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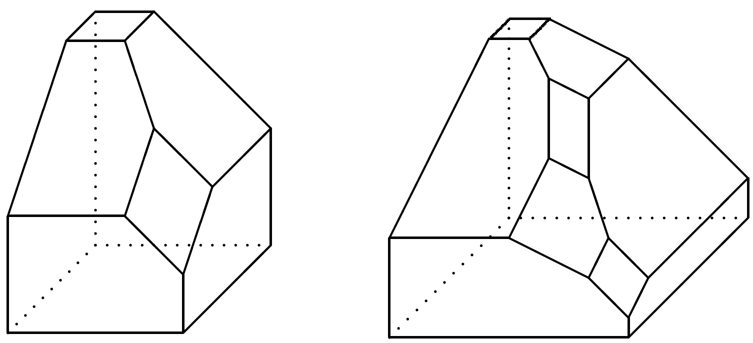
Polytopal Realizations of Generalized Associahedra

To Robert Moody on the occasion of his 60th birthday

Frédéric Chapoton, Sergey Fomin and Andrei Zelevinsky

Abstract. We prove polytopality of the generalized associahedra introduced in [5].



In [5], a complete simplicial fan was associated to an arbitrary finite root system. It was conjectured that this fan is the normal fan of a simple convex polytope (a generalized associahedron of the corresponding type). Here we prove this conjecture by explicitly exhibiting a family of such polytopal realizations (see Theorems 1.4–1.5 and Corollary 1.9 below).

The name "generalized associahedron" was chosen because for the type A_n the construction in [5] produces the n-dimensional associahedron (also known as the Stasheff polytope). Its face complex was introduced by J. Stasheff [13] as a basic tool for the study of homotopy associative H-spaces. The fact that this complex can be realized by a convex polytope was established much later in [8, 6]. Note that the realizations given in Corollary 1.9 are new even in this classical case.

The face complex of a generalized associahedron of type B_n (or C_n) is another familiar polytope: the n-dimensional "cyclohedron." It was first introduced by R. Bott and C. Taubes [1] (and given its name by J. Stasheff [14]) in connection with the study of link invariants; an alternative combinatorial construction was independently given by R. Simion [11, 12]. Polytopal realizations of cyclohedra were constructed explicitly by M. Markl [9] (*cf.* also [14, Appendix B]) and R. Simion [12]; again, our construction in Corollary 1.9 gives a new family of such realizations.

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Associahedra of types A and B have a number of remarkable connections with algebraic geometry [6], topology [13], moduli spaces, knots and operads [1, 3], combinatorics [10], *etc.* It would be interesting to extend these connections to the type D and the exceptional types.

As explained in [5], the construction of generalized associahedra given there was motivated by the theory of cluster algebras, introduced in [4] as a device for studying dual canonical bases and total positivity in semisimple Lie groups. This motivation remained a driving force for the present paper as well; although cluster algebras are not mentioned below, some of the present constructions and results will play an important role in a forthcoming sequel to [4]; this especially applies to Theorem 1.14.

The paper is organized as follows. In order to make it self-contained, we begin Section 1 by recalling the necessary background from [5]; in particular, we reproduce the construction of generalized associahedra as simplicial fans. We then state our main results. Section 2 describes their proof modulo three key statements: Theorem 1.14, Theorem 1.17, and Lemma 2.5. These are proved, respectively, in Sections 3, 4 and 5.

Acknowledgments We are grateful to Bernd Sturmfels and Jürgen Bokowski for supplying us, on the prehistoric stage of this project, in June 2000, with the evidence that the generalized associahedron of type D_4 can indeed be realized as a convex polytope. Additional experimental evidence (including the exceptional types E_6, E_7, E_8) was later obtained with the help of the software `porta`.

We are happy to dedicate this paper to Robert Moody, and are grateful for his support and encouragement since the early stages of this project. Some of the ideas presented here were reported for the first time at the conference "Aspects of Symmetry" held in his honor at The Banff Centre in August 2001.

1 Main Results

Let Φ be a rank n finite root system with the set of simple roots $\Pi = \{\alpha_i : i \in I\}$ and the set of positive roots $\Phi_{>0}$. Let $Q = \mathbb{Z}\Pi$ denote the root lattice and $Q_{\mathbb{R}}$ the ambient real vector space. Let W be the Weyl group of Φ. It is generated by the simple reflections s_i, $i \in I$; they act on simple roots by

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i,$$

where $A = (a_{ij})_{i,j\in I}$ is the Cartan matrix of Φ. Let $w_\circ$ denote the longest element of W.

Without loss of generality, from this point on we assume that Φ is irreducible. Then the Coxeter graph associated to Φ is a tree; recall that this graph has the index set I as the set of vertices, with i and j joined by an edge whenever $a_{ij} < 0$. In particular, the Coxeter graph is bipartite; the two parts $I_+, I_- \subset I$ are determined uniquely up to renaming. The *sign function* $\varepsilon\colon I \to \{+, -\}$ is defined by

$$\varepsilon(i) = \begin{cases} + & \text{if } i \in I_+; \\ - & \text{if } i \in I_-. \end{cases}$$

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For $\alpha \in Q_{\mathbb{R}}$, we denote by $[\alpha : \alpha_i]$ the coefficient of α_i in the expansion of α in the basis Π. Let τ_+ and τ_- denote the piecewise-linear automorphisms of $Q_{\mathbb{R}}$ given by

$$[\tau_\varepsilon \alpha : \alpha_i] = \begin{cases} -[\alpha : \alpha_i] - \sum_{j \neq i} a_{ij} \max([\alpha : \alpha_j], 0) & \text{if } i \in I_\varepsilon; \\ [\alpha : \alpha_i] & \text{otherwise.} \end{cases} \tag{1.1}$$

Let $\Phi_{\geq -1} = \Phi_{>0} \cup (-\Pi)$. It is easy to see that each of τ_+ and τ_- is an involution that preserves the set $\Phi_{\geq -1}$. In fact, the action of τ_+ and τ_- in $\Phi_{\geq -1}$ can be described as follows:

$$\tau_\varepsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i, i \in I_{-\varepsilon}; \\ \prod_{i \in I_\varepsilon} s_i(\alpha) & \text{otherwise.} \end{cases} \tag{1.2}$$

(The product $\prod_{i \in I_\varepsilon} s_i$ is well defined since its factors commute). To illustrate, consider the type A_2, with $I_+ = \{1\}$ and $I_- = \{2\}$. Then

$$\begin{array}{ccccccccc} -\alpha_1 & \overset{\tau_+}{\longleftrightarrow} & \alpha_1 & \overset{\tau_-}{\longleftrightarrow} & \alpha_1 + \alpha_2 & \overset{\tau_+}{\longleftrightarrow} & \alpha_2 & \overset{\tau_-}{\longleftrightarrow} & -\alpha_2. \\ \circlearrowright & & & & & & & & \circlearrowright \\ \tau_- & & & & & & & & \tau_+ \end{array} \tag{1.3}$$

Theorem 1.1 **([5, Theorem 2.6])** *Every $\langle \tau_-, \tau_+ \rangle$-orbit in $\Phi_{\geq -1}$ has a non-empty intersection with $(-\Pi)$. Furthermore, the correspondence $\Omega \mapsto \Omega \cap (-\Pi)$ is a bijection between the $\langle \tau_-, \tau_+ \rangle$-orbits in $\Phi_{\geq -1}$ and the $\langle -w_\circ \rangle$-orbits in $(-\Pi)$.*

According to [5, Section 3.1], there exists a unique function $(\alpha, \beta) \mapsto (\alpha \parallel \beta)$ on $\Phi_{\geq -1} \times \Phi_{\geq -1}$ with nonnegative integer values, called the *compatibility degree*, such that

$$(-\alpha_i \parallel \alpha) = \max([\alpha : \alpha_i], 0) \tag{1.4}$$

for any $i \in I$ and $\alpha \in \Phi_{\geq -1}$, and

$$(\tau_\varepsilon \alpha \parallel \tau_\varepsilon \beta) = (\alpha \parallel \beta) \tag{1.5}$$

for any $\alpha, \beta \in \Phi_{\geq -1}$ and any sign ε. We say that α and β are *compatible* if $(\alpha \parallel \beta) = 0$. (This is equivalent to $(\beta \parallel \alpha) = 0$ by [5, Proposition 3.3.2].)

The simplicial complex $\Delta(\Phi)$ (a *generalized associahedron*) has $\Phi_{\geq -1}$ as the set of vertices; its simplices are the subsets of mutually compatible elements in $\Phi_{\geq -1}$. The maximal simplices of $\Delta(\Phi)$ are called *clusters*.

Theorem 1.2 **([5, Theorems 1.8, 1.10])** *All clusters are of the same size n, i.e., the simplicial complex $\Delta(\Phi)$ is pure of dimension $n - 1$. Moreover, each cluster is a $\mathbb{Z}$-basis of the root lattice Q. The simplicial cones generated by the clusters form a complete simplicial fan in $Q_{\mathbb{R}}$: the interiors of these cones are mutually disjoint, and they tile the entire space $Q_{\mathbb{R}}$.*

Corollary 1.3 (**[5, Theorem 3.11]**) *Every vector γ in the root lattice Q has a unique cluster expansion, i.e., can be expressed uniquely as a nonnegative linear combination of mutually compatible roots from $\Phi_{\geq -1}$.*

An efficient algorithm for computation of cluster expansions is presented in Section 5.4.

By a common abuse of notation, we denote the simplicial fan in Theorem 1.2 by $\Delta(\Phi)$, since it provides a geometric realization for the (spherical) simplicial complex $\Delta(\Phi)$.

Our main result is the following theorem that confirms Conjecture 1.12 from [5].

Theorem 1.4 *The simplicial fan $\Delta(\Phi)$ is the normal fan of a simple n-dimensional convex polytope.*

The type A_2 case of Theorem 1.4 is illustrated in Figure 1.

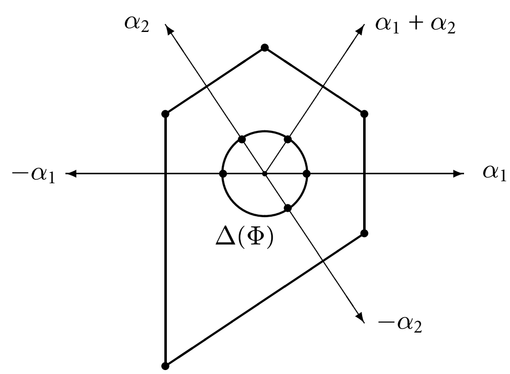

Figure 1: The complex $\Delta(\Phi)$ and the corresponding polytope in type A_2

Theorem 1.4 implies in particular the following statement conjectured in [5, Conjecture 1.13]: the complex $\Delta(\Phi)$, viewed as a poset under reverse inclusion, is the face lattice of a simple n-dimensional convex polytope. As pointed out in [5], the type A and type B (or, equivalently, type C) cases of this statement were known before: the corresponding polytopes are, respectively, the Stasheff polytope, or *associahedron* (see, *e.g.*, [13, 8] or [6, Chapter 7]) and the Bott-Taubes polytope, or *cyclohedron* (see [1, 9, 12]).

To make Theorem 1.4 more specific, let us recall some terminology and notation related to normal fans of convex polytopes (*cf.*, *e.g.*, [15, Example 7.3]); we shall only need the special case when a polytope is *simple*. Let P be a full-dimensional simple convex polytope in a real vector space V of dimension n. The *support function* of P is a real-valued function F on the dual vector space V^* given by

$$F(\gamma) = \max_{\varphi \in P} \langle \gamma, \varphi \rangle. \tag{1.6}$$

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The *normal fan* $\mathcal{N}(P)$ is a complete simplicial fan in the dual space V^* whose maximal (*i.e.*, full-dimensional) cones are the *domains of linearity* for F. More precisely, these cones correspond to the vertices of P as follows: each vertex φ of P gives rise to the cone

$$\{\gamma \in V^* : F(\gamma) = \langle \gamma, \varphi \rangle\}.$$

We prove Theorem 1.4 by explicitly describing a class of support functions F whose domains of linearity are the maximal cones of the fan $\Delta(\Phi)$. Any such function is uniquely determined by its restriction to a set of representatives of 1-dimensional cones in $\Delta(\Phi)$. A natural choice of such a set is $\Phi_{\geq -1}$. By Theorem 1.1, $\langle \tau_+, \tau_- \rangle$-invariant functions on $\Phi_{\geq -1}$ are naturally identified with $\langle -w_\circ \rangle$-invariant functions on $-\Pi$. With this in mind, we state the following refinement of Theorem 1.4.

Theorem 1.5 *Suppose that a function* $F\colon -\Pi \to \mathbb{R}$ *satisfies two conditions:*

$$F\big(w_\circ(\alpha_i)\big) = F(-\alpha_i) \quad \textit{for all } i \in I; \tag{1.7}$$

$$\sum_{i \in I} a_{ij} F(-\alpha_i) > 0 \quad \textit{for all } j \in I. \tag{1.8}$$

Then its unique $\langle \tau_+, \tau_- \rangle$*-invariant extension to* $\Phi_{\geq -1}$ *(also denoted by* F*) extends by linearity to the support function of a simple convex polytope with normal fan* $\Delta(\Phi)$*.*

Let the $\alpha_i^\vee$, for $i \in I$, be the simple coroots for Φ, *i.e.*, the simple roots of the dual root system $\Phi^\vee$ in $Q_\mathbb{R}^*$; recall that they are given by $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$.

Remark 1.6 For any $\langle -w_\circ \rangle$-invariant regular dominant coweight $\lambda^\vee \in Q_\mathbb{R}^*$, the function $F(-\alpha_i) = [\lambda^\vee : \alpha_i^\vee]$ satisfies conditions (1.7)–(1.8). Indeed, the left-hand side in (1.8) is then equal to $\langle \lambda^\vee, \alpha_j \rangle$, and the positivity of all these values is precisely what makes $\lambda^\vee$ regular dominant. In particular, one can take $\lambda^\vee = \rho^\vee$, the half-sum of all positive coroots; for this choice, the left-hand side of each inequality in (1.8) is equal to 1.

Remark 1.7 In view of [7, Theorem 4.3], condition (1.8) in Theorem 1.5 implies that $F(-\alpha_i) > 0$ for all $i \in I$.

Remark 1.8 One can prove the following converse of Theorem 1.5: if a $\langle \tau_+, \tau_- \rangle$-invariant function F on $\Phi_{\geq -1}$ extends by linearity to the support function of a simple convex polytope with normal fan $\Delta(\Phi)$, then its restriction to $(-\Pi)$ satisfies conditions (1.7)–(1.8).

The definition (1.6) of a support function implies the following explicit geometric realizations of generalized associahedra.

Corollary 1.9 *Let $F\colon -\Pi \to \mathbb{R}$ be a function satisfying the conditions in Theorem 1.5. Then its unique $\langle \tau_+, \tau_- \rangle$-invariant extension to $\Phi_{\geq -1}$ (also denoted by F) defines the following simple convex polytope P in $Q^*_{\mathbb{R}}$ whose normal fan is $\Delta(\Phi)$ and whose support function is (the piecewise-linear extension of) F:*

1. *For a cluster C in $\Phi_{\geq -1}$, let $\varphi_C \in Q^*_{\mathbb{R}}$ be the (unique) linear form such that $F(\alpha) = \langle \varphi_C, \alpha \rangle$ for $\alpha \in C$. The vertices of P are the points φ_C for all clusters C.*
2. *The minimal system of linear inequalities defining P is*

$$\langle \varphi, \alpha \rangle \leq F(\alpha), \quad \textit{for all } \alpha \in \Phi_{\geq -1}. \tag{1.9}$$

Remark 1.10 Let us represent a point $\varphi \in Q^*_{\mathbb{R}}$ by an n-tuple $(z_j = \langle \varphi, \alpha_j \rangle)_{j \in I}$. In these coordinates, Corollary 1.9.2 takes the following form: the generalized associahedron is given inside the real affine space $\mathbb{R}^I$ by the set of linear inequalities

$$\sum_j [\alpha : \alpha_j] z_j \leq F(\alpha), \quad \text{for all } \alpha \in \Phi_{\geq -1}. \tag{1.10}$$

(*cf.* (1.9)). Here, as before, F is an arbitrary $\langle \tau_+, \tau_- \rangle$-invariant function on $\Phi_{\geq -1}$ satisfying (1.8).

In the examples below, we use the numeration of simple roots from [2].

Example 1.11 In type A_2, there is only one $\langle \tau_+, \tau_- \rangle$-orbit, so F must be constant; say, $F(\alpha) = c$ for all α. Condition (1.8) requires that $c > 0$. Inequalities (1.10) then become

$$\max(-z_1, -z_2, z_1, z_2, z_1 + z_2) \leq c,$$

defining a pentagon (*cf.* Figure 1).

In type A_3, there are two $\langle \tau_+, \tau_- \rangle$-orbits:

$$\{-\alpha_1, -\alpha_3, \alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\} \quad \text{and} \quad \{-\alpha_2, \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\};$$

let c_1 and c_2 be the corresponding values of F. Condition (1.8) takes the form

$$0 < c_1 < c_2 < 2c_1.$$

For c_1 and c_2 satisfying these inequalities, the associahedron of type A_3 can be defined by the corresponding version of (1.10):

$$\max(-z_1, -z_3, z_1, z_3, z_1 + z_2, z_2 + z_3) \leq c_1$$
$$\max(-z_2, z_2, z_1 + z_2 + z_3) \leq c_2, .$$

In particular, choosing $c_i = [\rho^\vee : \alpha_i^\vee]$ (see Remark 1.6), we obtain $c_1 = 3/2$ and $c_2 = 2$. The corresponding polytope is shown in Figure 2, where we marked each visible facet by the (positive) root α that defines the corresponding supporting hyperplane $\sum_j [\alpha : \alpha_j] z_j = F(\alpha)$. (The hidden facets correspond in the same way to the negative simple roots.)

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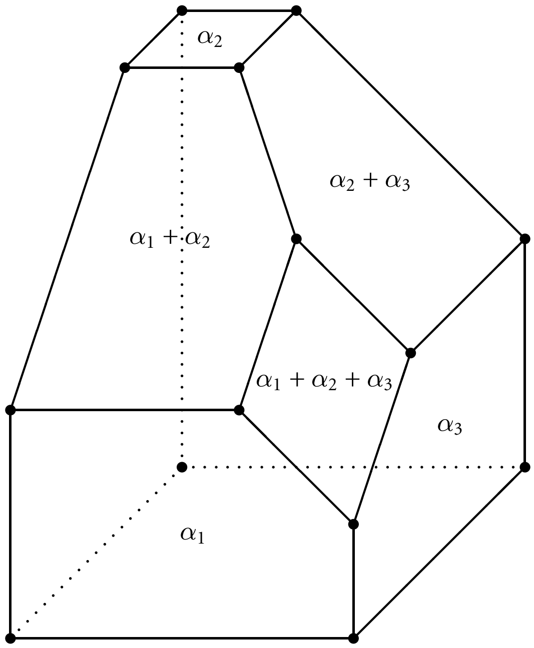

Figure 2: The type A_3 associahedron

Example 1.12 In type C_2, there are two $\langle \tau_+, \tau_- \rangle$-orbits: $\{-\alpha_1, \alpha_1, \alpha_1 + \alpha_2\}$ and $\{-\alpha_2, \alpha_2, 2\alpha_1 + \alpha_2\}$; let c_1 and c_2 be the corresponding values of F. Condition (1.8) takes the form

$$0 < c_1 < c_2 < 2c_1.$$

The generalized associahedron of type C_2 (the rank 2 *cyclohedron*) is the hexagon

$$\max(-z_1, z_1, z_1 + z_2) \leq c_1,$$
$$\max(-z_2, z_2, 2z_1 + z_2) \leq c_2.$$

In type C_3, there are three $\langle \tau_+, \tau_- \rangle$-orbits:

$$\{-\alpha_1, \alpha_1, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\},$$
$$\{-\alpha_2, \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\},$$
$$\{-\alpha_3, \alpha_3, 2\alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3\}.$$

Let c_1, c_2, and c_3 be the corresponding values of F. Condition (1.8) takes the form

$$c_2 < 2c_1, \quad c_1 + c_3 < 2c_2, \quad c_2 < c_3.$$

The generalized associahedron of type C_3 (the rank 3 cyclohedron) is then given by the inequalities

$$\max(-z_1, z_1, z_1 + z_2, z_2 + z_3) \leq c_1$$
$$\max(-z_2, z_2, z_1 + z_2 + z_3, z_1 + 2z_2 + z_3) \leq c_2.$$
$$\max(-z_3, z_3, 2z_2 + z_3, 2z_1 + 2z_2 + z_3) \leq c_3.$$

The choice $c_i = [\rho^\vee : \alpha_i^\vee]$ leads to $(c_1, c_2, c_3) = (5/2, 4, 9/2)$. This polytope is shown in Figure 3, where we followed the same conventions as in Figure 2.

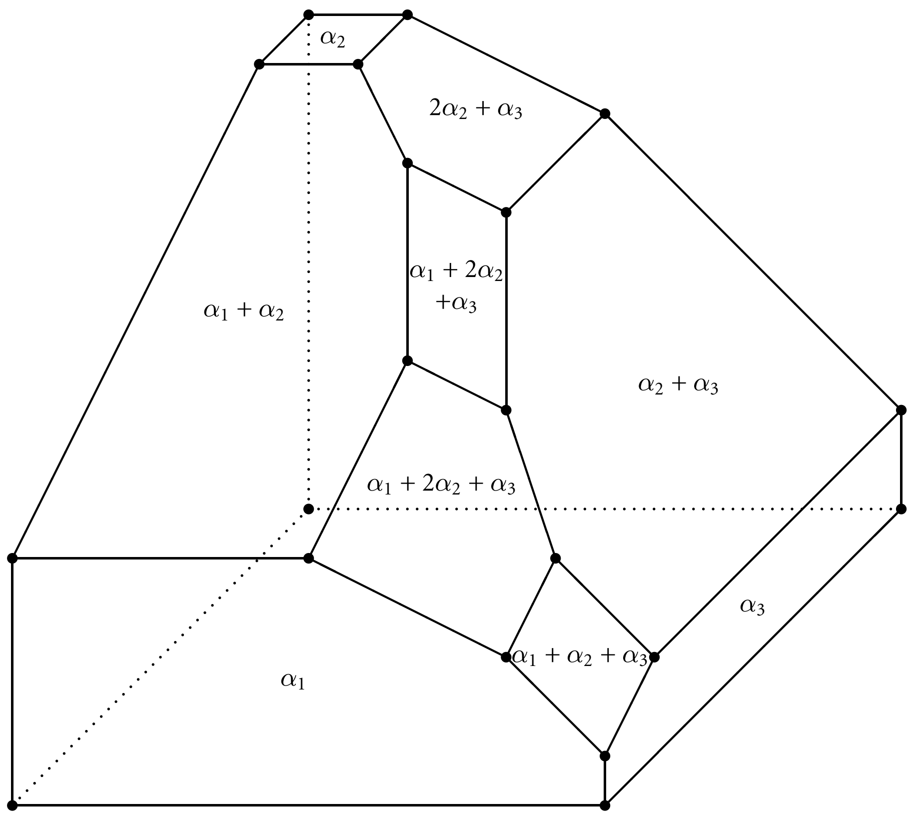

Figure 3: The type C_3 generalized associahedron (cyclohedron)

Our proof of Theorem 1.5 relies on the following results that we find of independent interest.

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Proposition 1.13 *Let $\mathcal{C} = \mathbb{R}_{\geq 0}C$ be the maximal cone of $\Delta(\Phi)$ generated by a cluster C. Every piecewise-linear transformation $\sigma \in \langle \tau_+, \tau_- \rangle$ restricts to a linear transformation on $\mathcal{C}$, sending this cone bijectively to the cone $\sigma(\mathcal{C}) = \mathbb{R}_{\geq 0}\sigma(C)$. Consequently, if $\gamma \in Q$ has the cluster expansion $\gamma = \sum_\beta m_\beta \beta$, then $\sigma(\gamma)$ has the cluster expansion $\sigma(\gamma) = \sum_\beta m_\beta \sigma(\beta)$.*

Proof It suffices to prove this for the generators $\sigma = \tau_\varepsilon$. Then the claim follows from the definition (1.1) once we notice that the components $[\gamma : \alpha_i]$ do not change sign when γ runs over $\mathcal{C}$. ■

Theorem 1.14 *Suppose that $n > 1$, and let α and α' be two elements of $\Phi_{\geq -1}$ such that $(\alpha \parallel \alpha') = (\alpha' \parallel \alpha) = 1$. Then the set*

$$E(\alpha, \alpha') = \left\{ \sigma\left(\sigma^{-1}(\alpha) + \sigma^{-1}(\alpha')\right) : \sigma \in \langle \tau_+, \tau_- \rangle \right\} \tag{1.11}$$

consists of two elements of Q, one of which is $\alpha + \alpha'$, and another will be denoted by $\alpha \uplus \alpha'$. In the special case where $\alpha' = -\alpha_j$, $j \in I$, we have

$$\begin{aligned}(-\alpha_j) \uplus \alpha &= \tau_{-\varepsilon(j)}\left(-\alpha_j + \tau_{-\varepsilon(j)}(\alpha)\right) \\ &= \alpha - \alpha_j + \sum_{i \neq j} a_{ij}\alpha_i.\end{aligned} \tag{1.12}$$

Remark 1.15 If $n = 1$, *i.e.*, Φ is of type A_1 with a unique simple root α_1, then $\{\alpha, \alpha'\} = \{-\alpha_1, \alpha_1\}$, and the group $\langle \tau_+, \tau_- \rangle$ is just the Weyl group $W = \langle s_1 \rangle$. Thus, in this case, the set in Theorem 1.14 consists of a single element $\alpha + \alpha' = 0$. It is then natural to set $\alpha \uplus \alpha' = 0$ as well.

Remark 1.16 For $\alpha' = -\alpha_j \in -\Pi$, the condition $(\alpha \parallel \alpha') = (\alpha' \parallel \alpha) = 1$ is equivalent to

$$[\alpha : \alpha_j] = [\alpha^\vee : \alpha_j^\vee] = 1, \tag{1.13}$$

where $\alpha^\vee$ is the coroot corresponding to α under the natural bijection between Φ and the dual system $\Phi^\vee$. This follows from (1.4) and the property $(\alpha \parallel \beta) = (\beta^\vee \parallel \alpha^\vee)$ established in [5, Proposition 3.3.1].

Theorem 1.17 *Let $\alpha, \alpha' \in \Phi_{\geq -1}$ be such that $(\alpha \parallel \alpha') = (\alpha' \parallel \alpha) = 1$. Assume that $[\alpha \uplus \alpha' : \alpha_i] > 0$ for some $i \in I$. Then $[\alpha + \alpha' : \alpha_i] > 0$.*

2 Proof of Theorem 1.5 (General Layout)

The proof of Theorem 1.5 presented in this section depends on Theorem 1.14, Theorem 1.17, and Lemma 2.5, which will be proved in Sections 3, 4 and 5, respectively.

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2.1 Generalities on Normal Fans

As before, let V be an n-dimensional vector space over $\mathbb{R}$. Not every complete simplicial fan Δ in the dual space V^* is the normal fan of a simple polytope in V. The following lemma provides a useful criterion, which will be our tool in proving Theorem 1.5.

Lemma 2.1 *Let Δ be a complete simplicial fan in V^*. Let $F\colon V^* \to \mathbb{R}$ be a continuous function which is linear within each maximal cone of Δ; as such, F is uniquely determined by its values on a set S of representatives of* 1*-dimensional cones in Δ. Then the following are equivalent:*

(i) *Δ is the normal fan $\mathcal{N}(P)$ of a (unique) full-dimensional simple convex polytope P in V with the support function F.*

(ii) *The values $F(\alpha)$, $\alpha \in S$ satisfy the following system of linear inequalities. For each pair of adjacent maximal cones $\mathcal{C}$ and $\mathcal{C}'$ in Δ, denote $\{\alpha\} = (S \cap \mathcal{C}) - \mathcal{C}'$ and $\{\alpha'\} = (S \cap \mathcal{C}') - \mathcal{C}$, and write the unique (up to a nonzero real multiple) linear dependence between the elements of $S \cap (\mathcal{C} \cup \mathcal{C}')$ in the form*

$$m_\alpha \alpha + m_{\alpha'} \alpha' - \sum_{\beta \in S \cap \mathcal{C} \cap \mathcal{C}'} m_\beta \beta = 0, \tag{2.1}$$

where m_α and $m_{\alpha'}$ are positive real numbers. Then

$$m_\alpha F(\alpha) + m_{\alpha'} F(\alpha') - \sum_{\beta \in S \cap \mathcal{C} \cap \mathcal{C}'} m_\beta F(\beta) > 0. \tag{2.2}$$

Proof Within each maximal cone $\mathcal{C}$ of Δ, the function F is given by $F(\gamma) = \langle \gamma, \varphi_{\mathcal{C}} \rangle$, for some (unique) $\varphi_{\mathcal{C}} \in V$. In view of (1.6), condition (i) is equivalent to

(i′) $F(\gamma) > \langle \gamma, \varphi_{\mathcal{C}} \rangle$ for all maximal cones $\mathcal{C}$ in Δ and all $\gamma \in V^* - \mathcal{C}$.

It is clear that (i′) $\Rightarrow$ (ii): the inequality (2.2) is a special case of the inequality in (i′) for $\gamma = \alpha'$.

To show that (ii) $\Rightarrow$ (i′), take a maximal cone $\mathcal{C}$ in Δ and a point $\gamma \in V^* - \mathcal{C}$. For dimension reasons, there exists a line segment L joining γ with some interior point of $\mathcal{C}$ and not crossing any cone of codimension two or more in Δ. Let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m = \mathcal{C}$ be all maximal cones consecutively crossed by L (so that $\gamma \in \mathcal{C}_1$). Condition (ii) then implies that $\langle \delta, \varphi_{\mathcal{C}_k} \rangle > \langle \delta, \varphi_{\mathcal{C}_{k+1}} \rangle$ for $k = 1, \dots, m-1$ and $\delta \in L \cap (\mathcal{C}_k - \mathcal{C}_{k+1})$. Looking at the restrictions of the linear forms $\varphi_{\mathcal{C}_k}$ onto L, we conclude that

$$F(\gamma) = \langle \gamma, \varphi_{\mathcal{C}_1} \rangle > \langle \gamma, \varphi_{\mathcal{C}_2} \rangle > \cdots > \langle \gamma, \varphi_{\mathcal{C}} \rangle,$$

implying (i′). ∎

Thus a complete simplicial fan Δ is a normal fan for some polytope P if and only if there exists a function $F\colon S \to \mathbb{R}_{>0}$ defined on a given set S of representatives of 1-dimensional cones in Δ that satisfies the linear inequalities in part (ii) of Lemma 2.1.

2.2 Dependences (2.1) and Cluster Expansions

In what follows, Δ will be the fan $\Delta(\Phi)$, with the set of representatives $S = \Phi_{\geq -1}$. To deduce Theorem 1.5 from Lemma 2.1, we need to describe explicitly all linear dependences of the form (2.1).

Lemma 2.2 *Each of the dependences (2.1) for the simplicial fan $\Delta(\Phi)$ can be normalized (that is, multiplied by a positive constant) to have the following features: $(\alpha \parallel \alpha') = (\alpha' \parallel \alpha) = 1$, $m_\alpha = m_{\alpha'} = 1$, and all coefficients m_β are nonnegative integers, i.e., the dependence expresses the cluster expansion of $\alpha + \alpha'$.*

Proof Let $\mathcal{C}$ and $\mathcal{C}'$ be two adjacent maximal cones in $\Delta(\Phi)$ generated by the clusters C and C', respectively. Let $\{\alpha\} = C - C'$ and $\{\alpha'\} = C' - C$. Consider the corresponding dependence (2.1). Since all the participating vectors lie in the lattice Q, we can normalize it so that the coefficients become relatively prime integers. We claim that this normalization has all the features listed in the lemma. First, let us show that $m_\alpha = m_{\alpha'} = 1$. Indeed, by Theorem 1.2, the cluster C is a $\mathbb{Z}$-basis of the root lattice Q, so all the coefficients in the expansion

$$\alpha' = -\frac{m_\alpha}{m_{\alpha'}}\alpha + \sum_{\beta \in C \cap C'} \frac{m_\beta}{m_{\alpha'}}\beta$$

are integers, implying that $m_{\alpha'} = 1$. In view of the obvious symmetry between α and α', we have $m_\alpha = 1$ as well.

Next, let us show that $(\alpha \parallel \alpha') = 1$ (implying that $(\alpha' \parallel \alpha) = 1$, by the above-mentioned symmetry). Since every transformation in the group $\langle \tau_+, \tau_- \rangle$ preserves the compatibility degree and sends clusters to clusters, Theorem 1.1 allows us to assume without loss of generality that $\alpha = -\alpha_i \in -\Pi$. In view of (1.4), we need to show that $[\alpha' - \alpha_i : \alpha_i] = 0$. This follows from the fact that α_i does not occur in any root $\beta \in C \cap C'$ appearing in (2.1); indeed, every such β is compatible with $\alpha = -\alpha_i$.

It remains to show that all coefficients m_β are nonnegative (note that this property does not hold for arbitrary simplicial fans). We will deduce this from the following result.

Lemma 2.3 *Let $\alpha, \alpha' \in \Phi_{\geq -1}$ be such that $(\alpha \parallel \alpha') = (\alpha' \parallel \alpha) = 1$. Then every root β that appears with a positive coefficient in the cluster expansion of $\alpha + \alpha'$ is compatible with both α and α', and also with any root which is itself compatible with both α and α'.*

Proof Let $\gamma \in \Phi_{\geq -1}$ be either one of α and α', or be compatible with α and α'. We need to show that β and γ are compatible. To do this, choose $\sigma \in \langle \tau_+, \tau_- \rangle$ so that $\sigma(\gamma) = -\alpha_i$ for some $i \in I$. It suffices to show that $\sigma(\beta)$ is compatible with $-\alpha_i$. By Proposition 1.13, the root $\sigma(\beta)$ appears with positive coefficient in the cluster expansion of $\sigma(\alpha + \alpha')$. Note that by Theorem 1.14, $\sigma(\alpha + \alpha')$ is equal to either $\sigma(\alpha) + \sigma(\alpha')$ or $\sigma(\alpha) \uplus \sigma(\alpha')$. We need to consider the following two cases:

Case 1 γ is one of α and α', say $\gamma = \alpha$. Then $\sigma(\alpha) = -\alpha_i$, and $[\sigma(\alpha') : \alpha_i] = 1$. Thus $[\sigma(\alpha)+\sigma(\alpha') : \alpha_i] = 0$. By Theorem 1.17, this implies $[\sigma(\alpha)\uplus\sigma(\alpha') : \alpha_i] \leq 0$. It follows that $[\sigma(\beta) : \alpha_i] \leq 0$; hence $\sigma(\beta)$ is compatible with $-\alpha_i$, as desired.

Case 2 γ is compatible with α and α'. Then both $\sigma(\alpha)$ and $\sigma(\alpha')$ are compatible with $-\alpha_i$, that is, $[\sigma(\alpha) : \alpha_i] \leq 0$ and $[\sigma(\alpha') : \alpha_i] \leq 0$. The same argument as in Case 1 then shows that $\sigma(\beta)$ is compatible with $-\alpha_i$, and we are done. ■

Let us finish the proof of Lemma 2.2. By Lemma 2.3, each β that appears with a positive coefficient in the cluster expansion of $\alpha + \alpha'$ is compatible with every root in C and with every root in C'. Hence $\beta \in C \cap C'$, so the cluster expansion of $\alpha + \alpha'$ does indeed coincide with the dependence (2.1). Lemma 2.2 is proved. ■

2.3 Completing the Proof of Theorem 1.5

Combining Lemmas 2.1 and 2.2, we see that Theorem 1.5 becomes a consequence of the following statement.

Lemma 2.4 *Let F be a $\langle\tau_+, \tau_-\rangle$-invariant function on $\Phi_{\geq -1}$ satisfying condition (1.8) in Theorem 1.5. Let $\alpha, \alpha' \in \Phi_{\geq -1}$ be such that $(\alpha \parallel \alpha') = (\alpha' \parallel \alpha) = 1$, and let*

$$\alpha + \alpha' = \sum_\beta m_\beta \beta \tag{2.3}$$

be the cluster expansion of $\alpha + \alpha'$. Then

$$F(\alpha) + F(\alpha') - \sum_\beta m_\beta F(\beta) > 0. \tag{2.4}$$

Using $\langle\tau_+, \tau_-\rangle$-invariance of F, we can substantially reduce the list of linear inequalities to be checked. Namely, we claim that each inequality of the form (2.4) appears already in the special case where $\alpha' \in -\Pi$. Indeed, take any pair (α, α') as in Lemma 2.4. By Theorem 1.1, there exists $\sigma \in \langle\tau_+, \tau_-\rangle$ such that $\sigma(\alpha') = -\alpha_j$ for some $j \in I$. By Proposition 1.13,

$$\sigma(\alpha + \alpha') = \sum_\beta m_\beta \sigma(\beta)$$

is the cluster expansion of $\sigma(\alpha + \alpha')$. On the other hand, Theorem 1.14 implies that

$$\begin{aligned} \sigma(\alpha + \alpha') &\in \{\sigma(\alpha) + \sigma(\alpha'), \sigma(\alpha) \uplus \sigma(\alpha')\} \\ &\in \left\{ \sigma(\alpha) - \alpha_j, \tau_{-\varepsilon(j)}\big(\tau_{-\varepsilon(j)}\sigma(\alpha) - \alpha_j\big) \right\}. \end{aligned}$$

We thus have the following alternative: either

$$\sigma(\alpha) - \alpha_j = \sum_\beta m_\beta \sigma(\beta)$$

is the cluster expansion of $\sigma(\alpha) - \alpha_j$, or

$$\tau_{-\varepsilon(j)}\sigma(\alpha) - \alpha_j = \sum_\beta m_\beta \tau_{-\varepsilon(j)}\sigma(\beta)$$

is the cluster expansion of $\tau_{-\varepsilon(j)}\sigma(\alpha) - \alpha_j$. Each of these cluster expansions is of the form (2.3), and each gives rise to the same inequality (2.4) as the original pair $\{\alpha, \alpha'\}$. Our claim follows.

Summarizing, we have reduced Theorem 1.5 (modulo Theorems 1.14 and 1.17) to the following lemma.

Lemma 2.5 *Let $F\colon -\Pi \to \mathbb{R}$ be a function satisfying conditions (1.7)–(1.8). Let the roots $\alpha \in \Phi_{>0}$ and $\alpha_j \in \Pi$ satisfy (1.13), and let*

$$\alpha - \alpha_j = \sum_\beta m_\beta \beta \tag{2.5}$$

be the cluster expansion of $\alpha - \alpha_j$. Then

$$F(-\alpha_j) + F(\alpha) - \sum_{\beta \in \Phi_{>0}} m_\beta F(\beta) > 0. \tag{2.6}$$

Example 2.6 Take any $j \in I$ and set

$$\alpha = \tau_{-\varepsilon(j)}\tau_{\varepsilon(j)}(-\alpha_j) = \alpha_j - \sum_{i \neq j} a_{ij}\alpha_i.$$

Then (1.13) is satisfied (indeed, we have $\alpha^\vee = \alpha_j^\vee - \sum_{i\neq j} a_{ji}\alpha_i^\vee$). The corresponding inequality (2.6) is just (1.8), so Lemma 2.5 holds in this instance.

Lemma 2.5 is proved in Section 5.

3 Proof of Theorem 1.14

We denote by h the *Coxeter number*, *i.e.*, the order in W of the (Coxeter) element $\prod_{i\in I_-} s_i \cdot \prod_{i\in I_+} s_i$. We shall use the following result; although not explicitly stated in [5], it is an immediate consequence of [5, Theorem 1.4, Proposition 2.5].

Theorem 3.1 *We have*

$$\underbrace{\tau_-\tau_+\tau_-\cdots\tau_\mp\tau_\pm}_{h+2 \text{ factors}} = \underbrace{\tau_+\tau_-\tau_+\cdots\tau_\pm\tau_\mp}_{h+2 \text{ factors}} = -w_\circ. \tag{3.1}$$

Furthermore, if $i \in I_\varepsilon$ is such that $w_\circ(-\alpha_i) \neq \alpha_i$, then the $\langle \tau_-, \tau_+\rangle$-orbit of $-\alpha_i$ contains precisely h positive roots, and they are

$$\tau_\varepsilon(-\alpha_i), \tau_{-\varepsilon}\tau_\varepsilon(-\alpha_i), \ldots, \underbrace{\tau_\pm\cdots\tau_{-\varepsilon}\tau_\varepsilon}_{h \text{ factors}}(-\alpha_i);$$

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if $w_\circ(-\alpha_i) = \alpha_i$, then the $\langle\tau_-, \tau_+\rangle$-orbit of $-\alpha_i$ contains precisely $h/2$ positive roots, and they are

$$\tau_\varepsilon(-\alpha_i), \tau_{-\varepsilon}\tau_\varepsilon(-\alpha_i), \ldots, \underbrace{\tau_\pm \cdots \tau_{-\varepsilon}\tau_\varepsilon}_{h/2 \text{ factors}}(-\alpha_i).$$

For every $\sigma \in \langle\tau_+, \tau_-\rangle$, let us denote

$$\alpha +_\sigma \alpha' = \sigma(\sigma^{-1}\alpha + \sigma^{-1}\alpha'), \tag{3.2}$$

so that

$$E(\alpha, \alpha') = \{\alpha +_\sigma \alpha' : \sigma \in \langle\tau_+, \tau_-\rangle\}$$

(*cf.* (1.11)). This definition implies at once that

$$\alpha +_{\sigma_1\sigma_2} \alpha' = \sigma_1(\sigma_1^{-1}\alpha +_{\sigma_2} \sigma_1^{-1}\alpha') \tag{3.3}$$

for any σ_1 and σ_2. Consequently,

$$E(\sigma\alpha, \sigma\alpha') = \sigma\big(E(\alpha, \alpha')\big) \tag{3.4}$$

for any $\sigma \in \langle\tau_+, \tau_-\rangle$.

The following lemma is immediate from the definitions (1.1) and (1.2).

Lemma 3.2 *If both α and α' are positive roots, or both $\tau_\varepsilon\alpha$ and $\tau_\varepsilon\alpha'$ are positive roots for some sign ε, then $\alpha +_{\tau_\varepsilon} \alpha' = \alpha + \alpha'$.*

To prove the main statement of Theorem 1.14, we need to show that the set $E(\alpha, \alpha')$ consists of two elements whenever $\alpha, \alpha' \in \Phi_{\geq -1}$ satisfy $(\alpha \parallel \alpha') = (\alpha' \parallel \alpha) = 1$. In view of (3.4), we can assume without loss of generality that $\alpha' = -\alpha_j \in -\Pi$; then α is a positive root. We calculate

$$-\alpha_j +_{\tau_{-\varepsilon(j)}} \alpha = \alpha - \alpha_j + \sum_{i \neq j} a_{ij}\alpha_i, \tag{3.5}$$

implying $-\alpha_j +_{\tau_{-\varepsilon(j)}} \alpha \neq -\alpha_j + \alpha$ (here we use the condition $n > 1$) and proving (1.12). It remains to show that

$$E(-\alpha_j, \alpha) = \{-\alpha_j + \alpha, -\alpha_j +_{\tau_{-\varepsilon(j)}} \alpha\}. \tag{3.6}$$

Let us abbreviate

$$\sigma(\varepsilon; l) = \underbrace{\tau_\varepsilon\tau_{-\varepsilon}\tau_\varepsilon \cdots \tau_\pm}_{l \text{ factors}}.$$

We need to show that for any sign ε and any $l \geq 1$, we have

$$-\alpha_j +_{\sigma(\varepsilon;l)} \alpha \in \{-\alpha_j + \alpha, -\alpha_j +_{\tau_{-\varepsilon(j)}} \alpha\}. \tag{3.7}$$

We prove (3.7) by induction on l. The case $l = 1$ is clear since one checks easily that

$$-\alpha_j +_{\tau_{\varepsilon(j)}} \alpha = -\alpha_j + \alpha.$$

So we can assume that $l > 1$, and that our claim holds for all smaller values of l.

Let us dispose of the case $\varepsilon = -\varepsilon(j)$. We have

$$\sigma\big(-\varepsilon(j); l\big) = \tau_{-\varepsilon(j)}\sigma\big(\varepsilon(j); l-1\big).$$

Applying (3.3) to this factorization, we obtain

$$-\alpha_j +_{\sigma(-\varepsilon(j);l)} \alpha = \tau_{-\varepsilon(j)}(-\alpha_j +_{\sigma(\varepsilon(j);l-1)} \tau_{-\varepsilon(j)}\alpha). \tag{3.8}$$

By the induction assumption,

$$-\alpha_j +_{\sigma(\varepsilon(j);l-1)} \tau_{-\varepsilon(j)}\alpha \in \{-\alpha_j + \tau_{-\varepsilon(j)}\alpha, -\alpha_j +_{\tau_{-\varepsilon(j)}} \tau_{-\varepsilon(j)}\alpha\}$$

Applying $\tau_{-\varepsilon(j)}$ and using (3.8), we obtain (3.7).

Let us consider the case $\varepsilon = \varepsilon(j)$. If $l \geq h+2$, then we have $\sigma(\varepsilon; l) = \sigma_1\sigma_2$ with $\sigma_1 = \sigma(\varepsilon; l-h-2)$ and $\sigma_2 = -w_\circ$ (see (3.1)). Applying (3.3), we obtain

$$-\alpha_j +_{\sigma(\varepsilon;l)} \alpha = -\alpha_j +_{\sigma(\varepsilon;l-h-2)} \alpha;$$

here we use an obvious fact that $\beta +_\sigma \beta' = \beta + \beta'$ if σ is a linear transformation. Therefore, we can assume that $2 \leq l \leq h+1$. Then the same argument shows that

$$-\alpha_j +_{\sigma(\varepsilon;l)} \alpha = -\alpha_j +_{\sigma(-\varepsilon;h+2-l)} \alpha. \tag{3.9}$$

In particular, we have

$$-\alpha_j +_{\sigma(\varepsilon(j);h+1)} \alpha = -\alpha_j +_{\tau_{-\varepsilon(j)}} \alpha;$$

Therefore, (3.7) holds for $l = h+1$, and we can assume that $2 \leq l \leq h$.

Now it is time to use Lemma 3.2. Applying (3.3) to the factorization $\sigma(\varepsilon; l) = \sigma(\varepsilon; l-1)\tau_\pm$, we see that

$$-\alpha_j +_{\sigma(\varepsilon;l)} \alpha = -\alpha_j +_{\sigma(\varepsilon;l-1)} \alpha$$

whenever both $\sigma(\varepsilon; l-1)^{-1}(-\alpha_j)$ and $\sigma(\varepsilon; l-1)^{-1}(\alpha)$ are positive roots, or both $\sigma(\varepsilon; l)^{-1}(-\alpha_j)$ and $\sigma(\varepsilon; l)^{-1}(\alpha)$ are positive roots. Thus, we can assume that each of the pairs $\{\sigma(\varepsilon; l-1)^{-1}(-\alpha_j), \sigma(\varepsilon; l-1)^{-1}(\alpha)\}$ and $\{\sigma(\varepsilon; l)^{-1}(-\alpha_j), \sigma(\varepsilon; l)^{-1}(\alpha)\}$ contains a root from $-\Pi$. Since $2 \leq l \leq h$, Theorem 3.1 implies that both roots $\sigma(\varepsilon; l-1)^{-1}(-\alpha_j)$ and $\sigma(\varepsilon; l)^{-1}(-\alpha_j)$ are positive. Therefore, both $\sigma(\varepsilon; l-1)^{-1}(\alpha)$ and $\sigma(\varepsilon; l)^{-1}(\alpha)$ must belong to $-\Pi$. This is only possible when $\alpha = \sigma(\varepsilon; l-1)(-\alpha_i)$ for some $i \in I$, and the last factor in $\sigma(\varepsilon; l-1)$ is $\tau_{\varepsilon(i)}$. To complete the proof, it suffices to show that, in these particular circumstances, we have

$$-\alpha_j +_{\sigma(\varepsilon;l)} \alpha = -\alpha_j +_{\tau_{-\varepsilon}} \alpha \tag{3.10}$$

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(remember that $\varepsilon = \varepsilon(j)$). Using (3.9) and (3.3), we obtain

$$-\alpha_j +_{\sigma(\varepsilon;l)} \alpha = \tau_{-\varepsilon}(-\alpha_j +_{\sigma(\varepsilon;h+1-l)} \alpha'),$$

where

$$\alpha' = \tau_{-\varepsilon}(\alpha) = \sigma(-\varepsilon;l)(-\alpha_i).$$

Therefore, (3.10) can be rewritten as

$$-\alpha_j +_{\sigma(\varepsilon;h+1-l)} \alpha' = -\alpha_j + \alpha'.$$

We prove this by iterating Lemma 3.2: all we need is to show that all the roots $\sigma(\varepsilon;k)^{-1}(-\alpha_j)$ and $\sigma(\varepsilon;k)^{-1}(\alpha')$ for $1 \le k \le h-l$ are positive. The fact that every $\sigma(\varepsilon;k)^{-1}(-\alpha_j)$ is positive follows from the second part of Theorem 3.1. As for $\sigma(\varepsilon;k)^{-1}(\alpha')$, this root is equal to $\sigma\big(\varepsilon(i);k+l\big)^{-1}(-\alpha_i)$, so Theorem 3.1 assures its positivity as well. This completes the proof of Theorem 1.14. ■

4 Proof of Theorem 1.17

We proceed case by case. For the classical types, we use the planar geometric realizations of $\Phi_{\ge -1}$ given in [5, Section 3.5].

4.1 Type A_n

We identify the set I in a standard way with $[1,n] = \{1,\dots,n\}$. The positive roots are

$$\alpha[i,j] = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \tag{4.1}$$

for $1 \le i \le j \le n$.

Lemma 4.1 *Assume that $(-\alpha_j \parallel \alpha) = 1$; say, $\alpha = \alpha[i,k]$ for some i and k with $1 \le i \le j \le k \le n$. Then*

$$(-\alpha_j) + \alpha[i,k] = \alpha[i,j-1] + \alpha[j+1,k], \tag{4.2}$$

$$(-\alpha_j) \uplus \alpha[i,k] = \alpha[i,j-2] + \alpha[j+2,k], \tag{4.3}$$

with the convention that $\alpha[\ell,\ell-1] = 0$ and $\alpha[\ell,\ell-2] = -\alpha_{\ell-1}$ (the latter formula yields 0 for $\ell = 1$ or $\ell = n+2$). ■

Proof Formulas (4.2)–(4.3) are easily checked using (1.12) and (4.1). ■

As in [5, Section 3.5], we identify $\Phi_{\ge -1}$ with the set of all diagonals of a regular $(n+3)$-gon. Under this identification, the roots in $-\Pi$ correspond to the diagonals on the "snake" shown in Figure 4. Non-crossing diagonals represent compatible roots, while crossing diagonals correspond to roots whose compatibility degree is 1. (Here

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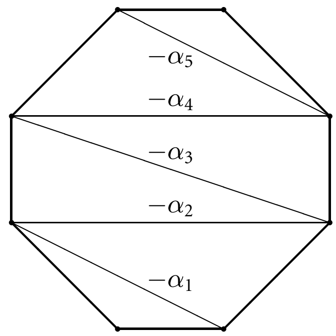

Figure 4: The "snake" in type A_5

and in the sequel, two diagonals are called *crossing* if they are distinct and have a common interior point.) Thus each root $\alpha[i, j]$ corresponds to the unique diagonal that crosses precisely the diagonals $-\alpha_i, -\alpha_{i+1}, \dots, -\alpha_j$ from the snake (*cf.* (1.4)). The group $\langle\tau_+, \tau_-\rangle$ becomes the group of all symmetries of the $(n+3)$-gon. See [5] for further details.

Lemma 4.2 *Suppose the roots $\alpha, \alpha' \in \Phi_{\geq -1}$ correspond to two crossing diagonals. Then one of the vectors $\alpha + \alpha'$ and $\alpha \uplus \alpha'$ has the cluster expansion $\beta_1 + \beta_3$, while another has the cluster expansion $\beta_2 + \beta_4$, where the roots $\beta_1, \dots, \beta_4$ correspond to the sides of the quadrilateral with diagonals α and α', as shown in Figure 5, with the convention that $\beta_i = 0$ if the corresponding side is not a diagonal.*

Furthermore, if $\alpha' \in -\Pi$ (say, $\alpha' = -\alpha_j$), then formulas (4.2)–(4.3) provide cluster expansions of $\alpha + \alpha'$ and $\alpha \uplus \alpha'$.

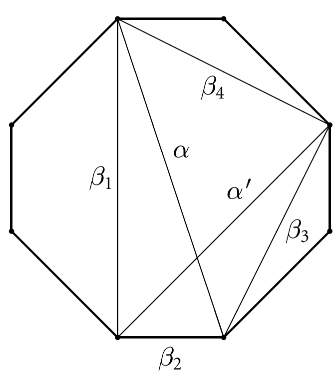

Figure 5: Lemma 4.2

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Proof Applying if needed a symmetry of the $(n+3)$-gon (which preserves cluster expansions by Proposition 1.13), we can assume that $\alpha' = -\alpha_j$ for some j, so that Lemma 4.1 applies. By inspection, the pairs $\{\alpha[i, j-1], \alpha[j+1, k]\}$ and $\{\alpha[i, j-2], \alpha[j+2, k]\}$ that appear in the right hand sides of (4.2) and (4.3), are precisely the pairs $\{\beta_1, \beta_3\}$ and $\{\beta_2, \beta_4\}$ in the lemma. In particular, the elements of each pair are non-crossing, so both (4.2) and (4.3) provide cluster expansions for respective left-hand sides. ■

To complete the proof of Theorem 1.17 for the type A_n, let α, α', and $\beta_1, \ldots, \beta_4$ have the same meaning as in Lemma 4.2, and suppose that $[\alpha \uplus \alpha' : \alpha_i] > 0$ for some $i \in I$. Taking into account the cluster expansion $\alpha \uplus \alpha' = \beta_2 + \beta_4$, we may assume that $[\beta_4 : \alpha_i] > 0$. Thus, the diagonal corresponding to $-\alpha_i$ crosses the one corresponding to β_4. Since β_4 corresponds to a side of the quadrilateral with diagonals α and α', it is geometrically obvious that the diagonal $-\alpha_i$ crosses at least one of α and α'. It follows that $[\alpha + \alpha' : \alpha_i] > 0$, and we are done. ■

4.2 Types B_n and C_n

Let Φ be a root system of type B_n or C_n. We identify the set I in a standard way with $[1, n]$. To treat both cases at the same time, we set $d = 1$ for Φ of type B_n, and $d = 2$ for Φ of type C_n. Our convention for the Cartan matrices is different from the one in [2] but agrees with that in [7]: we have $a_{n-1,n} = -d$ and $a_{n,n-1} = -2/d$. The positive roots of Φ can be found in [2]: they are

$$\alpha[i,k] = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k \quad (1 \le i \le k < n), \tag{4.4}$$

$$\alpha[i,k]_+ = \alpha_i + \cdots + \alpha_k + 2(\alpha_{k+1} + \cdots + \alpha_{n-1}) + \frac{2}{d}\alpha_n \quad (1 \le i \le k < n), \tag{4.5}$$

$$\alpha[i] = d(\alpha_i + \cdots + \alpha_{n-1}) + \alpha_n \quad (1 \le i \le n). \tag{4.6}$$

We can now formulate a type B_n/C_n counterpart of Lemma 4.1.

Lemma 4.3 *Suppose that $(-\alpha_j \parallel \alpha) = (\alpha \parallel -\alpha_j) = 1$. Then $-\alpha_j + \alpha$ and $-\alpha_j \uplus \alpha$ are given by one of the following formulas:*

$$(-\alpha_j) + \alpha[i,k] = \alpha[i, j-1] + \alpha[j+1, k] \quad (1 \le i \le j \le k < n), \tag{4.7}$$

$$(-\alpha_j) \uplus \alpha[i,k] = \alpha[i, j-2] + \alpha[j+2, k] \quad (1 \le i \le j \le k < n), \tag{4.8}$$

$$(-\alpha_j) + \alpha[i,k]_+ = \alpha[i, j-1] + \alpha[j+1, k]_+ \quad (1 \le i \le j \le k < n), \tag{4.9}$$

$$(-\alpha_j) \uplus \alpha[i,k]_+ = \alpha[i, j-2] + \alpha[j+2, k]_+ \quad (1 \le i \le j \le k < n), \tag{4.10}$$

$$(-\alpha_n) + \alpha[i] = d\alpha[i, n-1] \quad (1 \le i \le n), \tag{4.11}$$

$$(-\alpha_n) \uplus \alpha[i] = d\alpha[i, n-2] \quad (1 \le i \le n), \tag{4.12}$$

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where in the right-hand sides we use the following conventions:

$$\alpha[\ell, \ell-1] = 0 \quad (1 \le \ell \le n),$$
$$\alpha[\ell, \ell-2] = -\alpha_{\ell-1} \quad (1 < \ell \le n),$$
$$\alpha[1, -1] = 0,$$
$$\alpha[n+1, n-1] = -\frac{2}{d}\alpha_n,$$
$$\alpha[\ell, \ell-1]_+ = \frac{2}{d}\alpha[\ell] \quad (2 < \ell \le n),$$
$$\alpha[\ell, \ell-2]_+ = \alpha[\ell-1, \ell-1]_+ \quad (2 < \ell \le n),$$
$$\alpha[n+1, n-1]_+ = 0.$$

Proof The equalities (4.7)–(4.12) are checked using definitions (4.4)–(4.6) and formula (1.12). ■

Let Θ denote the 180° rotation of a regular $(2n+2)$-gon. There is a natural action of Θ on the diagonals of the $(2n+2)$-gon. Each orbit of this action is either a diameter (*i.e.*, a diagonal connecting antipodal vertices) or an unordered pair of centrally symmetric non-diameter diagonals of the $(2n+2)$-gon. Following [5], we identify $\Phi_{\ge -1}$ with the set of these orbits. Under this identification, each of the roots $-\alpha_i$ for $i = 1, \dots, n-1$ is represented by a pair of diagonals on the "snake" shown in Figure 6, whereas $-\alpha_n$ is identified with the only diameter on the snake. Two Θ-orbits represent compatible roots if and only if the diagonals they involve do not cross each other. More generally, for $\alpha, \beta \in \Phi_{\ge -1}$ in type B_n (resp., C_n), the compatibility degree $(\alpha \parallel \beta)$ is equal to the number of crossings of one of the diagonals representing α (resp., β) by the diagonals representing β (resp., α). Thus, each positive root $\beta = \sum_i b_i \alpha_i$ in type B_n (resp., C_n) is represented by the unique Θ-orbit such that every diagonal representing $-\alpha_i$ (resp., β) crosses the diagonals representing β (resp., $-\alpha_i$) at b_i points. In particular, the $n+1$ diameters of the $(2n+2)$-gon represent the roots $\alpha[i]$, $1 \le i \le n$, together with $-\alpha_n$. The group $\langle \tau_+, \tau_- \rangle$ is isomorphic to the quotient of the group of symmetries of the $(2n+2)$-gon modulo its center, which is generated by the involution Θ. See [5, Section 3.5] for a more detailed description of this construction.

We have the following type B_n/C_n analogue of Lemma 4.2.

Lemma 4.4 *Suppose the roots $\alpha, \alpha' \in \Phi_{\ge -1}$ are such that $(\alpha \parallel \alpha') = (\alpha' \parallel \alpha) = 1$. Then two cases are possible. (See Figure 7.)*

1. *Each of α and α' is represented by a pair of diagonals, and they cross at exactly two centrally symmetric points. Pick two crossing diagonals among these four, and let $\beta_1, \dots, \beta_4$ be the roots that correspond to the sides of the quadrilateral whose vertices are the endpoints of these diagonals. Then one of the vectors $\alpha + \alpha'$ and $\alpha \uplus \alpha'$ has the cluster expansion $b_1\beta_1 + b_2\beta_3$, while another has the cluster expansion $b_2\beta_2 + b_4\beta_4$, where $b_i = \frac{2}{d}$ if the corresponding side is a diameter, $b_i = 1$ if it is a non-diameter diagonal, and $b_i = 0$ if it lies on the perimeter.*

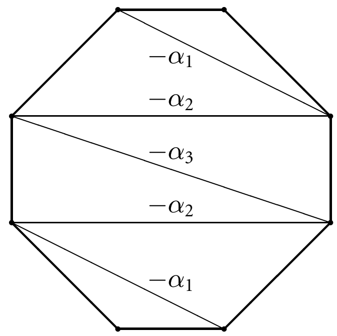

Figure 6: The "snake" for the types B_3 and C_3

2. *Each of α and α' is represented by a diameter. Let β_1 and β_2 be the roots that correspond to the pairs of opposite sides of the rectangle whose diagonals are these diameters. Then one of the vectors $\alpha + \alpha'$ and $\alpha \uplus \alpha'$ has the cluster expansion $d\beta_1$, while another has the cluster expansion $d\beta_2$.*

If, in addition, $\alpha' \in -\Pi$ (say, $\alpha' = -\alpha_j$), then formulas (4.7)–(4.12) provide the cluster expansions of $\alpha + \alpha'$ and $\alpha \uplus \alpha'$.

Proof The proof is analogous to the type A case, with Lemma 4.3 replacing Lemma 4.1. In fact, all we need is (4.7)–(4.8) and (4.11)–(4.12), since one can use the $\langle \tau_+, \tau_- \rangle$ action to transform any pair (α, α') in the lemma into one of these two special positions. ■

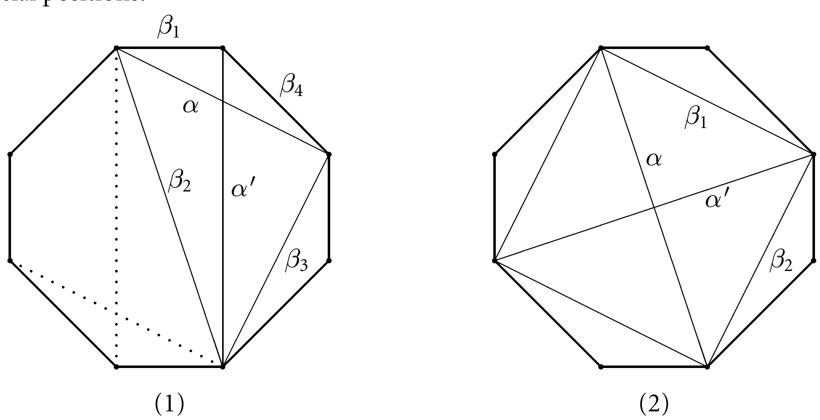

Figure 7: Lemma 4.4

The proof of Theorem 1.17 for the type B_n/C_n is exactly the same as in type A_n, with Lemma 4.4 playing the role of Lemma 4.2. ■

4.3 Type D_n

Let Φ be the root system of type D_n for some $n \geq 4$. We choose $I = [1, n-1] \cup \{\overline{n-1}\}$ as an indexing set; see Figure 8.

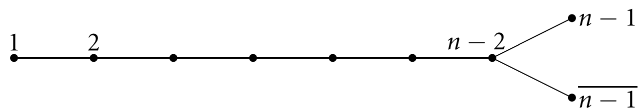

Figure 8: Coxeter graph of type D_n

The positive roots of Φ can be found in [2] (replace α_n by $\alpha_{\overline{n-1}}$); they are:

$$\alpha[i,k] = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k \quad (1 \leq i \leq k < n),$$

$$\alpha[i,k]_+ = \alpha_i + \cdots + \alpha_k + 2(\alpha_{k+1} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_{\overline{n-1}}$$
$$(1 \leq i \leq k < n-1),$$

$$\alpha[i,n-1]_+ = (\alpha_i + \cdots + \alpha_{n-2}) + \alpha_{\overline{n-1}} \quad (1 \leq i < n).$$

We next state the type D_n analogue of Lemmas 4.1 and 4.3. The proof is omitted.

Lemma 4.5 *Suppose that $(-\alpha_j \parallel \alpha) = 1$. Then $-\alpha_j + \alpha$ and $-\alpha_j \uplus \alpha$ are given by one of the following formulas, subject to conventions (4.25)–(4.34) below.*

Case 1 *For $1 \leq i \leq j \leq k < n$ with $j \leq n-2$,*

$$(-\alpha_j) + \alpha[i,k] = \alpha[i,j-1] + \alpha[j+1,k], \tag{4.13}$$

$$(-\alpha_j) \uplus \alpha[i,k] = \alpha[i,j-2] + \alpha[j+2,k], \tag{4.14}$$

$$(-\alpha_j) + \alpha[i,k]_+ = \alpha[i,j-1] + \alpha[j+1,k]_+, \tag{4.15}$$

$$(-\alpha_j) \uplus \alpha[i,k]_+ = \alpha[i,j-2] + \alpha[j+2,k]_+. \tag{4.16}$$

(Note that in some cases, the right-hand sides of the cluster expansions (4.14)–(4.16) may in effect involve three roots[1]*.)*

Case 2 *For $1 \leq i < n$,*

$$(-\alpha_{n-1}) + \alpha[i,n-1] = \alpha[i,n-2], \tag{4.17}$$

$$(-\alpha_{n-1}) \uplus \alpha[i,n-1] = \alpha[i,n-3], \tag{4.18}$$

$$(-\alpha_{\overline{n-1}}) + \alpha[i,n-1]_+ = \alpha[i,n-2], \tag{4.19}$$

$$(-\alpha_{\overline{n-1}}) \uplus \alpha[i,n-1]_+ = \alpha[i,n-3]. \tag{4.20}$$

[1] In the subcase $j = k = n-2$, the second term in the right-hand side of (4.14) is given by (4.33). Likewise, the subcase $j = k \leq n-2$ of (4.15) and the subcase $j+1 = k \leq n-2$ of (4.16) invoke (4.26).

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Case 3 *For* $1 \le i \le k \le n-2$,

$$(4.21)\quad (-\alpha_{n-1}) + \alpha[i,k]_+ = \begin{cases} \alpha[i,n-2] + \alpha[k+1,n-1]_+ & \text{if } n \not\equiv k \bmod 2; \\ \alpha[i,n-1]_+ + \alpha[k+1,n-2] & \text{if } n \equiv k \bmod 2; \end{cases}$$

$$(4.22)\quad (-\alpha_{n-1}) \uplus \alpha[i,k]_+ = \begin{cases} \alpha[k+1,n-3] + \alpha[i,n-1]_+ & \text{if } n \not\equiv k \bmod 2; \\ \alpha[k+1,n-1]_+ + \alpha[i,n-3] & \text{if } n \equiv k \bmod 2; \end{cases}$$

$$(4.23)\quad (-\alpha_{\overline{n-1}}) + \alpha[i,k]_+ = \begin{cases} \alpha[i,n-2] + \alpha[k+1,n-1] & \text{if } n \not\equiv k \bmod 2; \\ \alpha[i,n-1] + \alpha[k+1,n-2] & \text{if } n \equiv k \bmod 2; \end{cases}$$

$$(4.24)\quad (-\alpha_{\overline{n-1}}) \uplus \alpha[i,k]_+ = \begin{cases} \alpha[k+1,n-3] + \alpha[i,n-1] & \text{if } n \not\equiv k \bmod 2; \\ \alpha[k+1,n-1] + \alpha[i,n-3] & \text{if } n \equiv k \bmod 2. \end{cases}$$

Conventions used in formulas (4.13)–(4.24):

$$(4.25)\qquad \alpha[\ell,\ell-1] = 0 \quad (1 \le \ell < n),$$

$$(4.26)\qquad \alpha[\ell,\ell-1]_+ = \alpha[\ell,n-1] + \alpha[\ell,n-1]_+ \quad (1 < \ell < n),$$

$$(4.27)\qquad \alpha[n,n-1] = -\alpha_{\overline{n-1}}$$

$$(4.28)\qquad \alpha[n,n-1]_+ = -\alpha_{n-1},$$

$$(4.29)\qquad \alpha[\ell,\ell-2] = -\alpha_{\ell-1} \quad (1 < \ell \le n),$$

$$(4.30)\qquad \alpha[\ell,\ell-2]_+ = \alpha[\ell-1,\ell-1]_+ \quad (2 < \ell < n),$$

$$(4.31)\qquad \alpha[1,-1] = 0,$$

$$(4.32)\qquad \alpha[n+1,n-1] = 0,$$

$$(4.33)\qquad \alpha[n,n-2] = (-\alpha_{n-1}) + (-\alpha_{\overline{n-1}}),$$

$$(4.34)\qquad \alpha[n,n-2]_+ = 0.$$

Let us recall the geometric representation of the set $\Phi_{\ge -1}$ given in [5, Section 3.5]. Consider the set of diagonals in a regular $2n$-gon, in which each diameter can be of one of two different "colors." The 180° rotation Θ naturally acts on this set. We represent each root in $\Phi_{\ge -1}$ by a Θ-orbit. The negative simple roots form a "type D snake" shown in Figure 9. Two Θ-orbits represent compatible roots if and only if the diagonals they involve do not cross each other; here we use the following convention:

(4.35) diameters of the same color do not cross each other.

More generally, for $\alpha, \beta \in \Phi_{\ge -1}$, the compatibility degree $(\alpha \parallel \beta)$ is equal to the number of Θ-orbits in the set of crossing points between the diagonals representing α and β (again, with the convention (4.35)). Each positive root $\beta = \sum_i b_i \alpha_i$ is then represented by the unique Θ-orbit such that the diagonals representing β cross the diagonals representing $-\alpha_i$ at b_i pairs of centrally symmetric points (counting an intersection of two diameters of different color and location as one such pair). In particular, the $2n$ colored diameters of the $2n$-gon represent the roots $\alpha[i, n-1]$ and $\alpha[i, n-1]_+$, for $1 \le i < n$, together with $-\alpha_{n-1}$ and $-\alpha_{\overline{n-1}}$. Under this identification, the element $\tau_-\tau_+$ acts by rotating the $2n$-gon $\frac{180^\circ}{n}$ degrees and changing the colors of all diameters. See [5, Section 3.5] for further details.

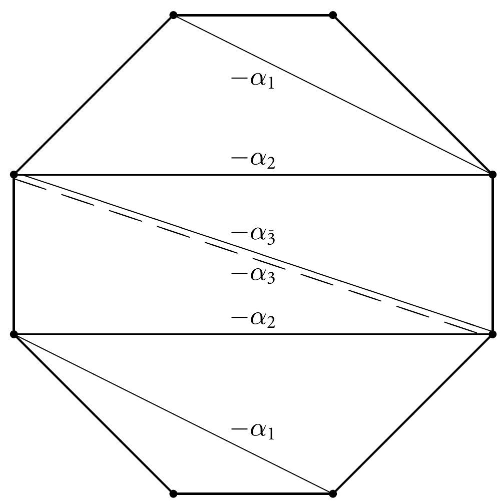

Figure 9: Representing the roots in $-\Pi$ for the type D_4

The type D_n analogue of Lemmas 4.2 and 4.4 is stated below.

Lemma 4.6 *Suppose the roots $\alpha, \alpha' \in \Phi_{\ge -1}$ are such that $(\alpha \parallel \alpha') = 1$. Then four cases are possible. (Refer to Figure 10.)*

1. *Each of α and α' is represented by a pair of diagonals, and they cross at exactly two centrally symmetric points. Pick two crossing diagonals among these four, and consider the quadrilateral whose vertices are their endpoints. First assume that none of the sides of this quadrilateral is a diameter. Let $\beta_1, \dots, \beta_4$ be the corresponding roots. Then one of the vectors $\alpha + \alpha'$ and $\alpha \uplus \alpha'$ has the cluster expansion $\beta_1 + \beta_3$, while another has the cluster expansion $\beta_2 + \beta_4$. (Throughout this lemma, we use the convention that $\beta_i = 0$ if the corresponding side lies on the perimeter.)*
2. *Same situation as above, except one of the sides of the quadrilateral is a diameter. Let $\beta_1, \beta_1' \in \Phi_{\ge -1}$ be the two roots associated with this diameter, and let $\beta_2, \beta_3, \beta_4$ correspond to the remaining sides. Then one of the vectors $\alpha + \alpha'$ and $\alpha \uplus \alpha'$ has the cluster expansion $\beta_1 + \beta_1' + \beta_3$, while another has the cluster expansion $\beta_2 + \beta_4$.*

3. *The roots α and α' are represented by diameters of different color and location. Let β_1 and β_2 be the roots that correspond to the pairs of opposite sides of the rectangle whose diagonals are these diameters. Then one of the vectors $\alpha + \alpha'$ and $\alpha \uplus \alpha'$ has the cluster expansion β_1, while another has the cluster expansion β_2.*
4. *One of α and α' (say, α) is represented by a diameter $[T, T']$ (of any color), while another (say, α') is represented by a pair of diagonals $[P, Q]$ and $[P', Q']$, so that the counter-clockwise order of these six points is P, T, Q, P', T', Q'. Let β_1 and β_2 be the roots that correspond to the diameters $[P, P']$ and $[Q, Q']$ and have the same color as α, and let β_3 and β_4 correspond to $[T, P]$ and $[T, Q]$, respectively. Then one of the vectors $\alpha + \alpha'$ and $\alpha \uplus \alpha'$ has the cluster expansion $\beta_1 + \beta_3$, while another has the cluster expansion $\beta_2 + \beta_4$.*

If, in addition, $\alpha' \in -\Pi$ (say, $\alpha' = -\alpha_j$), then formulas (4.13)–(4.24) provide the cluster expansions of $\alpha + \alpha'$ and $\alpha \uplus \alpha'$.

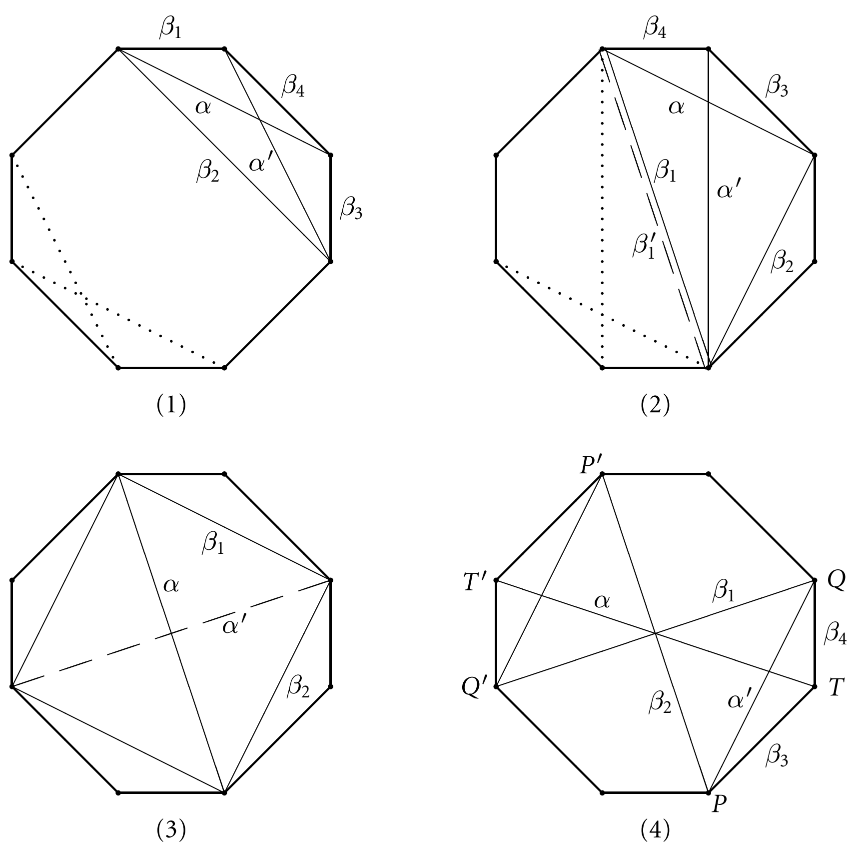

Figure 10: Lemma 4.6

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The proofs of Lemma 4.6 and the type D_n case of Theorem 1.17 follow the lines of their counterparts in types ABC. The details are left to the reader.

4.4 Exceptional Types

For the exceptional types E_6, E_7, E_8, F_4, and G_2, Theorem 1.17 can be verified on a computer without much difficulty; we in particular used `Maple`. The sets $E(\alpha, \alpha')$ are constructed recursively within each orbit

$$\big\{ (\alpha, \alpha') = \big(\sigma(-\alpha_j), \sigma(\beta)\big) : \sigma \in \langle \tau_+, \tau_- \rangle \big\}$$

(for β and α_j satisfying $[\beta : \alpha_j] = [\beta^\vee : \alpha_j^\vee] = 1$), starting with

$$E(-\alpha_j, \beta) = \Big\{ -\alpha_j + \beta, -\alpha_j + \beta + \sum_{i \neq j} a_{ij}\alpha_i \Big\}$$

(*cf.* (1.12)) and using (3.4). One then checks the statement of Theorem 1.17 directly for each pair $E(\alpha, \alpha') = \{\alpha + \alpha', \alpha \uplus \alpha'\}$. ■

5 Proof of Lemma 2.5

We prove Lemma 2.5 case by case. For the classical types $ABCD$, we directly prove the (seemingly) more general Lemma 2.4.

5.1 Type A_n

We follow the conventions of Section 4.1. In the geometric model described there, a $\langle \tau_+, \tau_- \rangle$-invariant function F of Lemma 2.4 becomes a function on diagonals of a regular $(n+3)$-gon that is invariant under the symmetries of the latter. One can view such an F as a function $f \colon \{1, \dots, n\} \to \mathbb{R}$ satisfying

$$f(n+1-i) = f(i). \tag{5.1}$$

In other words, we use $f(i)$ as a shorthand for $F(-\alpha_i) = F(\alpha_i)$, that is, for the value of F at diagonals that connect vertices $i+1$ steps apart. Condition (1.8) takes the form

$$2f(j) - f(j-1) - f(j+1) > 0, \tag{5.2}$$

for $1 \le j \le n$, with the conventions

$$f(0) = f(n+1) = 0. \tag{5.3}$$

To rephrase, f is a *strictly concave* function on $\{0, 1, \dots, n+1\}$ satisfying (5.1) and (5.3). Under these assumptions, we need to prove the type A_n version of (2.4). In view of Lemma 4.2, and using its notation, it is enough to show that

$$F(\alpha) + F(\alpha') > F(\beta_2) + F(\beta_4) \tag{5.4}$$

for any pair of crossing diagonals representing roots α and α' (*cf.* Figure 5). Equivalently, we need to show that

$$f(i_1 + i_2 + 1) + f(i_2 + i_3 + 1) > f(i_2) + f(i_4) \tag{5.5}$$

for any positive integers i_1, i_2, i_3, i_4 satisfying

$$(i_1 + 1) + (i_2 + 1) + (i_3 + 1) + (i_4 + 1) = n + 3. \tag{5.6}$$

The concavity condition (5.2) implies that

$$f(x) + f(y) > f(x - t) + f(y + t)$$

for $x < y$ and $t > 0$. Combining this with (5.1) and (5.6), we obtain

$$f(i_1 + i_2 + 1) + f(i_2 + i_3 + 1) > f(i_2) + f(i_1 + i_2 + i_3 + 2) = f(i_2) + f(i_4),$$

as desired.

5.2 Types B_n and C_n

We follow the conventions of Section 4.2. The proof is similar to the type A_n. For a $\langle \tau_+, \tau_- \rangle$-invariant function F on $\Phi_{\geq -1}$, we define a function $f\colon \{0, \ldots, 2n\} \to \mathbb{R}$ by

$$f(i) = \begin{cases} F(-\alpha_i) & \text{if } 1 \leq i \leq n-1; \\ \frac{2}{d} F(-\alpha_n) & \text{if } i = n; \\ F(-\alpha_{2n-i}) & \text{if } n+1 \leq i \leq 2n-1; \\ 0 & \text{if } i = 0 \text{ or } i = 2n. \end{cases} \tag{5.7}$$

Condition (1.8) can then be rewritten as

$$2f(j) - f(j-1) - f(j+1) > 0, \tag{5.8}$$

for $1 \leq j \leq 2n-1$. Thus, f satisfies the same conditions (5.1)–(5.3) as before, with n replaced by $2n-1$.

The roots in $\Phi_{\geq -1}$ can be represented by the Θ-orbits of diagonals of a regular $(2n+2)$-gon. (Recall that Θ is the 180° degree rotation.) Then F becomes a function on diagonals of the $(2n+2)$-gon that is invariant under its symmetries. The type B_n version of (2.4) can be restated, by virtue of Lemma 4.4 and using its notation, as follows: in a situation of Figure 7(1), we have the inequality

$$F(\alpha) + F(\alpha') > b_2 F(\beta_2) + b_4 F(\beta_4), \tag{5.9}$$

whereas in a situation of Figure 7(2), we have

$$F(\alpha) + F(\alpha') > dF(\beta_1). \tag{5.10}$$

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One easily checks that (5.9) would follow if we show that

$$f(i_1 + i_2 + 1) + f(i_2 + i_3 + 1) > f(i_2) + f(i_4)$$

for any positive integers i_1, i_2, i_3, i_4 satisfying

$$(i_1 + 1) + (i_2 + 1) + (i_3 + 1) + (i_4 + 1) = 2n + 2;$$

this is proved in the same way as (5.5). Finally, (5.10) can be restated as $f(n) > f(i)$, for $i < n$, which follows from concavity of f together with the symmetry condition $f(i) = f(2n - i)$.

5.3 Type D_n

We follow the conventions of Section 4.3. The proof is similar to the types ABC, with (5.7) replaced by

$$f(i) = \begin{cases} F(-\alpha_i) & \text{if } 1 \leq i \leq n-2; \\ F(-\alpha_{n-1}) + F(-\alpha_{\overline{n-1}}) & \text{if } i = n-1; \\ F(-\alpha_{2n-2-i}) & \text{if } n \leq i \leq 2n-3; \\ 0 & \text{if } i = 0 \text{ or } i = 2n-2. \end{cases}$$

Details are left to the reader.

5.4 Calculation of Cluster Expansions

For $\gamma \in Q$, let

$$\gamma_+ = \sum_{[\gamma:\alpha_i]>0} [\gamma : \alpha_i]\alpha_i.$$

Also, let $K(\gamma)$ denote the set of nonzero terms $m_\beta \beta$ contributing to the cluster expansion $\gamma = \sum_\beta m_\beta \beta$ of γ.

Lemma 5.1 *For any $\gamma \in Q$ and any sign ε, we have*

$$K(\gamma) = \{(-[\gamma : \alpha_i])(-\alpha_i) : [\gamma : \alpha_i] < 0\} \cup \tau_\varepsilon\Big(K\big(\tau_\varepsilon(\gamma_+)\big)\Big). \tag{5.11}$$

Proof Follows from Proposition 1.13 together with [5, Lemma 3.12] (which is in turn an easy consequence of (1.4)). ∎

Lemma 5.1 enables us to efficiently compute cluster expansions, by recursively applying (5.11) with $\varepsilon = -1, -1, 1, -1, 1, \ldots$, until we hit $K(0) = \varnothing$. The fact that this computation terminates follows from Theorems 1.1 and 3.1; in fact, the depth of recursion is at most h.

5.5 Exceptional Types

We describe the verification of Lemma 2.5 for type E_6 only; other exceptional types are treated in a similar way, and in fact are easier to handle since the involution $\alpha \mapsto -w_\circ(\alpha)$ is trivial.

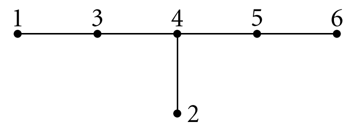

Figure 11: Coxeter graph of type E_6

We use the numeration of roots shown in Figure 11. The involution $-w_\circ$ interchanges α_1 with α_6, and α_3 with α_5, and fixes α_2 and α_4. We denote $F(-\alpha_1) = F(-\alpha_6) = f_1$, $F(-\alpha_2) = f_2$, $F(-\alpha_3) = F(-\alpha_5) = f_3$, and $F(-\alpha_4) = f_4$. The inequalities (1.8) take the form

$$
\begin{aligned}
2f_1 - f_3 &> 0,\\
2f_2 - f_4 &> 0,\\
-f_1 + 2f_3 - f_4 &> 0,\\
-f_2 - 2f_3 + 2f_4 &> 0.
\end{aligned}
\tag{5.12}
$$

We need to show that these linear inequalities imply every inequality

$$
c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4 > 0 \tag{5.13}
$$

on the list of the type E_6 versions of the inequalities (2.6). Equivalently, we need to show that the parameters c_1, c_2, c_3, c_4 of each inequality (5.13) satisfy

$$
\begin{aligned}
2c_1 + 2c_2 + 3c_3 + 4c_4 &\geq 0,\\
c_1 + 2c_2 + 2c_3 + 3c_4 &\geq 0,\\
3c_1 + 4c_2 + 6c_3 + 8c_4 &\geq 0,\\
2c_1 + 3c_2 + 4c_3 + 6c_4 &\geq 0.
\end{aligned}
\tag{5.14}
$$

(The coefficient matrix in (5.14) is the transposed inverse of the matrix in (5.12), so the left-hand sides in (5.14) are the coefficients in the expansion of the left-hand side of (5.13) as a linear combination of the left-hand sides in (5.12).)

We find the cluster expansions (2.5) using the algorithm described in Section 5.4. We then produce the corresponding inequalities (2.6) (note that the values $F(\beta)$ appearing in (2.6) are obtained as a byproduct of the same algorithm), and verify the conditions (5.14) in each instance. To illustrate, consider the following example:

$$
\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \quad j = 3.
$$

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α	j	cluster expansion of $\alpha - \alpha_j$
$[1,1,1,1,1,1]$	1	$[0,1,1,1,1,1]$
	2	$[1,0,1,1,1,1]$
	3	$[1,0,0,0,0,0] + [0,1,0,1,1,1]$
	4	$[0,0,0,0,1,1] + [1,0,1,0,0,0] + [0,1,0,0,0,0]$
	5	$[0,0,0,0,0,1] + [1,1,1,1,0,0]$
	6	$[1,1,1,1,1,0]$
$[1,1,1,2,1,1]$	1	$[0,1,1,2,1,1]$
	2	$[1,0,1,1,0,0] + [0,0,0,1,1,1]$
	3	$[1,0,0,0,0,0] + [0,1,0,1,0,0] + [0,0,0,1,1,1]$
	5	$[0,0,0,0,0,1] + [0,1,0,1,0,0] + [1,0,1,1,0,0]$
	6	$[1,1,1,2,1,0]$
$[1,1,2,2,1,1]$	1	$[0,0,1,1,1,1] + [0,1,1,1,0,0]$
	2	$[0,0,1,1,0,0] + [1,0,1,1,1,1]$
	5	$[0,0,0,0,0,1] + [0,0,1,1,0,0] + [1,1,1,1,0,0]$
	6	$[1,1,2,2,1,0]$
$[1,1,1,2,2,1]$	1	$[0,1,1,2,2,1]$
	2	$[0,0,0,1,1,0] + [1,0,1,1,1,1]$
	3	$[1,0,0,0,0,0] + [0,0,0,1,1,0] + [0,1,0,1,1,1]$
	6	$[1,0,1,1,1,0] + [0,1,0,1,1,0]$
$[1,1,2,2,2,1]$	1	$[0,0,1,1,1,0] + [0,1,1,1,1,1]$
	2	$[1,0,1,1,1,0] + [0,0,1,1,1,1]$
	6	$[0,0,1,1,1,0] + [1,1,1,1,1,0]$
$[1,1,2,3,2,1]$	1	$[0,1,1,2,1,0] + [0,0,1,1,1,1]$
	2	$[0,0,1,1,0,0] + [0,0,0,1,1,0] + [1,0,1,1,1,1]$
	6	$[0,1,1,2,1,0] + [1,0,1,1,1,0]$
$[1,2,2,3,2,1]$	1	$[0,1,1,2,2,1] + [0,1,1,1,0,0]$
	6	$[1,1,2,2,1,0] + [0,1,0,1,1,0]$

Figure 12: Cluster expansions (2.5) in type E_6

We compute the cluster expansion of $\alpha - \alpha_j$ (*cf.* line 18 in Figure 12) as

$$\alpha - \alpha_j = (\alpha_1) + (\alpha_4 + \alpha_5) + (\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6),$$

with

$$F(\alpha_1) = F(\alpha_4 + \alpha_5) = F(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6) = f_1.$$

Since $F(\alpha) = f_3$, the corresponding inequality (2.6) (or (5.13)) is

$$-3f_1 + 2f_3 > 0.$$

Thus, in this case, $c_1 = -3$, $c_2 = 0$, $c_3 = 2$, $c_4 = 0$, and conditions (5.14) hold. The verifications based on the procedures described above were performed for all

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exceptional types using `Maple` and altogether took a few minutes of processor time. This completes our proof of Lemma 2.5. ■

We conclude by providing the list of "most interesting" instances of cluster expansions (2.5) in type E_6. More precisely: note that in view of [5, Proposition 3.3.3], the cluster expansion of a vector $\gamma \in Q$ that belongs to a root sublattice generated by a proper subset of simple roots coincides with the cluster expansion of γ with respect to the corresponding root subsystem. Thus, such cluster expansions already appear in smaller rank. Consequently, we only list the cluster expansions (2.5) for the roots α (in type E_6) that have full support. See Figure 12, where notation $[b_1, \dots, b_6]$ is used to denote a root $b_1\alpha_1 + \cdots + b_6\alpha_6$. Similar tables in types E_7 and E_8 have 56 and 121 rows, respectively, and are omitted due to space limitations.

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