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Optimal Hardy-weights for the (p, A)-Laplacian with a potential term

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We construct new optimal L^p Hardy-type inequalities for elliptic Schrödinger-type operators with a potential term.

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1. Introduction

For any $\xi \in \mathbb{R}^n$ and a positive definite matrix $A \in \mathbb{R}^{n\times n}$, let $|\xi|_A := \sqrt{\langle A\xi, \xi\rangle}$, where $\langle\cdot, \cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$. Consider a second order half-linear operator of the form

$$Q_{p,A,V}(u) := -\mathrm{div}\left(|\nabla u|_A^{p-2}A\nabla u\right) + V|u|^{p-2}u$$

defined in a domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, and assume that the equation $Q_{p,A,V}(u) = 0$ admits a positive solution in Ω. We are interested to find an optimal weight function $W \gneqq 0$ (see definition 2.29) such that the equation $Q_{p,A,V-W}(u) = 0$ admits a positive solution in Ω. Equivalently [19, theorem 4.3], we are interested to find an optimal weight function $W \gneqq 0$ such that the following Hardy-type inequality is satisfied:

$$\int_\Omega (|\nabla\phi|_A^p + V|\phi|^p)\,\mathrm{d}x \geqslant \int_\Omega W|\phi|^p\,\mathrm{d}x \quad \forall\phi \in C_0^\infty(\Omega). \tag{1.1}$$

In some definite sense, an optimal weight $W \gneqq 0$ is 'as large as possible' nonnegative function such that (1.1) is satisfied for all nonnegative $\phi \in C_0^\infty(\Omega)$.

The search for Hardy-type inequalities with optimal weight function W was originally proposed by Agmon, who raised this problem in connection with his theory of exponential decay of Schrödinger eigenfunctions [1, p. 6]. In the past four decades, the problem of improving Hardy-type inequalities has engaged many authors. In particular, Hardy-type inequalities were established for a vast class of operators (e.g., elliptic operators, Schrödinger operators on graphs, fractional differential

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equations) with different types of boundary conditions, see [**2**–**4**, **6**, **8**–**10**, **14**, **22**]. In [**9**], Devyver and Pinchover studied the problem of *optimal* weights for the operator $Q_{p,A,V}$. However, they managed to find optimal weights only in the case where A is the identity matrix and $V=0$. They proved (under certain assumptions) that the p-Laplace operator, $-\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, admits an optimal Hardy-weight. More specifically, it is proved that if $1<p\leqslant n$, then $W=(\frac{p-1}{p})^p\left|\frac{\nabla G}{G}\right|^p$ an optimal Hardy-weight, where G is the associated positive minimal Green function with singularity at 0. For $p>n$, several cases should be considered, depending on the behaviour of a positive p-harmonic function with singularity at 0.

In the present paper we make a nontrivial progress towards the study of (1.1) in the case where A is not necessarily the identity matrix, and V is a slowly growing potential function. Our main result reads as follows.

THEOREM 1.1. *Let $\Omega\subset\mathbb{R}^n$, $n\geqslant 2$, be a domain and $x_0\in\Omega$. Let $Q_{p,A,V}$ be a subcritical operator in Ω satisfying assumptions 2.8 in Ω. Suppose that $Q_{p,A,V}$ admits a (nonnegative) Green potential, $G_\varphi(x)$, in Ω (see definition 2.22) satisfying*

$$\lim_{x\to\overline{\infty}} G_\varphi(x)=0;\quad \int_\Omega VG_\varphi(x)^{p-1}\,\mathrm{d}x<0;\ \int_\Omega |V|G_\varphi(x)^{p-1}\,\mathrm{d}x<\infty,$$

where $\overline{\infty}$ denotes the ideal point in the one-point compactification of Ω. Then the operator $Q_{p,A,V/c_p}$ admits an optimal Hardy-weight in Ω, where $c_p=(p/(p-1))^{p-1}$.

As a corollary of the proof of theorem 1.1 we obtain the following result.

COROLLARY 1.2. *Let $\Omega\subset\mathbb{R}^n$, $n\geqslant 2$, be a domain and $x_0\in K\Subset\Omega$. Let $Q_{p,A,V}$ be a subcritical operator in Ω satisfying assumptions 2.8 with $V\leqslant 0$ in Ω. Suppose that $Q_{p,A,V}$ admits a positive minimal Green function $G(x)$ in $\Omega\setminus\{x_0\}$ (see definition 2.22) satisfying*

$$\lim_{x\to\overline{\infty}} G(x)=0,\ \textit{and}\ \int_{\Omega\setminus K}|V||G(x)|^{p-1}\,\mathrm{d}x<\infty, \tag{1.2}$$

where $\overline{\infty}$ denotes the ideal point in the one-point compactification of Ω. Then the operator $Q_{p,A,V/c_p}$ admits an optimal Hardy-weight in Ω, where $c_p=(p/(p-1))^{p-1}$.

The paper is organized as follows. In § 2, we introduce the necessary notation and recall some previously obtained results needed in the present paper. We proceed in § 3, with proving essential results needed for the proof of theorem 1.1, and then we prove theorem 1.1 and corollary 1.2.

2. Preliminaries

Let $\Omega\subset\mathbb{R}^n$ be a domain, and let $1<p<\infty$. Throughout the paper we use the following notation and conventions:

- For any $R>0$ and $x\in\mathbb{R}^n$, we denote by $B_R(x)$ the open ball of radius R centred at x, and $B_R^+(0)=\{x\in B_R(0):x_n>0\}$.

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- We write $\Omega_1 \Subset \Omega_2$ if Ω_2 is open in Ω, the set $\overline{\Omega_1}$ is compact, and $\overline{\Omega_1} \subset \Omega_2$.
- C refers to a positive constant which may vary from line to line.
- Let g_1, g_2 be two positive functions defined in Ω. We use the notation $g_1 \asymp g_2$ in Ω if there exists a positive constant C such that
$$C^{-1}g_2(x) \leqslant g_1(x) \leqslant Cg_2(x) \qquad \text{for all } x \in \Omega.$$
- Let g_1, g_2 be two positive functions defined in Ω, and let $x_0 \in \Omega$. We use the notation $g_1 \sim g_2$ near x_0 if there exists a positive constant C such that
$$\lim_{x\to x_0} \frac{g_1(x)}{g_2(x)} = C.$$
- The gradient of a function f will be denoted by ∇f.
- χ_B denotes the characteristic function of a set $B \subset \mathbb{R}^n$.
- For any $1 \leqslant p \leqslant \infty$, p' is the Hölder conjugate exponent of p satisfying $p' = p/(p-1)$.
- For $1 \leqslant p < n$, $p^* := np/(n-p)$ is its Sobolev critical exponent.
- For a real valued function W, we write $W \gneqq 0$ in Ω if $W \geqslant 0$ in Ω and $\sup_\Omega W > 0$.
- For a symmetric positive definite $A \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$, we denote $\Delta_{p,A}(u) := \operatorname{div}(|\nabla u|_A^{p-2} A\nabla u)$ is the (p, A)-Laplace operator.
- For a real valued function u and $1 < p < \infty$, $\mathcal{I}_p(u) := |u|^{p-2}u$.
- $\overline{\infty}$ denotes the ideal point in the one-point compactification of Ω.
- $\mathbb{R}_+$ denotes the interval $(0, \infty)$.
- $d_\Omega = \operatorname{dist}(\cdot, \partial\Omega) : \Omega \to (0, \infty)$ is the distance function to $\partial\Omega$.
- $\operatorname{diam}(\Omega)$ denotes the diameter of Ω.
- $\operatorname{supp}(u)$ denotes the support of the function u.
- $\mathcal{H}^l$, $1 \leqslant l \leqslant n$, denotes the l-dimensional Hausdorff measure on $\mathbb{R}^n$.

2.1. Gauss–Green formula

We continue with several definitions and results concerning the Gauss–Green theorem [7].

Definition 2.1. *Let $D \subset \mathbb{R}^n$ be an open set.*

(i) *We denote by $\mathcal{M}(D)$ the space of all signed Radon measures μ on D such that $\int_D d|\mu| < \infty$.*

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(ii) *A vector field $F \in L^\infty(D, \mathbb{R}^n)$ is called a divergence measure field, written as $F \in \mathcal{DM}^\infty(D)$, if $\mathrm{div}(F) = \mu \in \mathcal{M}(D)$, i.e., there exists $\mu \in \mathcal{M}(D)$ such that*

$$\int_D \phi \,\mathrm{d}\mu = -\int_D \nabla\phi \cdot F \,\mathrm{d}x \quad \forall\, \phi \in C_0^\infty(D).$$

(iii) *We say that a vector field $F \in L^\infty_{loc}(D, \mathbb{R}^n)$ belongs to $\mathcal{DM}^\infty_{loc}(D)$ if for any open subset $E \Subset D$, we have $F \in \mathcal{DM}^\infty(E)$.*

Definition 2.2 cf. [7. *and [11, section 5]] Let $D \subset \mathbb{R}^n$ be an open set. A function $f \in L^1(D)$ has a bounded variation in D if*

$$\sup\left\{\int_D f \,\mathrm{div}(\phi)\,\mathrm{d}x : \phi \in C_0^1(D, \mathbb{R}^n), |\phi| \leqslant 1\right\} < \infty.$$

Denote by $\mathrm{BV}(D)$ the space of all functions $f \in L^1(D)$ having bounded variation.

Definition 2.3. *Let $D \subset \mathbb{R}^n$ be an open set. A measurable subset $E \subset \mathbb{R}^n$ is said to be a set of finite perimeter in D if $\chi_E \in \mathrm{BV}(D)$.*

Proposition 2.4 [11, theorem 5.9, p. 212]. *Let $E \Subset \mathbb{R}^n$ and let $0 \leqslant f \in \mathrm{BV}(E) \cap C^1(E)$. Then, for a.e. $t \in [0, \infty)$ the set $\{x \in E : f(x) > t\}$ has finite perimeter. In particular, for a.e. $0 \leqslant t_1 < t_2$ the set $\{x \in E : t_1 < f(x) < t_2\}$ has finite perimeter.*

We proceed with the following Gauss–Green theorem of divergence measure fields over sets of finite perimeter (see [7, theorems 5.2 and 7.2] and [9, proposition 3.1]).

Lemma 2.5. *Let $D \subset \mathbb{R}^n$ be an open set. Suppose that $F \in \mathcal{DM}^\infty_{loc}(D)$ with $\mathrm{div}(F) = \mu \in \mathcal{M}(D)$. Let $E \Subset D$ be a set of finite perimeter satisfying*

- $\partial E = (\bigcup_{k\in\mathbb{N}} D_k) \cup N$,
- *for each $k \in \mathbb{N}$, D_k is $(n-1)$- dimensional C^1 surface, and $\mathcal{H}^{n-1}(N) = 0$.*

Then,

$$\int_E \mathrm{div}(F)\,\mathrm{d}x = \int_{\partial E} F \cdot \vec{n}\,\mathrm{d}\mathcal{H}^{n-1},$$

where $\vec{n}$ is a classical outer unit normal to ∂E which is defined $\mathcal{H}^{n-1}$-a.e. on ∂E.

2.2. Local Morrey spaces

In the present subsection we introduce a certain class of Morrey spaces that depend on the index p, where $1 < p < \infty$.

Definition 2.6. *Let $q \in [1, \infty]$ and $\omega \Subset \mathbb{R}^n$. For a measurable, real valued function f defined in ω, we set*

$$\|f\|_{M^q(\omega)} := \sup_{\substack{y\in\omega \\ r<\mathrm{diam}(\omega)}} \frac{1}{r^{n/q'}} \int_{\omega\cap B_r(y)} |f|\,\mathrm{d}x.$$

We write $f \in M^q_{loc}(\Omega)$ if for any $\omega \Subset \Omega$ we have $\|f\|_{M^q(\omega)} < \infty$.

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Next, we define a special local Morrey space $M^q_{\text{loc}}(p;\Omega)$ which depends on the values of the exponent p.

Definition 2.7. *For $p \neq n$, we define*

$$M^q_{loc}(p;\Omega) := \begin{cases} M^q_{loc}(\Omega) \text{ with } q > n/p & \text{if } p < n \\ L^1_{loc}(\Omega) & \text{if } p > n, \end{cases}$$

while for $p = n$, $f \in M^q_{loc}(n;\Omega)$ means that for some $q > n$ and any $\omega \Subset \Omega$ we have

$$\|f\|_{M^q_{n;\omega}} := \sup_{\substack{y\in\omega \\ r<\text{diam}(\omega)}} \varphi_q(r) \int_{\omega\cap B_r(y)} |f| \,\mathrm{d}x < \infty,$$

where $\varphi_q(r) := \log(\text{diam}(\omega)/r)^{q/n'}$ and $0 < r < \text{diam}(\omega)$.

For the regularity theory of equations with coefficients in Morrey spaces we refer the reader to [18, 19].

We associate to any domain $\Omega \subset \mathbb{R}^n$ an *exhaustion*, i.e. a sequence of smooth, precompact domains $\{\Omega_j\}_{j=1}^\infty$ such that $\Omega_1 \neq \emptyset$, $\Omega_j \Subset \Omega_{j+1}$ and $\bigcup_{j=1}^\infty \Omega_j = \Omega$.

2.3. Criticality theory for $Q_{p,A,V}$

Let $1 < p < \infty$, and consider the operator

$$Q_{p,A,V}(u) := -\Delta_{p,A}(u) + V\mathcal{I}_p(u), \tag{2.1}$$

defined on a domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, where $\Delta_{p,A} := \text{div}(|\nabla u|_A^{p-2} A\nabla u)$ and $\mathcal{I}_p(u) := |u|^{p-2}u$. Unless otherwise stated, we always assume that the matrix A and the potential function V satisfy the following regularity assumptions:

Assumption 2.8.

- $A(x) = (a^{ij}(x))_{i,j=1}^n \in C^\alpha_{\text{loc}}(\Omega, \mathbb{R}^{n^2})$ is a symmetric positive definite matrix which is locally uniformly elliptic, that is, for any compact $K \Subset \Omega$ there exists $\Theta_K > 0$ such that

$$\Theta_K^{-1} \sum_{i=1}^n \xi_i^2 \leqslant \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \leqslant \Theta_K \sum_{i=1}^n \xi_i^2 \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in K.$$

- $V \in M^q_{\text{loc}}(p;\Omega)$ is a real valued function.

The associated energy functional for the operator $Q_{p,A,V}$ in Ω is defined by

$$\mathcal{Q}^\Omega_{p,A,V}(\phi) := \int_\Omega (|\nabla\phi|_A^p + V|\phi|^p) \,\mathrm{d}x \qquad \phi \in C_0^\infty(\Omega).$$

Definition 2.9. *We say that $u \in W^{1,p}_{loc}(\Omega)$ is a (weak) solution (resp. supersolution) of $Q_{p,A,V}(u) = 0$ in Ω if for any $\phi \in C_0^\infty(\Omega)$ (resp. $0 \leqslant \phi \in C_0^\infty(\Omega)$)*

$$\int_\Omega |\nabla u|_A^{p-2} A\nabla u \cdot \nabla\phi \,\mathrm{d}x + \int_\Omega V|u|^{p-2}u\phi \,\mathrm{d}x = 0 \text{ (resp. } \geqslant 0).$$

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It should be noted that the above definition makes sense due to the following Morrey–Adams theorem (see, e.g., [19, theorem 2.4] and references therein).

THEOREM 2.10 Morrey–Adams theorem. *Let $\omega \Subset \mathbb{R}^n$ and $V \in M^q(p;\omega)$.*

(i) *There exists a constant $C = C(n, p, q) > 0$ such that for any $\delta > 0$*

$$\int_\omega |V||u|^p \,\mathrm{d}x \leqslant \delta \|\nabla u\|^p_{L^p(\omega,\mathbb{R}^n)} + \frac{C}{\delta^{n/(pq-n)}} \|V\|^{pq/(pq-n)}_{M^q(p;\omega)} \|u\|^p_{L^p(\omega)} \quad \forall u \in W^{1,p}_0(\omega).$$

(ii) *For any $\omega' \Subset \omega$ with Lipschitz boundary, there exists a positive constant $C = C(n, p, q, \omega', \omega, \delta, \|V\|_{M^q(p;\omega)})$ and δ_0 such that for $0 < \delta \leqslant \delta_0$*

$$\int_{\omega'} |V||u|^p \,\mathrm{d}x \leqslant \delta \|\nabla u\|^p_{L^p(\omega',\mathbb{R}^n)} + C\|u\|^p_{L^p(\omega')} \quad \forall u \in W^{1,p}(\omega').$$

We denote the set of all positive solutions (resp., supersolutions) of $Q_{p,A,V}(u) = 0$ in Ω by $\mathcal{C}^{Q_{p,A,V}}(\Omega)$ (resp., $\mathcal{K}^{Q_{p,A,V}}(\Omega)$). We say that the operator $Q_{p,A,V}$ is *nonnegative* (in short $Q_{p,A,V} \geqslant 0$) in Ω if $\mathcal{C}^{Q_{p,A,V}} \neq \emptyset$.

REMARK 2.11. A weak (super)solution of the equation $-\Delta_{p,A}(u) = 0$ in Ω is said to be a (p, A)-*(super)harmonic function* in Ω.

It is well known that under assumptions 2.8 any positive solution of the equation $Q_{p,A,V}(u) = 0$ in Ω belongs to $C^{1,\alpha}(\Omega)$ (see, e.g. [19, remark 1.1]). Furthermore, the following Harnack convergence principle holds true.

PROPOSITION 2.12 Harnack convergence principle [13, proposition 2.7]. *Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be an exhaustion of Ω. Assume that $\{A_k\}_{k\in\mathbb{N}}$ is a sequence of symmetric and locally uniformly positive definite matrices such that the local ellipticity constants do not depend on k, and $\{A_k\}_{k\in\mathbb{N}} \subset L^\infty_{loc}(\Omega_k, \mathbb{R}^{n^2})$ converges weakly in $L^\infty_{loc}(\Omega, \mathbb{R}^{n^2})$ to a matrix $A \in L^\infty_{loc}(\Omega, \mathbb{R}^{n^2})$. Assume further that $\{V_k\}_{k\in\mathbb{N}} \subset M^q_{loc}(p;\Omega_k)$ converges weakly in $M^q_{loc}(p;\Omega)$ to $V \in M^q_{loc}(p;\Omega)$. For each k, let v_k be a positive solution of the equation $Q_{p,A_k,V_k}(u) = 0$ in Ω_k such that $v_k(x_0) = 1$, where x_0 is a fixed reference point in Ω_1. Then there exists $0 < \beta < 1$ such that, up to a subsequence, $\{v_k\}_{k\in\mathbb{N}}$ converges weakly in $W^{1,p}_{loc}(\Omega)$ and in $C^\beta_{loc}(\Omega)$ to a positive weak solution v of the equation $Q_{p,A,V}(u) = 0$ in Ω.*

DEFINITION 2.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. A principal eigenvalue of $Q_{p,A,V}$ in Ω is an eigenvalue λ of the problem*

$$\begin{cases} Q_{p,A,V}(u) = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with a nonzero nonnegative u which is called a principal eigenfunction.

PROPOSITION 2.14 [19, theorem 3.9]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and assume that A is a uniformly elliptic, bounded matrix in Ω, and $V \in M^q(p;\Omega)$. Then, the operator $Q_{p,A,V}$ admits a unique principal eigenvalue $\lambda_1(\Omega)$. Moreover,*

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λ_1 is simple and its principal eigenfunction is the minimizer of the Rayleigh–Ritz variational problem

$$\lambda_1(\Omega) = \min_{u \in W_0^{1,p} \setminus \{0\}} \frac{\mathcal{Q}^{\Omega}_{p,A,V}(u)}{\|u\|^p_{L^p(\Omega)}}.$$

The following well-known Allegretto–Piepenbrink theorem (in short, the **AP** theorem) connects between the nonnegativity of $Q_{p,A,V}$ and the nonnegativity of its associated energy functional $\mathcal{Q}^{\Omega}_{p,A,V}$ [**19**, theorem 4.3].

THEOREM 2.15 **AP** theorem. *The following assertions are equivalent.*

(i) $\mathcal{Q}^{\Omega}_{p,A,V}(\phi) \geqslant 0$ *for all* $\phi \in C_0^\infty(\Omega)$.

(ii) $\mathcal{C}^{Q_{p,A,V}}(\Omega) \neq \emptyset$.

(iii) $\mathcal{K}^{Q_{p,A,V}}(\Omega) \neq \emptyset$.

DEFINITION 2.16. *Assume that $Q_{p,A,V} \geqslant 0$ in Ω. We say that $Q_{p,A,V}$ is subcritical in Ω if there exists $0 \lneqq W \in M^q_{loc}(p;\Omega)$ such that $Q_{p,A,V-W} \geqslant 0$ in Ω. We say that $Q_{p,A,V}$ is critical in Ω if for all $0 \lneqq W \in M^q_{loc}(p;\Omega)$ the equation $Q_{p,A,V-W}(u) = 0$ does not admit a positive solution in Ω.*

DEFINITION 2.17. *Let ω be a bounded Lipschitz domain. We say that $Q_{p,A,V}$ satisfies the (generalized) weak maximum principle in ω if for any $u \in W^{1,p}(\omega)$ satisfying $Q_{p,A,V}(u) \geqslant 0$ in ω and $u \geqslant 0$ on $\partial\omega$, we have $u \geqslant 0$ in ω.*

We say that $Q_{p,A,V}$ satisfies the strong maximum principle in ω if for any $u \in W^{1,p}(\omega)$ satisfying $Q_{p,A,V}(u) \geqslant 0$ in ω and $u \geqslant 0$ on $\partial\omega$, either $u = 0$, or $u > 0$ in ω.

LEMMA 2.18 [**19**, theorem 3.10]. *Let Ω be a bounded Lipschitz domain, and assume that A is a uniformly elliptic, bounded matrix in Ω, and $V \in M^q(p;\Omega)$. Then the following assertions are equivalent.*

(i) *$Q_{p,A,V}$ satisfies the (generalized) weak maximum principle in Ω.*

(ii) *$Q_{p,A,V}$ satisfies the strong maximum principle in Ω.*

(iii) *The equation $Q_{p,A,V}(u) = 0$ admits a positive supersolution in $W_0^{1,p}(\Omega)$ which is not a solution.*

(iv) *The equation $Q_{p,A,V}(u) = 0$ admits a positive supersolution in $W^{1,p}(\Omega)$ which is not a solution.*

(v) $\lambda_1(\Omega) > 0$.

(vi) *For any $0 \leqslant g \in L^{p'}(\Omega)$, there exists a unique nonnegative solution in $W_0^{1,p}(\Omega)$ of $Q_{p,A,V}(u) = g$.*

COROLLARY 2.19. *If there exists a weak positive (super)solution of $Q_{p,A,V}(u) = 0$ in a domain $\Omega \subset \mathbb{R}^n$, then $\lambda_1(\Omega') > 0$ for any bounded Lipschitz subdomain $\Omega' \Subset \Omega$.*

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Definition 2.20. *Let K_0 be a compact subset of Ω. A positive solution u of $Q_{p,A,V}(u) = 0$ in $\Omega \setminus K_0$ is said to be a positive solution of minimal growth in a neighbourhood of infinity in Ω, and denoted by $u \in \mathcal{MG}_{A,V,\Omega;K_0}$, if for any smooth compact subset K of Ω with $K_0 \Subset \mathrm{int}(K)$, and any positive supersolution $v \in C(\Omega \setminus K)$ of $Q_{p,A,V}(w) = 0$ in $\Omega \setminus K$, we have*

$$u \leqslant v \text{ on } \partial K \Longrightarrow u \leqslant v \text{ in } \Omega \setminus K.$$

If $u \in \mathcal{MG}_{A,V,\Omega;\emptyset}$, then u is called an Agmon ground state of $Q_{p,A,V}$ in Ω.

Lemma 2.21 [**13**, proposition 3.17]. *Let $V \in M^q_{loc}(p;\Omega)$, and suppose that $Q_{p,A,V} \geqslant 0$ in Ω. Then for any $x_0 \in \Omega$ the equation $Q_{p,A,V}(w) = 0$ admits a unique (up to a multiplicative constant) solution $u \in \mathcal{MG}_{A,V,\Omega;\{x_0\}}$.*

Definition 2.22. *A function $u \in \mathcal{MG}_{A,V,\Omega;\{x_0\}}$ having a nonremovable singularity at x_0 is called a minimal positive Green function of $Q_{p,A,V}$ in Ω with singularity at x_0. We denote such a function by $G^{\Omega}_{Q_{p,A,V}}(x, x_0)$.*

Lemma 2.23 [**19**, theorem 5.9]. *Suppose that $Q_{p,A,V} \geqslant 0$ in Ω. Then $Q_{p,A,V}$ is critical in Ω if and only if the equation $Q_{p,A,V} = 0$ admits a ground state in Ω.*

Definition 2.24. *A sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$ is called a null-sequence with respect to a nonnegative operator $Q_{p,A,V}$ in Ω if*

(i) *$\phi_k \geqslant 0$ for all $k \in \mathbb{N}$,*

(ii) *there exists a fixed open set $B \Subset \Omega$ such that $\|\phi_k\|_{L^p(B)} \asymp 1$ for all $k \in N$,*

(iii) $\lim_{k\to\infty} \mathcal{Q}^{\Omega}_{p,A,V}(\phi_k) = 0$.

Lemma 2.25 [**19**, theorem 4.15]. *A nonnegative operator $Q_{p,A,V}$ is critical in Ω if and only if $Q_{p,A,V}$ admits a null-sequence in Ω.*

The next lemma shows that the energy functional $\mathcal{Q}^{\Omega}_{p,A,V}$ is equivalent to a *simplified energy* that does not explicitly depend on V and contains only nonnegative terms.

Lemma 2.26 [**20**, lemma 3.4]. *Let $v \in \mathcal{C}^{Q_{p,A,V}}(\Omega)$. Then, for any $0 \leqslant u \in W^{1,p}_{loc}(\Omega)$ having compact support in Ω, and such that $w := u/v \in L^\infty_{loc}(\Omega)$, we have*

$$\mathcal{Q}^{\Omega}_{p,A,V}(u) \asymp \mathcal{Q}^{\Omega}_{\mathrm{sim},p,A,V}(w) := \int_\Omega v^2 |\nabla w|^2_A \left(w|\nabla v|_A + v|\nabla w|_A\right)^{p-2} \,\mathrm{d}x. \tag{2.2}$$

Remark 2.27. Lemma 2.26 is proved in [**20**] for the case $V \in L^\infty_{\mathrm{loc}}(\Omega)$. However, the proof is purely algebraic and therefore, holds for $V \in M^q_{\mathrm{loc}}(p;\Omega)$ as well.

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As a corollary of (2.2) and Hölder's inequality we obtain the following.

Corollary 2.28. *Let $v \in \mathcal{C}^{Q_{p,A,V}}(\Omega) \cap C^{1,\alpha}_{loc}(\Omega)$ and let $X(w) := \int_\Omega v^p |\nabla w|_A^p \, \mathrm{d}x$ and $Y(w) := \int_\Omega |w|^p |\nabla v|_A^p \, \mathrm{d}x$. Then, for any continuous function $w \in W^{1,p}(\Omega)$ having compact support in Ω, the following assertions hold true.*

(i) $\mathcal{Q}^\Omega_{p,A,V}(vw) \asymp \mathcal{Q}^\Omega_{\mathrm{sim},\,p,\,A,\,V}(w)$.

(ii) $\mathcal{Q}^\Omega_{\mathrm{sim},\,p,\,A,\,V}(w) \leqslant \begin{cases} CX(w) & 1 < p \leqslant 2, \\ C[X(w) + X(w)^{2/p} Y(w)^{\frac{p-2}{p}}] & p > 2. \end{cases}$

2.4. Optimal Hardy-weights

Let $\overline{\infty}$ denote the ideal point in the one-point compactification of Ω. Let us define the notion of an optimal Hardy-weight for the operator $Q_{p,A,V}$.

Definition 2.29 [**9**]. *Suppose that $Q_{p,A,V}$ is subcritical in Ω. We say that $0 \lneqq W$ is an optimal Hardy-weight of $Q_{p,A,V}$ in Ω if the following two assertions are satisfied:*

(i) ***Criticality:*** *$Q_{p,A,V-W}$ is critical in Ω.*

(ii) ***Null-criticality with respect to*** *W: $\int_\Omega |\psi|^p W \, \mathrm{d}x = \infty$, where ψ is the (Agmon) ground state of $Q_{p,A,V-W}$ in Ω.*

Remark 2.30. Let us discuss definition 2.29. Suppose that $Q_{p,A,V}$ is subcritical in a domain Ω containing x_0, and let $x_0 \in K \Subset \Omega$. Then, for any $0 \lneqq W \in C_0^\infty(\Omega)$ there exists $\tau > 0$ such that $Q_{p,A,V-\tau W}$ is critical in Ω (see, e.g. [**21**, proposition 4.4] and [**19**]). On the other hand, the ground state of $Q_{p,A,V-\tau W}$, ϕ, satisfies

$$\phi \asymp G^\Omega_{Q_{p,A,V}}(x, x_0) \text{ in } \Omega \setminus K.$$

Therefore, there are infinity many weight functions $0 \lneqq W \in C_0^\infty(\Omega)$ such that $Q_{p,A,V-W}$ is critical in Ω. Obviously, for such a weight W, the operator $Q_{p,A,V-W}$ is not null-critical with respect to W.

Definition 2.31. *We say that a Hardy-weight W is optimal at infinity in Ω if for any $K \Subset \Omega$, we have*

$$\sup\{\lambda \in \mathbb{R} \mid Q_{p,A,V-\lambda W} \geqslant 0 \text{ in } \Omega \setminus K\} = 1.$$

Remark 2.32. The definition of an optimal Hardy-weight in [**8**] includes the requirement that W should be optimal at infinity. But, it is proved in [**15**] that if $Q - W$ is null-critical with respect to W in Ω, then $Q - W$ is optimal at infinity. The same proof applies under the assumptions considered in the present paper, hence, in definition 2.29 we avoid the requirement of optimality at infinity.

The following coarea formula is a direct consequence of [**9**, proposition 3.1].

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Lemma 2.33 Coarea formula. *Let Ω be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and $G \in C^{1,\alpha}(\Omega)$ is a positive (p, A)-harmonic function in $\Omega^* := \Omega \setminus \{0\}$. Assume that for any $0 < t_1 < t_2 < \infty$, the set $\mathcal{A} := \{x \in \Omega^* \mid t_1 < G(x) < t_2\}$ is bounded. Let $h \in C^2(0, \infty)$ be a positive function satisfying $h'(s) > 0$ for all $s > 0$, and denote $v := h(G)$. Then there exists $C > 0$ such that for any locally bounded real measurable function f such that $f(v)$ has a compact support in Ω^*, we have*

$$\int_{\Omega^*} f(v)|\nabla v|_A^p \mathrm{d}x = C \int_{h(\inf_{\Omega^*} G)}^{h(\sup_{\Omega^*} G)} \frac{f(\tau)}{((h^{-1})'(\tau))^{p-1}} \,\mathrm{d}\tau. \tag{2.3}$$

Proof. Since $G \in C^{1,\alpha}(\Omega^*)$ and $1 < p$, then $\frac{|\nabla G|_A^p}{|\nabla G|} \in L^1_{\mathrm{loc}}(\Omega^*)$ and we may use the (classical) coarea formula ([7, theorem 2.32]) to obtain for $v = h(G)$

$$\begin{aligned} &\int_{\Omega^*} f(v)|\nabla v|_A^p \,\mathrm{d}x = \int_{\Omega^*} f(h(v))|h'(G)|^p \frac{|\nabla G|_A^p}{|\nabla G|}|\nabla G| \,\mathrm{d}x = \\ &\int_{\mathbb{R}_+} f(h(t))h'(t)^p \int_{\{G=t\}} \frac{|\nabla G|_A^p}{|\nabla G|} \,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{2.4}$$

By (a generalized) Sard's theorem for $C^{1,\alpha}$ functions [5, theorem 1.2],

$$\mathcal{H}^{n-1}\left(\{G = t\} \cap \mathrm{Crit}(G)\right) = 0.$$

The fact that $G \in C^{1,\alpha}$ and proposition 2.4 imply that (for a.e. $t_1 < t_2$) the set $\mathcal{A} := \{t_1 < G < t_2\}$ has a finite perimeter. In particular, $\nabla G \neq 0$ and $\vec{n}$ is well defined on $\partial\mathcal{A}$, $\mathcal{H}^{n-1}$-a.e.. Let $\partial_+ = \{x \in \overline{\mathcal{A}} : G(x) = t_2\}$ and $\partial_- = \{x \in \overline{\mathcal{A}} : G(x) = t_1\}$. The Gauss–Green theorem (lemma 2.5) implies that

$$\begin{aligned} 0 &= -\int_{\mathcal{A}} \mathrm{div}(|\nabla G|_A^{p-2} A\nabla G) \,\mathrm{d}x = \int_{\partial_+} + \int_{\partial_-} |\nabla G|_A^{p-2} A\nabla G \cdot \vec{n} \,\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\partial_+} |\nabla G|_A^{p-2} A\nabla G \cdot \frac{\nabla G}{|\nabla G|} \,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial_-} |\nabla G|_A^{p-2} A\nabla G \cdot \frac{\nabla G}{|\nabla G|} \,\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\{G=t_2\}} \frac{|\nabla G|_A^p}{|\nabla G|} \,\mathrm{d}\mathcal{H}^{n-1} - \int_{\{G=t_1\}} \frac{|\nabla G|_A^p}{|\nabla G|} \,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

In particular, for any $t > 0$, $\int_{\{G=t\}} \frac{|\nabla G|_A^p}{|\nabla G|} \,\mathrm{d}\mathcal{H}^{n-1} = C$. By (2.4),

$$\int_{\Omega^*} f(v)|\nabla v|_A^p \,\mathrm{d}x = C \int_{\mathbb{R}_+} f(h(t))h'(t)^p \,\mathrm{d}t.$$

The change of the variable $h(t) = \tau$ then implies (2.3). $\square$

The following theorem is proved in [9] for the case $A = \mathbf{1}$. However, it can be easily checked that the validity of lemma 2.33 for a general matrix A satisfying assumptions 2.8, gives rise to the following theorem.

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THEOREM 2.34 [**9**, theorem 1.5]. *Let $\overline{\infty}$ denote the ideal point in the one point compactification of Ω. Suppose that $-\Delta_{p,A}$ is subcritical in Ω, and admits a positive $(p,\, A)$-harmonic function $G(x)$ in $\Omega^* := \Omega \setminus \{0\}$ satisfying one of the following conditions (2.5),(2.6)*:

$$1 < p \leqslant n, \quad \lim_{x\to 0} G(x) = \infty, \textit{ and } \lim_{x\to\overline{\infty}} G(x) = 0, \tag{2.5}$$

$$p > n, \quad \lim_{x\to 0} G(x) = \gamma \geqslant 0, \textit{ and } \lim_{x\to\overline{\infty}} G(x) = \begin{cases} \infty & \textit{if } \gamma = 0, \\ 0 & \textit{if } \gamma > 0. \end{cases} \tag{2.6}$$

Define a positive function v and a nonnegative weight W on Ω^* as follows:

(i) *If either (2.5) is satisfied, or (2.6) is satisfied with $\gamma = 0$, then*

$$v := G^{(p-1)/p}, \ \textit{ and } W := \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla G}{G}\right|_A^p.$$

(ii) *If (2.6) is satisfied with $\gamma > 0$, then $v := [G(\gamma - G)]^{(p-1)/p}$, and*

$$W := \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla G}{G(\gamma - G)}\right|_A^p |\gamma - 2G|^{p-2}[2(p-2)G(\gamma - G) + \gamma^2].$$

Then the following Hardy-type inequality holds in Ω^* :

$$\int_{\Omega^*} |\nabla\phi|_A^p \,\mathrm{d}x \geqslant \int_{\Omega^*} W|\phi|^p \,\mathrm{d}x, \quad \forall\ \phi \in C_0^\infty(\Omega^*), \tag{2.7}$$

and W is an optimal Hardy-weight of $-\Delta_{p,A}$ in Ω^*. Moreover, up to a multiplicative constant, v is the ground state of $-\Delta_{p,A} - W\mathcal{I}_p$ in Ω^*.

The following simple observation concerns the existence of optimal Hardy-weights for a 'small perturbation' of an operator with a given optimal Hardy-weight.

LEMMA 2.35. *Assume that $Q_{p,A,V}$ is subcritical in Ω and admits an optimal Hardy-weight W in $\Omega^* := \Omega \setminus \{0\}$. Let $V_1 \in M^q_{loc}(p;\Omega)$ satisfy $V_1 \geqslant -\varepsilon W$ for some $0 \leqslant \varepsilon < 1$ and $q > n/p$. Then $W + V_1$ is an optimal Hardy-weight for $Q_{p,A,V+V_1}$ in Ω^*.*

Proof. Consider the function $W + V_1$. Then, $Q_{p,A,V+V_1} - (W + V_1)\mathcal{I}_p = Q_{p,A,V} - W\mathcal{I}_p$ is a critical operator in Ω^*.

Obviously, $W + V_1 \gneqq 0$, and the ground state ψ of $Q_{p,A,V} - W\mathcal{I}_p$ in Ω^* is the ground state of $Q_{p,A,V+V_1} - (W + V_1)\mathcal{I}_p$ in Ω^*. Moreover,

$$\int_{\Omega^*} (W + V_1)|\psi|^p \,\mathrm{d}x \geqslant (1-\varepsilon)\int_{\Omega^*} W|\psi|^p \,\mathrm{d}x = \infty,$$

implying that $Q_{p,A,V+V_1-(W+V_1)}$ is null-critical in Ω^* with respect to $W + V_1$. In particular, $W + V_1$ is an optimal Hardy-weight of $Q_{p,A,V+V_1}$ in Ω^*. □

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3. Optimal Hardy-weights for indefinite potentials

Lemma 2.35 obviously applies when $V_1 \geqslant 0$. The main goal in the current section is to obtain optimal Hardy-weights for a general subcritical operator $Q_{p,A,V}$ in a domain Ω, without assuming $V = 0$ in Ω. In particular, we prove theorem 1.1.

First, we recall the following weak comparison principle [**19**, theorem 5.3].

LEMMA 3.1 Weak comparison principle. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Assume that A is a uniformly elliptic and bounded matrix in Ω, $V \in M^q(p;\Omega)$ and $0 \leqslant g \in L^\infty(\Omega)$. Assume further that $\lambda_1(\Omega) > 0$, where $\lambda_1(\Omega)$ is the principal eigenvalue of the operator $Q_{p,A,V}$. Let $u_2 \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ be a (weak) solution of*

$$\begin{cases} Q_{p,A,V}(u_2) = g & \text{in } \Omega, \\ u_2 > 0 & \text{on } \partial\Omega. \end{cases}$$

If $u_1 \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{cases} Q_{p,A,V}(u_1) \leqslant Q_{p,A,V}(u_2) & \text{in } \Omega, \\ u_1 \leqslant u_2 & \text{on } \partial\Omega, \end{cases}$$

then $u_1 \leqslant u_2$ in Ω.

In the following lemma we generalize the notion of Green potential for $Q_{p,A,V}$.

LEMMA 3.2. *Assume that $Q_{p,A,V}$ is subcritical in Ω, and let $0 \lneqq \varphi \in C_0^\infty(\Omega)$. Then there exists a positive function $G_\varphi \in W^{1,p}_{loc}(\Omega)$, such that G_φ is a positive solution of minimal growth at infinity and satisfies $Q_{p,A,V}(G_\varphi) = \varphi$ in Ω.*

Proof. / Fix $0 \lneqq \varphi \in C_0^\infty(\Omega)$, and let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a smooth exhaustion of Ω with $\operatorname{supp}(\varphi) \Subset \Omega_1$. Lemma 2.18 implies that there exists a unique positive solution $G^k \in W^{1,p}(\Omega_k)$ to the problem

$$\begin{cases} -\Delta_{p,A}(w) + (V + \frac{1}{k})|w|^{p-2}w = \varphi & \text{in } \Omega_k, \\ w = 0 & \text{on } \partial\Omega_k. \end{cases}$$

By the weak comparison principle (lemma 3.1), $\{G^k\}_{k\in\mathbb{N}}$ is a monotone increasing sequence of functions. Assume first that the sequence $\{G^k\}_{k\in\mathbb{N}}$ is not locally uniformly bounded in Ω, and let $x_1 \in \Omega_2 \setminus \Omega_1$. By Harnack's convergence principle there exists a subsequence of $\{z_k(x) := G^k(x)/G^k(x_1)\}_{k\in\mathbb{N}}$ which converges locally uniformly to a positive solution G, of the equation $Q_{p,A,V}(u) = 0$ in Ω. Therefore, G is a positive solution of the equation $Q_{p,A,V}(u) = 0$ in Ω which clearly has minimal growth in a neighborhood of infinity in Ω, i.e., G is a ground state. This is a contradiction to the subcriticality of the operator $Q_{p,A,V}$ in Ω.

Consequently, Harnack inequality ([**19**, theorem 2.7]) implies that the sequence $\{G^k\}_{k\in\mathbb{N}}$ is locally uniformly bounded in Ω. By Harnack convergence principle and the strong maximum principle, it converges locally uniformly (up to a subsequence) to a positive solution, G_φ, of the equation $Q_{p,A,V}(u) = \varphi$ in Ω. In fact, [**16**, theorem 5.3] implies that there exists $0 < \alpha < 1$ such that $G_\varphi \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$. $\square$

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Definition 3.3. *Let $0 \lneq \varphi \in C_0^\infty(\Omega)$. A positive solution $u \in G_\varphi \in \mathcal{MG}_{A,V,\Omega,supp\,(\varphi)}$ that satisfies $Q_{p,A,V}(u) = \varphi$ in Ω, is called a Green potential of $Q_{p,A,V}$ in Ω with a density φ.*

We proceed with the following technical proposition (cf. [9, lemma 2.10]).

Proposition 3.4. *Let $f(t) \in C^2(\mathbb{R}_+)$ satisfying f, f', $-f'' > 0$. Then, for all $0 \leqslant u \in C^1(\Omega)$*

$$Q_{p,A,V}(f(u)) = -\Delta_p^{1D}(f)(u)|\nabla u|_A^p + (f'(u))^{p-1} \times \left(-\Delta_{p,A}(u) + V\left(\frac{f(u)}{f'(u)u}\right)^{p-1}|u|^{p-1}\right)$$

in the weak sense. Here $-\Delta_p^{1D} f(t) := -(|f'(t)|^{p-2}f'(t))'$ is the one-dimensional p-Laplacian.

Proof. By [9, lemma 2.10] (which clearly holds for the (p, A)-Laplacian), we have:

$$-\Delta_{p,A}(f(u)) = -|f'(u)|^{p-2}\left[(p-1)f''(u)|\nabla u|_A^p + f'(u)\Delta_{p,A}(u)\right] \tag{3.1}$$

in the weak sense. Since $f \in C^2$, f, f', $-f'' > 0$ we have

$$-|f'(u)|^{p-2}(p-1)f''(u)|\nabla u|_A^p = -\frac{d}{dt}[|f'(t)|^{p-1}](u)|\nabla u|_A^p = -\Delta_p^{1D}(f)(u)|\nabla u|_A^p,$$

and together with (3.1) the proposition is proved. □

Remark 3.5. We remark that if $f(t) = t^{\frac{p-1}{p}}$, then

$$-\Delta_p^{1D}(f(t)) - \left(\frac{p-1}{p}\right)^p \frac{f(t)^{p-1}}{t^p} = 0, \text{ and } c_p := \left(\frac{f(u)}{f'(u)u}\right)^{p-1} = \left(\frac{p}{p-1}\right)^{p-1} > 1.$$

Lemma 3.2 and proposition 3.4 imply:

Corollary 3.6. *Assume that Q_{p,A,c_pV} is subcritical in Ω. For $0 \lneq \varphi \in C_0^\infty(\Omega)$, let G_φ be a Green potential satisfying $Q_{p,A,c_pV}(G_\varphi) = \varphi$ in Ω, and let $f(t) = t^{\frac{p-1}{p}}$. Then,*

$$Q_{p,A,V}(f(G_\varphi)) = -\Delta_p^{1D}(f)(G_\varphi)|\nabla G_\varphi|_A^p + (f'(G_\varphi))^{p-1}\varphi \gneq 0. \tag{3.2}$$

In particular, $f(G_\varphi)$ is a positive solution of the equation $Q_{p,A,V-W}(v) = 0$, where

$$W = \frac{Q_{p,A,V}(f(G_\varphi))}{f(G_\varphi)^{p-1}}, \text{ and } W = \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla G_\varphi}{G_\varphi}\right|_A^p \text{ in } \Omega \setminus \operatorname{supp}(\varphi).$$

The following lemma is a generalization of lemma 2.33 to the case $V \neq 0$.

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Lemma 3.7. *Assume that $Q_{p,A,V}$ is subcritical in Ω, and let $G_\varphi \in C^{1,\alpha}_{loc}(\Omega)$ be a Green potential (with respect to $0 \lneq \varphi \in C_0^\infty(\Omega)$), and assume that*

$$\lim_{x\to\overline{\infty}} G_\varphi = 0; \quad \int_\Omega V G_\varphi^{p-1}\,\mathrm{d}x < 0; \ \int_\Omega |V||G_\varphi|^{p-1}\,\mathrm{d}x < \infty. \tag{3.3}$$

Then, there exists $0 < M_\varphi < \sup_\Omega G_\varphi$ such that for almost every $0 < t < M_\varphi$, satisfying

$$supp(\varphi) \Subset \Omega_t := \{x \in \Omega : G_\varphi(x) > t\},$$

there exists $C > 0$, independent of t, such that

$$C^{-1} \leqslant \int_{G_\varphi = t} |\nabla G_\varphi|_A^{p-1}\,\mathrm{d}\sigma_A \leqslant C, \tag{3.4}$$

where $\mathrm{d}\sigma_A = \frac{|\nabla G_\varphi|_A}{|\nabla G_\varphi|}\,\mathrm{d}\mathcal{H}^{n-1}$, $\mathcal{H}^{n-1}$-a.e.

Proof. The assumption $\lim_{x\to\overline{\infty}} G_\varphi = 0$, and proposition imply that for a.e. $t > 0$ the set Ω_t has finite perimeter. Furthermore, (3.3) implies that $|V|G_\varphi^{p-1} \in \mathcal{M}(\Omega')$. Finally, Sard's theorem for $C^{1,\alpha}$-functions implies that the conditions in Gauss–Green theorem (lemma 2.5) are satisfied in Ω'. Hence,

$$\begin{aligned}\int_{\Omega_t} (\varphi - V|G_\varphi|^{p-2}G_\varphi)\,\mathrm{d}x &= -\int_{\Omega_t} \operatorname{div}(|\nabla G_\varphi|_A^{p-2} A\nabla G_\varphi)\,\mathrm{d}x \\ &= -\int_{\partial\Omega_t} |\nabla G_\varphi|_A^{p-2} A\nabla G_\varphi \cdot \vec{n}\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

The assumptions $\lim_{x\to\overline{\infty}} G_\varphi = 0$, and $\int_\Omega V G_\varphi^{p-1}\,\mathrm{d}x < 0$ imply that there exists a sufficiently small $M_\varphi > 0$ such that for a.e $0 < t < M_\varphi$,

$$\int_{\Omega_t} \left(\varphi - V|G_\varphi|^{p-2}G_\varphi\right)\mathrm{d}x \leqslant \int_\Omega \left(\varphi + |V||G_\varphi|^{p-1}\right)\mathrm{d}x \leqslant C. \tag{3.5}$$

Moreover, the assumption $\operatorname{supp}(\varphi) \Subset \Omega_t$ implies

$$C^{-1} \leqslant \int_\Omega \varphi\,\mathrm{d}x = \int_{\Omega_t} \varphi\,\mathrm{d}x \leqslant \int_{\Omega_t} \left(\varphi - V|G_\varphi|^{p-2}G_\varphi\right)\mathrm{d}x.$$

Consequently,

$$\int_{\Omega_t} \left(\varphi - V|G_\varphi|^{p-2}G_\varphi\right)\mathrm{d}x \asymp C,$$

and C does not depend on t. Sard's theorem for $C^{1,\alpha}$ functions implies that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega'$, $|\nabla G(x)| \neq 0$. Furthermore, the definition of Ω' implies that $G_\varphi \geqslant t$ in Ω', and hence, $\vec{n} = -\frac{\nabla G_\varphi}{|\nabla G_\varphi|}$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega'$. Therefore,

$$-\int_{\partial\Omega_t} |\nabla G_\varphi|_A^{p-2} A\nabla G_\varphi \cdot \vec{n}\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial\Omega_t} |\nabla G_\varphi|_A^{p-1} \frac{|\nabla G_\varphi|_A}{|\nabla G_\varphi|}\,\mathrm{d}\mathcal{H}^{n-1} \asymp C.$$

□

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Remark 3.8. Th e assumption $\int_\Omega VG_\varphi^{p-1}\,\mathrm{d}x < 0$ in lemma 3.7 is needed for arguing (3.5). In particular, the lemma still holds once assuming instead that $V \leqslant 0$ in Ω.

We proceed with the following lemma.

Lemma 3.9 cf. [9, propositions 5.1 and 5.5]. *Let $0 \lneqq \varphi \in C_0^\infty(\Omega)$, and assume that Q_{p,A,c_pV} is subcritical in Ω. Let $G_\varphi \in C^{1,\alpha}_{loc}(\Omega)$ be a Green potential satisfying $Q_{p,A,c_pV}(G_\varphi) = \varphi$ in Ω, and assume that*

$$\lim_{x\to\overline{\infty}} G_\varphi = 0; \quad \int_\Omega VG_\varphi^{p-1}\,\mathrm{d}x < 0; \quad \int_\Omega |V||G_\varphi|^{p-1}\,\mathrm{d}x < \infty.$$

Consider the function $f(t) = t^{\frac{p-1}{p}}$, and let

$$W := \frac{Q_{p,A,V}(f(G_\varphi))}{f(G_\varphi)^{p-1}}.$$

Then $Q_{p,A,V-W}$ is critical in Ω, with a ground state $f(G_\varphi)$ and $\int_\Omega Wf(G_\varphi)^p\,\mathrm{d}x = \infty$. Hence, W is an optimal Hardy-weight for $Q_{p,A,V}$ in Ω.

Proof. **Criticality:** Notice that $c_p > 1$. and therefore, $Q_{p,A,V}$ is subcritical in Ω [19, Corollary 4.17]. Let M_φ be given by lemma 3.7, and let $K \Subset \Omega$ be a precompact smooth subdomain satisfying $\operatorname{supp}\varphi \Subset K$, $\max_{\Omega\setminus K} G_\varphi < M_\varphi$ and $G_\varphi < 1$ for all $x \in \Omega \setminus K$. Assume without loss of generality that $\inf_K G_\varphi \geqslant 1$.

For each $k \in \mathbb{N}$, consider the function $\phi_k(f(G_\varphi))$, where $f(t) = t^{\frac{p-1}{p}}$ and

$$\phi_k(t) = \begin{cases} 0 & 0 \leqslant t \leqslant \frac{1}{k^2}, \\ 2 + \frac{\log t}{\log k} & \frac{1}{k^2} \leqslant t \leqslant \frac{1}{k}, \\ 1 & \frac{1}{k} \leqslant t \leqslant k, \\ 2 - \frac{\log t}{\log k} & k \leqslant t \leqslant k^2, \\ 0 & t \geqslant k^2. \end{cases}$$

We claim that $u_k = \phi_k(f(G_\varphi))f(G_\varphi)$ is a null-sequence of $Q_{p,A,V-W}$ in Ω. Indeed, by (2.2), $\mathcal{Q}_{\mathrm{sim}}(w) \asymp \mathcal{Q}(wf(G_\varphi)) = \mathcal{Q}(u)$, where

$$\mathcal{Q}(u) = \int_\Omega \left(|\nabla u|_A^p + (V - W)|u|^p\right)\mathrm{d}x,$$

and

$$\mathcal{Q}_{\mathrm{sim}}(w) = \int_\Omega f(G_\varphi)^2|\nabla w|_A^2\left(w|\nabla(f(G_\varphi))|_A + f(G_\varphi)|\nabla w|_A\right)^{p-2}\ \mathrm{d}x.$$

Moreover, by corollary 2.28 we have

$$\mathcal{Q}_{\mathrm{sim}}(w) \leqslant \begin{cases} CX(w) & 1 < p \leqslant 2, \\ C\left[X(w) + X(w)^{2/p}Y(w)^{\frac{p-2}{p}}\right] & p > 2, \end{cases}$$

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where

$$X(w)=\int_{\Omega}|\nabla w|_A^p f(G_\varphi)^p\,\mathrm{d}x,\quad Y(w)=\int_{\Omega}|w|^p|\nabla(f(G_\varphi))|_A^p\,\mathrm{d}x.$$

By the (classical) coarea formula ([7, theorem 2.32]),

$$\begin{aligned}X(\phi_k(f(G_\varphi)))&=\int_{\Omega\setminus K} f(G_\varphi)^p|\phi_k'(f(G_\varphi))|^p|f'(G_\varphi)|^p|\nabla G_\varphi|_A^p\,\mathrm{d}x\\&=\int_0^{\max_{\Omega\setminus K}G_\varphi} f(t)^p|\phi_k'(f(t))|^p f'(t)^p\,\mathrm{d}t\int_{G_\varphi=t}|\nabla G_\varphi|_A^{p-1}\mathrm{d}\sigma_A.\end{aligned}$$

By lemma 3.7, for a.e. $0<t<\max\limits_{\Omega\setminus K}G_\varphi$ we have $\int_{G_\varphi=t}|\nabla G_\varphi|_A^{p-1}\mathrm{d}\sigma_A\asymp 1$. Moreover,

$$\begin{aligned}\int_0^{\max_{\Omega\setminus K}G_\varphi} f(t)^p|\phi_k'(f(t))|^p f'(t)^p\,\mathrm{d}t&=C(p)\int_0^{f(\max_{\Omega\setminus K}G_\varphi)}\frac{|s\phi_k'(s)|^p}{s}\,\mathrm{d}s\\&=\frac{C(p)}{\log^p k}\int_{\frac{1}{k^2}}^{\frac{1}{k}}\frac{1}{s}\,\mathrm{d}s\asymp\left(\frac{1}{\log k}\right)^{p-1}.\end{aligned}$$

Consequently, $X(\phi_n(f(G_\varphi)))\asymp(\dfrac{1}{\log k})^{p-1}$. By a similar calculation,

$$Y(\phi_k(f(G_\varphi)))=\int_{\Omega\setminus K}|\phi_k(f(G_\varphi))|^p f'(G_\varphi)^p|\nabla G_\varphi|_A^p\,\mathrm{d}x\asymp\int_0^1|\phi_k(f(t))|^p f'(t)^p\,\mathrm{d}t\asymp$$
$$\int_0^{f(1)}|\phi_k(s)|^p\frac{ds}{s}=\int_{1/k^2}^{1/k}\left(2+\frac{\log s}{\log k}\right)\frac{1}{s}\,\mathrm{d}s+\int_{1/k}^1\frac{1}{s}\,\mathrm{d}s\asymp\int_{1/k}^1\frac{1}{s}\,\mathrm{d}s\asymp\log k.$$

It follows that $\mathcal{Q}_{\mathrm{sim}}(w_k)=\mathcal{Q}_{\mathrm{sim}}(\phi_k(f(G_\varphi)))\to 0$ as $k\to\infty$, and therefore,

$$\mathcal{Q}(u_k)=\mathcal{Q}(\phi_k(f(G_\varphi)f(G_\varphi)))\to 0\text{ as }k\to\infty.$$

Let us specialize $\varepsilon_0>0$ such that the set $B=\{x\in\Omega:\varepsilon_0/2<f(G_\varphi)<\varepsilon_0\}$ is nonempty, bounded, and contained in $\Omega\setminus K$. Therefore,

$$\int_B|u_k|^p\,\mathrm{d}x=\int_B|\phi_k(f(G_\varphi))|^p f(G_\varphi)^p\,\mathrm{d}x\asymp 1.\tag{3.6}$$

Thus, the sequence $\{u_k\}$ is a null-sequence, and in light of lemma 2.25, $Q_{p,A,V-W}$ is critical in Ω.

Null-criticality: Let $K\Subset\Omega$ be a precompact smooth subdomain as in the first part of the proof.

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For almost every $0 < \tau < 1$ we consider the set $\Omega_\tau := \{x \in \Omega \mid \tau < G_\varphi < \min_K G_\varphi\}$ which has finite perimeter. Recall that

$$W = \left(\frac{p-1}{p}\right)^p \frac{|\nabla G_\varphi|_A^p}{G_\varphi^p} \text{ in } \Omega_\xi.$$

By the (classical) coarea formula and (3.4),

$$\int_{\Omega_\tau} W(f(G_\varphi))^p \, \mathrm{d}x = \left(\frac{p-1}{p}\right)^p \int_{\Omega_\tau} \frac{|\nabla G_\varphi|_A^p}{G_\varphi^p} (f(G_\varphi))^p \, \mathrm{d}x$$
$$= \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}_+} \left(\frac{f(t)}{t}\right)^p \mathrm{d}t \int_{G_\varphi = t} |\nabla G_\varphi|_A^{p-1} \mathrm{d}\sigma_A \asymp C \int_\tau^{\min_K G_\varphi} \left(\frac{f(t)}{t}\right)^p \mathrm{d}t.$$

By letting $\tau \to 0$ we obtain that $\int_{\Omega \setminus K} W f(G_\varphi)^p \, \mathrm{d}x = \infty$. □

Remark 3.10. Remark 3.8 implies that lemma 3.9 still holds if one assumes $V \leqslant 0$ in Ω instead of the assumption $\int_\Omega V G_\varphi^{p-1} \, \mathrm{d}x < 0$.

Proof of theorem 1.1. Notice that $c_p > 1$, and hence $Q_{p,A,V/c_p}$ is subcritical in Ω. Let G_φ be the Green potential of $Q_{p,A,V}$, given by lemma 3.2. By lemma 3.9, the operator $Q_{p,A,V/c_p}$ admits an optimal Hardy-weight in Ω. □

Proof of corollary 1.2. Notice that $c_p > 1$, and hence $Q_{p,A,V/c_p}$ is subcritical in Ω. Let G_φ be the Green potential of $Q_{p,A,V}$, given by lemma 3.2. By the minimal growth property of G_φ, for any $x_0 \in K \Subset \Omega$, $G_\varphi \leqslant CG$ in $\Omega \setminus K$, and in particular,

$$\lim_{x \to \overline{\infty}} G_\varphi = 0, \quad \int_\Omega |V||G_\varphi|^{p-1} \, \mathrm{d}x < \infty.$$

By lemma 3.9 and remark 3.10, the operator $Q_{p,A,V/c_p}$ admits an optimal Hardy-weight in Ω. □

Corollary 1.2 and the following remark give rise to new optimal Hardy-type inequalities in the smooth case.

Remark 3.11. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $Q_{p,A,V}$ be a subcritical operator in Ω satisfying assumptions 2.8. Assume further that $V \leqslant 0$ in Ω. Then, there exists $K \Subset \Omega$ and $x_0 \in \mathrm{int} K \Subset \Omega$, such that the operator $Q_{p,A,V}$ admits a positive solution $G(x)$ in $\Omega \setminus \{x_0\}$ satisfying (1.2) in each of the following cases :

- A is a constant, symmetric, positive definite matrix; $V \in L^\infty(\Omega)$; Ω is a bounded $C^{1,\alpha}$ domain and $\lambda_1(\Omega) > 0$ [**17**].
- A is a constant, symmetric, positive definite matrix; $V \in C_0^\infty(\mathbb{R}^n)$; $\Omega = \mathbb{R}^n$ [**12**, **13**].

In particular, theorem 1.1 can be applied in each of the latter cases.

Remark 3.12. Combining oheorem 1.1 and lemma 2.35, we obtain optimal Hardy-weights for a wide family of operators $Q_{p,A,V}$ with indefinite potentials V.

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