\sigma_1}^{\sigma} d\eta(s) \right] x(t+\sigma) + \int_{\sigma_1}^{\sigma} x(t+s)\, d\eta(s) \\ &= -\int_{\sigma}^{\sigma_2} x(t+s)\, d\eta(s) - \left[\int_{\sigma_1}^{\sigma} d\eta(s) \right] x(t+\sigma) \\ &< 0 \end{aligned}$$

and hence z_2 is strictly decreasing on $[0, \infty)$. On the other hand, for every $t \geq 0$, we get

$$\int_{\sigma_1}^{\sigma} \left[\int_{t+s}^{t+\sigma} x(r)\, dr \right] d\eta(s) < \left[\int_{\sigma_1}^{\sigma} (\sigma - s)\, d\eta(s) \right] x(t+\sigma_1)$$

and consequently

$$\lim_{t \to \infty} \int_{\sigma_1}^{\sigma} \left[\int_{t+s}^{t+\sigma} x(r)\, dr \right] d\eta(s) = 0. \tag{19}$$

This together with the fact that $\lim_{t \to \infty} y^{(n-1)}(t) = 0$ ensure that $\lim_{t \to \infty} z_2(t) = 0$. Thus, since z_2 is strictly decreasing on $[0, \infty)$, we conclude that z_2 is positive on the interval $[0, \infty)$. So, for every $t \geq 0$, we obtain

$$\begin{aligned} 0 < z_2(t) &= y^{(n-1)}(t) - \int_{\sigma_1}^{\sigma} \left[\int_{t+s}^{t+\sigma} x(r)\, dr \right] d\eta(s) \\ &< y^{(n-1)}(t) - \left[\int_{\sigma_1}^{\sigma} (\sigma - s)\, d\eta(s) \right] x(t+\sigma) \end{aligned}$$

and hence

$$y^{(n-1)}(t) > A_2 x(t+\sigma) \text{ for all } t \geq 0, \tag{20}$$

where $A_2 = \int_{\sigma_1}^{\sigma} (\sigma - s)\, d\eta(s) > 0$. Assume first that $n = 1$. If $\delta \in \{0, -1\}$, then (20) gives

$$x(t) \geq x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) > A_2 x(t+\sigma) \text{ for } t \geq 0$$

and consequently

$$x(t) < \frac{1}{A_2} x(t-\sigma) \text{ for every } t \geq 0,$$

which means that (14) holds with

$$T = 0, \quad M = 1/A_2 > 0, \text{ and } \xi = -\sigma > 0.$$

If $\delta = +1$ and $\tau_1 > 0$, then from (20) we obtain for $t \geq 0$

$$\begin{aligned} A_2 x(t+\sigma) < y(t) &= x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) \\ &< \Big[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t) \end{aligned}$$

and therefore

$$x(t) < \frac{1 + \int_{\tau_1}^{\tau_2} d\mu(s)}{A_2} x(t-\sigma) \text{ for } t \geq 0,$$

i.e. (14) is true with

$$T = 0, \quad M = \frac{1 + \int_{\tau_1}^{\tau_2} d\mu(s)}{A_2} > 0, \text{ and } \xi = -\sigma > 0.$$

If $\delta = +1$ and $\tau_1 < 0$, then from (20) it follows that

$$\begin{aligned} A_2 x(t+\sigma) < y(t) &= x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) \\ &< \Big[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t+\tau_1) \end{aligned}$$

for all $t \geq 0$, which gives

$$x(t) < \frac{1 + \int_{\tau_1}^{\tau_2} d\mu(s)}{A_2} x(t-\sigma+\tau_1), \quad t \geq 0.$$

Thus (14) is also true with

$$T = 0, \quad M = \frac{1 + \int_{\tau_1}^{\tau_2} d\mu(s)}{A_2} > 0, \text{ and } \xi = -\sigma + \tau_1 > 0.$$

Next, let us examine the case where $n > 1$. Consider an arbitrary positive number ϑ. From (18) and (20) it follows that for every $t \geq T_0$

$$\begin{aligned} -y^{(n-2)}(t) > y^{(n-2)}\Big(t + \frac{\vartheta}{n-1}\Big) - y^{(n-2)}(t) &= \int_t^{t+\vartheta/(n-1)} y^{(n-1)}(s)\, ds \\ > A_2 \int_t^{t+\vartheta/(n-1)} x(s+\sigma)\, ds &> A_2 \frac{\vartheta}{n-1} x\Big(t + \frac{\vartheta}{n-1} + \sigma\Big), \end{aligned}$$

i.e.

$$-y^{(n-2)}(t) > A_2 \frac{\vartheta}{n-1} x\Big(t + \frac{\vartheta}{n-1} + \sigma\Big) \text{ for } t \geq T_0.$$

By using the same arguments, we can obtain

(21) $$(-1)^{n+1} y(t) > A_2 \Big(\frac{\vartheta}{n-1}\Big)^{n-1} x(t + \vartheta + \sigma) \text{ for all } t \geq T_0.$$

If n is even, $\delta = -1$, and $\tau_1 < 0$, then we choose the number ϑ so that $0 < \vartheta < \tau_1 - \sigma$ and from (21) we obtain for $t \geq T_0$

$$A_2\Big(\frac{\vartheta}{n-1}\Big)^{n-1} x(t+\vartheta+\sigma) < -y(t) = -x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\,d\mu(s) < \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t+\tau_1)$$

and consequently

$$x(t) < \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big]\Big[A_2\Big(\frac{\vartheta}{n-1}\Big)^{n-1}\Big]^{-1} x(t-\vartheta-\sigma+\tau_1) \text{ for } t \geq T_0.$$

This means that (14) is true with

$$T = T_0 \geq 0, \quad M = \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big]\Big[A_2\Big(\frac{\vartheta}{n-1}\Big)^{n-1}\Big]^{-1} > 0, \text{ and } \xi = -\vartheta - \sigma + \tau_1 > 0.$$

In the case where n is odd and $\delta \in \{0, -1\}$ we take $\vartheta = -\frac{\sigma}{2} > 0$ and from (21) we obtain for $t \geq T_0$

$$x(t) \geq x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\,d\mu(s) = y(t) > A_2\Big(\frac{-\sigma/2}{n-1}\Big)^{n-1} x\Big(t+\frac{\sigma}{2}\Big),$$

which gives

$$x(t) < \Big[A_2\Big(\frac{-\sigma/2}{n-1}\Big)^{n-1}\Big]^{-1} x\Big(t-\frac{\sigma}{2}\Big) \text{ for } t \geq T_0.$$

Hence, (14) holds with

$$T = T_0 \geq 0, \quad M = \Big[A_2\Big(\frac{-\sigma/2}{n-1}\Big)^{n-1}\Big]^{-1} > 0, \text{ and } \xi = -\frac{\sigma}{2} > 0.$$

For n odd, $\delta = +1$, and $\tau_1 > 0$, we set $\vartheta = -\frac{\sigma}{2} > 0$ and from (21) we get for every $t \geq T_0$

$$A_2\Big(\frac{-\sigma/2}{n-1}\Big)^{n-1} x\Big(t+\frac{\sigma}{2}\Big) < y(t) = x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\,d\mu(s) < \Big[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t)$$

and hence

$$x(t) < \Big[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\Big]\Big[A_2\Big(\frac{-\sigma/2}{n-1}\Big)^{n-1}\Big]^{-1} x\Big(t-\frac{\sigma}{2}\Big), \quad t \geq T_0.$$

So, (14) is fulfilled with

$$T = T_0 \geq 0, \quad M = \Big[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\Big]\Big[A_2\Big(\frac{-\sigma/2}{n-1}\Big)^{n-1}\Big]^{-1} > 0, \text{ and } \xi = -\frac{\sigma}{2} > 0.$$

Finally, let n be odd, $\delta = +1$, and $\tau_1 < 0$. Suppose that the number is ϑ is chosen so that $0 < \vartheta < \tau_1 - \sigma$. From (21) we obtain for every $t \geq T_0$

$$A_2\Big(\frac{\vartheta}{n-1}\Big)^{n-1} x(t+\vartheta+\sigma) < y(t) = x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\,d\mu(s) < \Big[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t+\tau_1)$$

and therefore

$$x(t) < \left[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\right]\left[A_2\left(\frac{\vartheta}{n-1}\right)^{n-1}\right]^{1} x(t - \vartheta - \sigma + \tau_1) \text{ for } t \geq T_0$$

That is, (14) holds if we take

$$T = T_0 \geq 0, \quad M = \left[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\right]\left[A_2\left(\frac{\vartheta}{n-1}\right)^{n-1}\right]^{1} > 0, \text{ and } \xi = -\vartheta - \sigma + \tau_1 > 0$$

CASE III $\zeta = -1$ and $L = \infty$ From (15) we obtain for $t \geq 0$

$$y^{(n)}(t) = \int_{\sigma_1}^{\sigma_2} x(t+s)\, d\eta(s) > \left[\int_{\sigma_1}^{\sigma_2} d\eta(s)\right] x(t + \sigma_1),$$

which ensures that $\lim_{t\to\infty} y^{(n)}(t) = \infty$ Thus, $y^{(n)}$ is positive on the interval $[0, \infty)$ It follows easily that $\lim_{t\to\infty} y^{(i)}(t) = \infty$ ($i = 0, 1, \quad , n-1$) and so

(22) $$y^{(i)}(t) > 0 \text{ for every } t \geq T_0 \quad (i = 0, 1, \quad , n-1)$$

If $\delta = -1$ and $\tau_2 > 0$, then we can choose a positive number τ such that $\tau_1 < \tau < \tau_2$ So, from (22) we obtain for $t \geq T_0$

$$0 < y(t) = x(t) - \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) < x(t) - \left[\int_{\tau}^{\tau_2} d\mu(s)\right] x(t + \tau)$$

Therefore

$$x(t) > \left[\int_{\tau}^{\tau_2} d\mu(s)\right] x(t + \tau) \text{ for all } t \geq T_0,$$

which shows that (13) is true for

$$T = T_0 \geq 0, \quad M = \int_{\tau}^{\tau_2} d\mu(s) > 0, \text{ and } \xi = \tau > 0$$

In what follows, we will assume that $\tau_2 < 0$ when $\delta = -1$ By the assumptions (x), (xi), and (xii)$'$, we conclude that $\sigma_2 > 0$ and $\tau_2 < \sigma_2$ (If $\delta = 0$, then τ_2 can be chosen arbitrarily and so we can assume that $\tau_2 < \sigma_2$) Thus, we can choose a positive number σ with $\sigma_1 < \sigma < \sigma_2$ and $\tau_2 < \sigma < \sigma_2$ We put

$$z_3(t) = y^{(n-1)}(t) - \int_{\sigma}^{\sigma_2} \left[\int_{t+\sigma}^{t+s} x(r)\, dr\right] d\eta(s) \text{ for } t \geq 0$$

Then, by taking into account (15), we have for every $t \geq 0$

$$\begin{aligned} z_3'(t) &= y^{(n)}(t) - \int_{\sigma}^{\sigma_2} [x(t+s) - x(t+\sigma)]\, d\eta(s) \\ &= \int_{\sigma_1}^{\sigma_2} x(t+s)\, d\eta(s) - \int_{\sigma}^{\sigma_2} x(t+s)\, d\eta(s) + \left[\int_{\sigma}^{\sigma_2} d\eta(s)\right] x(t+\sigma) \\ &= \int_{\sigma_1}^{\sigma} x(t+s)\, d\eta(s) + \left[\int_{\sigma}^{\sigma_2} d\eta(s)\right] x(t+\sigma) \\ &> \left[\int_{\sigma}^{\sigma_2} d\eta(s)\right] x(t+\sigma) \end{aligned}$$

It is now clear that $\lim_{t\to\infty} z_3'(t) = \infty$ and so we can find a $t_2 \geq T_0$ such that z_3 is positive on the interval $[t_2, \infty)$. Hence, for every $t \geq t_2$

$$0 < z_3(t) = y^{(n-1)}(t) - \int_\sigma^{\sigma_2}\left[\int_{t+\sigma}^{t+s} x(r)\,dr\right] d\eta(s)$$
$$< y^{(n-1)}(t) - \left[\int_\sigma^{\sigma_2}(s-\sigma)\,d\eta(s)\right]x(t+\sigma)$$

or

$$y^{(n-1)}(t) > A_1 x(t+\sigma) \text{ for all } t \geq t_2, \tag{23}$$

where $A_1 = \int_\sigma^{\sigma_2}(s-\sigma)\,d\eta(s) > 0$. Assume first that $n = 1$. If $\delta \in \{0, -1\}$, then (23) gives for $t \geq t_2$

$$x(t) \geq x(t) + \delta\int_{\tau_1}^{\tau_2} x(t+s)\,d\mu(s) = y(t) > A_1 x(t+\sigma)$$

and so (13) holds with

$$T = t_2 \geq 0, \quad M = A_1 > 0, \text{ and } \xi = \sigma > 0.$$

If $\delta = +1$, then we choose a positive number τ with $\tau_2 < \tau < \sigma$ and from (23) we obtain for every $t \geq t_2$

$$A_1 x(t+\sigma) < y(t) = x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\,d\mu(s) < \left[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\right]x(t+\tau).$$

Hence

$$x(t) > \frac{A_1}{1 + \int_{\tau_1}^{\tau_2} d\mu(s)}x(t-\tau+\sigma), \quad t \geq t_2 + \tau,$$

i.e. (13) holds again for

$$T = t_2 + \tau \geq 0, \quad M = \frac{A_1}{1 + \int_{\tau_1}^{\tau_2} d\mu(s)} > 0, \text{ and } \xi = -\tau + \sigma > 0.$$

Next, we will examine the case where $n > 1$. Consider an arbitrary positive number ϑ. From (22) and (23) we obtain for all $t \geq t_2$

$$y^{(n-2)}\left(t + \frac{\vartheta}{n-1}\right) > y^{(n-2)}\left(t + \frac{\vartheta}{n-1}\right) - y^{(n-2)}(t) = \int_t^{t+\vartheta/(n-1)} y^{(n-1)}(s)\,ds$$
$$> A_1\int_t^{t+\vartheta/(n-1)} x(s+\sigma)\,ds > A_1\frac{\vartheta}{n-1}x(t+\sigma),$$

i.e.

$$y^{(n-2)}(t) > A_1\frac{\vartheta}{n-1}x\left(t - \frac{\vartheta}{n-1} + \sigma\right) \text{ for } t \geq t_2 + \frac{\vartheta}{n-1}.$$

Following the same procedure, after $n-1$ steps we derive

$$y(t) > A_1\left(\frac{\vartheta}{n-1}\right)^{n-1} x(t-\vartheta+\sigma) \text{ for every } t \geq t_2 + \vartheta. \tag{24}$$

If $\delta \in \{0, -1\}$, then, for $\vartheta = \frac{\sigma}{2}$, (24) gives

$$x(t) \geq x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) = y(t) > A_1 \Big(\frac{\sigma/2}{n-1}\Big)^{n-1} x\Big(t + \frac{\sigma}{2}\Big)$$

for every $t \geq t_2 + \sigma/2$, and so (13) is true with

$$T = t_2 + \frac{\sigma}{2} \geq 0, \quad M = A_1 \Big(\frac{\sigma/2}{n-1}\Big)^{n-1} > 0, \text{ and } \xi = \frac{\sigma}{2} > 0$$

If $\delta = +1$, then we can choose a positive number τ with $\tau_2 < \tau < \sigma_2$ and next we can consider that ϑ satisfies $0 < \vartheta < -\tau + \sigma$ In this case from (24) it follows that for $t \geq t_2 + \vartheta$

$$A_1 \Big(\frac{\vartheta}{n-1}\Big)^{n-1} x(t - \vartheta + \sigma) < y(t) = x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) < \Big[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t+\tau),$$

which gives

$$x(t) > A_1 \Big(\frac{\vartheta}{n-1}\Big)^{n-1} \Big[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\Big]^{-1} x(t - \tau - \vartheta + \sigma) \text{ for } t \geq t_2 + \vartheta + \tau$$

This implies that (13) is true if we take

$$T = t_2 + \vartheta + \tau \geq 0, \quad M = A_1 \Big(\frac{\vartheta}{n-1}\Big)^{n-1} \Big[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\Big]^{1}, \text{ and } \xi = -\tau - \vartheta + \sigma > 0$$

CASE IV $\zeta = -1$ and $L = 0$ From (15) it follows that $y^{(n)}$ is positive on $[0, \infty)$ We also see that $\lim_{t\to\infty} y^{(i)}(t) = 0$ ($i = 0, 1, \quad , n-1$) and so we have

(25) $$(-1)^{n-i} y^{(i)}(t) > 0 \text{ for all } t \geq T_0 \quad (i = 0, 1, \quad , n-1)$$

Particularly, y is positive on $[T_0, \infty)$ if n is even, and y is negative on $[T_0, \infty)$ if n is odd

If $\delta \in \{0, +1\}$, then the definition of y guarantees that $y(t) > 0$ for $t \geq 0$, which contradicts (25) when n is odd Now let n be even, $\delta = -1$, and $\tau_1 < 0$ Then we can choose a $\tau < 0$ with $\tau_1 < \tau < \tau_2$ and next from (25) we get for $t \geq T_0$

$$0 < y(t) = x(t) - \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) < x(t) - \Big[\int_{\tau_1}^{\tau} d\mu(s)\Big] x(t+\tau)$$

Consequently

$$x(t) < \Big[\int_{\tau_1}^{\tau} d\mu(s)\Big]^{1} x(t - \tau) \text{ for all } t \geq T_0$$

Thus, (14) holds with

$$T = T_0 \geq 0, \quad M = \Big[\int_{\tau_1}^{\tau} d\mu(s)\Big]^{-1} > 0, \text{ and } \xi = -\tau > 0$$

Moreover, if n is odd, $\delta = -1$, and $\tau_1 > 0$, then (25) gives

$$0 > y(t) = x(t) - \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) > x(t) - \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t + \tau_1)$$

for every $t \geq T_0$. That is,

$$x(t) < \left[\int_{\tau_1}^{\tau_2} d\mu(s)\right] x(t+\tau_1) \text{ for } t \geq T_0,$$

which means that (14) is true for

$$T = T_0 \geq 0, \quad M = \int_{\tau_1}^{\tau_2} d\mu(s) > 0, \text{ and } \xi = \tau_1 > 0.$$

Now, it remains to concentrate our interest to the next cases:

$$\begin{cases} n \text{ is even and } \delta \in \{0, +1\}; \\ n \text{ is even, } \delta = -1, \text{ and } \tau_1 > 0; \\ n \text{ is odd, } \delta = -1, \text{ and } \tau_1 < 0; \end{cases}$$

By the assumptions (vi), (vii), (viii)$'$, and (ix)$'$, we can see that $\sigma_1 < 0$ and $\sigma_1 < \tau_1$. (If $\delta = 0$, then there is no loss of generality in assuming that $\sigma_1 < \tau_1$.) Thus, we can choose a $\sigma < 0$ with $\sigma_1 < \sigma < \tau_1$ and $\sigma < \sigma_2$. Define

$$z_4(t) = -y^{(n-1)}(t) - \int_{\sigma_1}^{\sigma}\left[\int_{t+s}^{t+\sigma} x(r)\,dr\right] d\eta(s) \text{ for } t \geq 0.$$

Taking into account (15), we obtain for $t \geq 0$

$$\begin{aligned} z_4'(t) &= -y^{(n)}(t) - \int_{\sigma_1}^{\sigma}[x(t+\sigma) - x(t+s)]\,d\eta(s) \\ &= -\int_{\sigma_1}^{\sigma_2} x(t+s)\,d\eta(s) - \left[\int_{\sigma_1}^{\sigma} d\eta(s)\right] x(t+\sigma) + \int_{\sigma_1}^{\sigma} x(t+s)\,d\eta(s) \\ &= -\int_{\sigma}^{\sigma_2} x(t+s)\,d\eta(s) - \left[\int_{\sigma_1}^{\sigma} d\eta(s)\right] x(t+\sigma) \\ &< 0. \end{aligned}$$

Hence, z_4 is strictly decreasing on $[0, \infty)$. But (19) and $\lim_{t\to\infty} y^{(n-1)}(t) = 0$ imply that $\lim_{t\to\infty} z_4(t) = 0$. So, z_4 is positive on the interval $[0, \infty)$. Thus, we have for $t \geq 0$

$$\begin{aligned} 0 < z_4(t) &= -y^{(n-1)}(t) - \int_{\sigma_1}^{\sigma}\left[\int_{t+s}^{t+\sigma} x(r)\,dr\right] d\eta(s) \\ &< -y^{(n-1)}(t) - \left[\int_{\sigma_1}^{\sigma}(\sigma - s)\,d\eta(s)\right] x(t+\sigma) \end{aligned}$$

and consequently

$$-y^{(n-1)}(t) > A_2 x(t+\sigma) \text{ for all } t \geq 0, \tag{26}$$

where $A_2 = \int_{\sigma_1}^{\sigma}(\sigma - s)\,d\eta(s) > 0$. Assume first that $n = 1$. Then we always have $\delta = -1$ and $\tau_1 < 0$ and so (26) gives for $t \geq 0$

$$A_2 x(t+\sigma) < -y(t) = -x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\,d\mu(s) < \left[\int_{\tau_1}^{\tau_2} d\mu(s)\right] x(t+\tau_1).$$

Consequently

$$x(t) < \frac{\int_{\tau_1}^{\tau_2} d\mu(s)}{A_2} x(t - \sigma + \tau_1) \text{ for all } t \geq 0$$

and (14) holds for

$$T = 0, \quad M = \frac{\int_{\tau_1}^{\tau_2} d\mu(s)}{A_2} > 0, \text{ and } \xi = -\sigma + \tau_1 > 0$$

Now, let $n > 1$ We consider an arbitrary positive number ϑ Using (25) and (26), we find for $t \geq 0$

$$\begin{aligned} y^{(n-2)}(t) &> -y^{(n-2)}\Big(t + \frac{\vartheta}{n-1}\Big) + y^{(n-2)}(t) = -\int_{T}^{t+\vartheta/(n-1)} y^{(n-1)}(s)\, ds \\ &> A_2 \int_t^{t+\vartheta/(n-1)} x(s+\sigma)\, ds > A_2 \frac{\vartheta}{n-1} x\Big(t + \frac{\vartheta}{n-1} + \sigma\Big) \end{aligned}$$

Following this procedure, after $n-1$ steps we derive

$$(-1)^n y(t) > A_2 \Big(\frac{\vartheta}{n-1}\Big)^{n-1} x(t + \vartheta + \sigma) \text{ for all } t \geq 0 \tag{27}$$

If n is odd, $\delta = -1$, and $\tau_1 < 0$, then (27) gives for $t \geq 0$

$$A_2 \Big(\frac{\vartheta}{n-1}\Big)^{n-1} x(t+\vartheta+\sigma) < -y(t) = -x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) < \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t+\tau_1)$$

Taking $\vartheta \in (0, -\sigma + \tau_1)$ we find that

$$x(t) < \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big]\Big[A_2 \Big(\frac{\vartheta}{n-1}\Big)^{n-1}\Big]^{-1} x(t - \vartheta - \sigma + \tau_1) \text{ for } t \geq 0$$

Thus, (14) is true for

$$T = 0, \quad M = \Big[\int_{\tau_1}^{\tau_2} d\mu(s)\Big]\Big[A_2 \Big(\frac{\vartheta}{n-1}\Big)^{n-1}\Big]^{-1} > 0, \text{ and } \xi = -\vartheta - \sigma + \tau_1 > 0$$

If n is even and $\delta \in \{0, -1\}$, then we take $\vartheta = -\frac{\sigma}{2} > 0$ and from (27) we get for $t \geq 0$

$$A_2 \Big(\frac{-\sigma/2}{n-1}\Big)^{n-1} x\Big(t + \frac{\sigma}{2}\Big) < y(t) = x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) \leq x(t)$$

This gives

$$x(t) < \Big[A_2 \Big(\frac{-\sigma/2}{n-1}\Big)^{n-1}\Big]^{-1} x\Big(t - \frac{\sigma}{2}\Big) \text{ for all } t \geq 0,$$

which shows that (14) holds with

$$T = 0, \quad M = \Big[A_2 \Big(\frac{-\sigma/2}{n-1}\Big)^{n-1}\Big]^{-1} > 0, \text{ and } \xi = -\frac{\sigma}{2} > 0$$

If n is even, $\delta = +1$, and $\tau_1 < 0$, then we can assume that $0 < \vartheta < -\sigma + \tau_1$ and from (27) we obtain

$$A_2 \Big(\frac{\vartheta}{n-1}\Big)^{n-1} x(t+\vartheta+\sigma) < y(t) = x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) < \Big[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\Big] x(t+\tau_1)$$

for all $t \geq 0$. This implies that

$$x(t) < \left[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\right]\left[A_2\left(\frac{\vartheta}{n-1}\right)^{n-1}\right]^{-1} x(t - \vartheta - \sigma + \tau_1) \text{ for } t \geq 0.$$

Clearly, (14) is true with

$$T = 0, \quad M = \left[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\right]\left[A_2\left(\frac{\vartheta}{n-1}\right)^{n-1}\right]^{-1} > 0, \text{ and } \xi = -\vartheta - \sigma + \tau_1 > 0.$$

Finally, assume that n is even, $\delta = +1$ and $\tau_1 > 0$ and set $\vartheta = -\frac{\sigma}{2} > 0$. Then from (27) it follows that

$$A_2\left(\frac{-\sigma/2}{n-1}\right)^{n-1} x\left(t + \frac{\sigma}{2}\right) < y(t) = x(t) + \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s) < \left[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\right] x(t)$$

for every $t \geq 0$. Consequently

$$x(t) < \left[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\right]\left[A_2\left(\frac{-\sigma/2}{n-1}\right)^{n-1}\right]^{-1} x\left(t - \frac{\sigma}{2}\right) \text{ for } t \geq 0,$$

which shows that (14) holds with

$$T = 0, \quad M = \left[1 + \int_{\tau_1}^{\tau_2} d\mu(s)\right]\left[A_2\left(\frac{-\sigma/2}{n-1}\right)^{n-1}\right]^{-1} > 0, \text{ and } \xi = -\frac{\sigma}{2} > 0.$$

The proof of the lemma is now complete.

3. **Proof of the main result.** We are now ready to give the proof of our theorem. Equivalently, we will show that there is a nonoscillatory solution of the differential equation (E) if and only if the characteristic equation $(*)$ has a real root.

Assume first that $(*)$ has a real root λ. Then (E) has the nonoscillatory solution $x(t) = e^{\lambda t}$, $t \geq \gamma$.

Assume, conversely, that there is a nonoscillatory solution x of the differential equation (E). By Lemma 5, the solution x can be supposed to be n-times continuously differentiable, positive and strictly monotone on the interval $[\gamma, \infty)$. Assume also, for the sake of contradiction, that the characteristic equation $(*)$ has no real roots. Lemma 3 ensures that for $\zeta = +1$ the implications (i)–(v) are true and for $\zeta = -1$ the implications (vi)–(xii) are also true. Furthermore, for $\zeta = +1$ the hypothesis (H_+) means that (iii)$'$–(v)$'$ are valid and for $\zeta = -1$ the assumption (H_-) ensures the validity of (viii)$'$, (ix)$'$, and (xii)$'$. Hence, we can apply Lemma 6 to conclude that the solution x is of exponential order c for some $c \in (-\infty, \infty)$ and that the Laplace transform X of x has abscissa of convergence $\alpha > -\infty$.

We now introduce the function u defined by

$$u(t) = x(t) + \delta \int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s), \quad t \geq 0,$$

which is n-times continuously differentiable on the interval $[0, \infty)$. We also consider the continuous function v which is defined on the interval $[0, \infty)$ as follows

$$v(t) = \int_{\sigma_1}^{\sigma_2} x(t+s)\, d\eta(s), \quad t \geq 0.$$

These functions satisfy $u^{(n)} = -\zeta v$. As the function x is of exponential order c, we can consider a positive constant C such that $x(t) \leq Ce^{ct}$ for all $t \geq \gamma$. Thus, for each $t \geq 0$, we have

$$|u(t)| \leq C\Big[1 + \int_{\tau_1}^{\tau_2} e^{cs}\, d\mu(s)\Big]e^{ct} \text{ and } v(t) \leq C\Big[\int_{\sigma_1}^{\sigma_2} e^{cs}\, d\eta(s)\Big]e^{ct},$$

which means that the functions u and v are also of exponential order c. Moreover, the function $u^{(n)}$ is of exponential order c. Without loss of generality, we can assume that $c \geq 1$. It is easy to see that, if φ is a continuously differentiable function on the interval $[0, \infty)$ such that φ' is of exponential order c (with $c \geq 1$), then φ is also of exponential order c. So, we can conclude that the functions $u^{(i)}$ $(i = 0, 1, \ldots, n)$ are of exponential order c. Let U, V and Z be the Laplace transforms of the functions u, v and $u^{(n)}$ respectively. Obviously, we have $Z = -\zeta V$. Moreover, one has

$$Z(\lambda) = \lambda^n U(\lambda) - \sum_{i=0}^{n-1} \lambda^i u^{(n-1-i)}(0) \text{ for } \operatorname{Re}\lambda > c$$

and consequently

$$-\zeta V(\lambda) = \lambda^n U(\lambda) - \sum_{i=0}^{n-1} \lambda^i u^{(n-1-i)}(0) \text{ for } \operatorname{Re}\lambda > c. \tag{28}$$

Now, for any λ with $\operatorname{Re}\lambda > c$, we obtain

$$\begin{aligned} U(\lambda) &= X(\lambda) + \delta \int_0^{\infty} e^{-\lambda t}\Big[\int_{\tau_1}^{\tau_2} x(t+s)\, d\mu(s)\Big]\, dt \\ &= X(\lambda) + \delta \int_{\tau_1}^{\tau_2}\Big[\int_0^{\infty} e^{-\lambda t} x(t+s)\, dt\Big]\, d\mu(s) \\ &= X(\lambda) + \delta \int_{\tau_1}^{\tau_2} e^{\lambda s}\Big[\int_s^{\infty} e^{-\lambda t} x(t)\, dt\Big]\, d\mu(s) \\ &= X(\lambda) + \delta \int_{\tau_1}^{\tau_2} e^{\lambda s}\Big[X(\lambda) - \int_0^{s} e^{-\lambda t} x(t)\, dt\Big]\, d\mu(s). \end{aligned}$$

So, we have for $\operatorname{Re}\lambda > c$

$$U(\lambda) = \Big[1 + \delta \int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s)\Big]X(\lambda) - \delta \int_{\tau_1}^{\tau_2} e^{\lambda s}\Big[\int_0^{s} e^{-\lambda t} x(t)\, dt\Big]\, d\mu(s).$$

In a similar way, we find for $\operatorname{Re}\lambda > c$

$$V(\lambda) = \Big[\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s)\Big]X(\lambda) - \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\Big[\int_0^{s} e^{-\lambda t} x(t)\, dt\Big]\, d\eta(s).$$

Thus, (28) gives

$$\begin{aligned} F(\lambda)X(\lambda) = \sum_{i=0}^{n-1} \lambda^i u^{(n-1-i)}(0) &+ \delta\lambda^n \int_{\tau_1}^{\tau_2} e^{\lambda s}\Big[\int_0^{s} e^{-\lambda t} x(t)\, dt\Big]\, d\mu(s) \\ &+ \zeta \int_{\sigma_1}^{\sigma_2} e^{\lambda s}\Big[\int_0^{s} e^{-\lambda t} x(t)\, dt\Big]\, d\eta(s) \equiv G(\lambda) \end{aligned} \tag{29}$$

for all λ with $\operatorname{Re}\lambda > c$. The functions $F \cdot X$ and G are obviously analytic in the half-plane $\operatorname{Re}\lambda > \alpha$. So, we can extend (29) to hold for every λ with $\operatorname{Re}\lambda > \alpha$. Finally, Lemma 1 guarantees that X has a nonremovable singularity at the real point $\lambda = \alpha$. This is a contradiction, since the equation $F(\lambda) = 0$ has no real roots.

The proof of the theorem is complete.

4. **Some applications: discussion.** Introduce the assumption

$$\text{(H)} \qquad \begin{cases} (\mathrm{H}_+) \text{ holds if } \zeta = +1 \\ (\mathrm{H}_-) \text{ holds if } \zeta = -1. \end{cases}$$

In our theorem, the hypothesis (H) has been used only in the proof of the fact that, if the characteristic equation $(*)$ has no real roots, then all solutions of the differential equation (E) are oscillatory. There are many interesting cases in which the restriction (H) imposed in the theorem can be removed, because of the fact that in these cases (H) is a consequence of the assumption that the characteristic equation has no real roots. Such a case is that of neutral difference-differential equations. The same is also true in the case where μ and η have continuous and positive derivatives on the intervals $[\tau_1, \tau_2]$ and $[\sigma_1, \sigma_2]$ respectively. These two special cases will be examined below.

Consider the neutral difference-differential equation

$$(\mathrm{E}_0) \qquad \frac{d^n}{dt^n}\Big[x(t) + \delta \sum_{i=1}^{m} p_i x(t+s_i)\Big] + \zeta \sum_{j=1}^{\ell} q_j x(t+r_j) = 0,$$

where $\delta \in \{0, +1, -1\}$, $\zeta \in \{+1, -1\}$, $m > 1$, $\ell > 1$, p_i $(i = 1, \ldots, m)$ and q_j $(j = 1, \ldots, \ell)$ are positive numbers, s_i $(i = 1, \ldots, m)$ are real numbers such that $s_1 < s_2 < \cdots < s_m$ and $s_1 s_m \neq 0$, and r_j $(j = 1, \ldots, \ell)$ are real numbers with $r_1 < r_2 < \cdots < r_\ell$. This equation is a special case of the differential equation (E). The characteristic equation of (E_0) is

$$(*)_0 \qquad F_0(\lambda) \equiv \lambda^n \Big(1 + \delta \sum_{i=1}^{m} p_i e^{\lambda s_i}\Big) + \zeta \sum_{j=1}^{\ell} q_j e^{\lambda r_j} = 0.$$

We will show that in the special case of the differential equation (E_0) the condition (H) follows from the assumption that the characteristic equation $(*)_0$ has no real roots. To this end, let us assume that $(*)_0$ has no real roots. Since $\zeta F_0(0) = \sum_{j=1}^{\ell} q_j > 0$, we must have

$$(30) \qquad \zeta F_0(\lambda) > 0 \text{ for all real numbers } \lambda.$$

Assume first that $\zeta = +1$. If n is odd, $\delta = +1$, $s_1 < 0$, and $s_1 = r_1$ (here we have $\tau_1 = s_1$ and $\sigma_1 = r_1$), then we can see that $F_0(-\infty) = -\infty$, which contradicts (30). If n is even, $\delta = -1$, $s_1 < 0$, and $s_1 = r_1$, then we have $F_0(-\infty) = -\infty$, a contradiction. In the case where $\delta = -1$, $s_m > 0$, and $s_m = r_\ell$ (we have here $\tau_2 = s_m$ and $\sigma_2 = r_\ell$) we can obtain $F_0(\infty) = -\infty$, which contradicts (30). Next, let us suppose that $\zeta = -1$. If n is odd, $\delta = -1$, $s_1 < 0$, and $s_1 = r_1$, then it follows that $F_0(-\infty) = \infty$, which contradicts (30). The same contradiction can be obtained when n is even, $\delta = +1$, $s_1 < 0$, and $s_1 = r_1$. Finally, if $\delta = +1$, $s_m > 0$, and $s_m = r_\ell$, then we get $F_0(\infty) = \infty$, which contradicts (30). Thus, our assertion has been proved. Now, from our theorem we obtain the following result (*cf.* [7]):

A necessary and sufficient condition for the oscillation of all solutions of (E_0) is that its characteristic equation $(*)_0$ has no real roots.

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Consider next the case where the functions μ and η have continuous and positive derivatives on the intervals $[\tau_1, \tau_2]$ and $[\sigma_1, \sigma_2]$ respectively Our purpose is to establish that in this case the assumption that the characteristic equation $(*)$ has no real roots implies again that condition (H) is fulfilled There are two positive constants α and β so that

$$\alpha \leq \mu'(s) \leq \beta \text{ for } s \in [\tau_1, \tau_2] \tag{31}$$

and

$$\alpha \leq \eta'(s) \leq \beta \text{ for } s \in [\sigma_1, \sigma_2] \tag{32}$$

Furthermore, for each $\lambda \neq 0$, we can apply the mean value theorem to obtain

$$\int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s) = \int_{\tau_1}^{\tau_2} e^{\lambda s} \mu'(s)\, ds = \mu'(s^*) \int_{\tau_1}^{\tau_2} e^{\lambda s}\, ds$$

and consequently

$$\int_{\tau_1}^{\tau_2} e^{\lambda s}\, d\mu(s) = \mu'(s^*) \frac{e^{\lambda\tau_2} - e^{\lambda\tau_1}}{\lambda}, \tag{33}$$

where the point $s^* \in [\tau_1, \tau_2]$ depends on λ In a similar way, for any $\lambda \neq 0$, we can find a point $\hat{s} \in [\sigma_1, \sigma_2]$ (which depends on λ) such that

$$\int_{\sigma_1}^{\sigma_2} e^{\lambda s}\, d\eta(s) = \eta'(\hat{s}) \frac{e^{\lambda\sigma_2} - e^{\lambda\sigma_1}}{\lambda} \tag{34}$$

Combining (31), (32), (33) and (34), we can conclude that there exist positive constants A_i and B_i $(i = 1, 2)$ such that

$$F(\lambda) \geq \lambda^n \left(1 + \delta A_1 \frac{e^{\lambda\tau_2} - e^{\lambda\tau_1}}{\lambda}\right) + \zeta A_2 \frac{e^{\lambda\sigma_2} - e^{\lambda\sigma_1}}{\lambda} \text{ for } \lambda \neq 0 \tag{35}$$

and

$$F(\lambda) \leq \lambda^n \left(1 + \delta B_1 \frac{e^{\lambda\tau_2} - e^{\lambda\tau_1}}{\lambda}\right) + \zeta B_2 \frac{e^{\lambda\sigma_2} - e^{\lambda\sigma_1}}{\lambda} \text{ for } \lambda \neq 0 \tag{36}$$

Now, let us assume that the characteristic equation $(*)$ has no real roots As $\zeta F(0) = \int_{\sigma_1}^{\sigma_2} d\eta(s) > 0$, we must have

$$\zeta F(\lambda) > 0 \text{ for all } \lambda \in (-\infty, \infty) \tag{37}$$

Consider first the case where $\zeta = +1$ If n is odd, $\delta = +1$, and $\tau_1 = \sigma_1 < 0$ or if n is even, $\delta = -1$, and $\tau_1 = \sigma_1 < 0$, then from (36) it follows that $F(-\infty) = -\infty$, which contradicts (37) Also, if $\delta = -1$ and $\tau_2 = \sigma_2 > 0$, then (36) gives $F(\infty) = -\infty$, which also contradicts (37) Next, we suppose that $\zeta = -1$ In the cases where n is odd, $\delta = -1$, and $\tau_1 = \sigma_1 < 0$ or n is even, $\delta = +1$, and $\tau_1 = \sigma_1 < 0$, we can obtain from (35) that $F(-\infty) = \infty$, which contradicts (37) Finally, for $\delta = +1$ and $\tau_2 = \sigma_2 > 0$, from (35) it follows that $F(\infty) = \infty$, a contradiction So, condition (H) is satisfied

There are cases of neutral differential equations of the form (E) for which the characteristic equation has no real roots while the assumption (H) fails Such a case is that of the neutral equation (*cf* [24])

$$\frac{d}{dt}\left[x(t) + \int_{\sigma}^{-e} x(t+s)\, ds\right] + 2x(t - \sigma) = 0 \quad (\sigma > e)$$

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