Now if $ad+bc$ is a multiple of P, $ac+bd$ cannot be, for, squaring and adding, we should have $P^2+4abcd$ a multiple of P, which is impossible, since a, b, c, d are prime to P.

Hence either $ad+bc$ and $ac-bd$ are multiples together, or

$ad-bc$ and $ac+bd$ are multiples together.

In either case, square and add. Then $(a^2+b^2)(c^2+d^2)$, *i.e.* $P^2=(r^2+s^2)P^2$, so that $r^2+s^2=1$.

Hence one of r or $s=0$, *i.e.* $ad=bc$ or $ac=bd$, both of which are impossible, by (i). Hence our supposition is wrong, and P can be expressed in one way only as the sum of two squares.

A. C. AITKEN.

Theorem regarding a regular polygon and a circle cutting its sides, with corollary and application to trigonometry.

1. *Theorem.*

If a circle cut all the sides (produced if necessary) of a regular polygon, the algebraic sum of the intercepts, on the sides, between the vertices and the circle is zero.

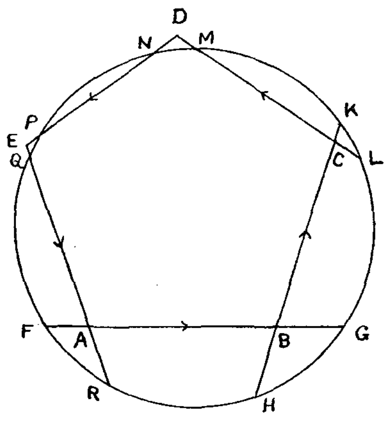

Fig. 1

Consider the case of a regular pentagon $ABCDE$ whose sides are cut by a circle as shown in Fig. 1. Let $AB=x$.

Then (Euclid III. 35, etc.) we have

$$\begin{aligned} AF(x+BG) &= AR(-x+EQ), \\ BH(x+CK) &= BG(-x+AF), \\ CL(x+DM) &= CK(-x+BH), \\ DN(x+EP) &= DM(-x+CL), \\ EQ(x+AR) &= EP(-x+DN). \end{aligned}$$

Adding these results and omitting terms which occur on both sides we get

$$x(AF+BH+CL+DN+EQ = -x(AR+BG+CK+DM+EP)$$

$$\therefore\ AF+BH+CL+DN+EQ+AR+BG+CK+DM+EP=0.$$

2. *Corollary.*

If through a point within a circle n directed chords are drawn so that each makes with the next an angle of $\frac{2\pi}{n}$, the algebraic sum of their segments, measured from the given point, is zero.

Consider first the case in which the number of chords is odd.

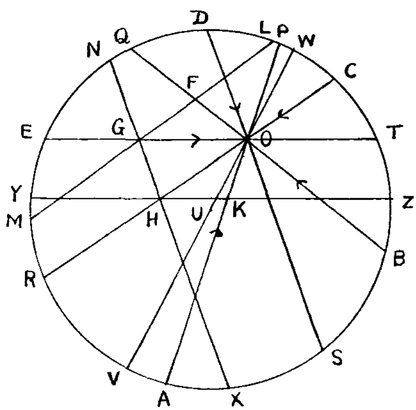

Fig 2

In Fig. 2 the five chords AP, BQ, CR, DS, ET intersect at O, each making an angle of $\frac{2\pi}{5}$ with the next. From OA cut off any length OK, through K draw YZ parallel to ET and let it cut OR at H, through H draw NX parallel to DS and let it cut OE at G, through G draw LM parallel to CR and let it cut OQ at F. Then $OFGHK$ is a regular pentagon, therefore by the foregoing theorem

$$KA + OP + OB + FQ + FL + GM + GN + HX + HY + KZ = 0.$$

Now, since chords CR and LM are parallel and OF and HG are equally inclined to them $FL + GM = OC + HR$, similarly $GN + HX = OD + OS$, and $HY + KZ = GE + OT$.

$$\therefore\ KA + OP + OB + FQ + OC + HR + OD + OS + GE + OT = 0.$$

Again $OK + OF = 0$, and $OH + OG = 0$,

$$\therefore\ (OK + KA) + OP + OB + (OF + FQ) + OC + (OH + HR) + OD + OS + (OG + GE) + OT = 0,$$

$$\therefore\ OA + OP + OB + OQ + OC + OR + OD + OS + OE + OT = 0.$$

If the number of chords is even they coincide in pairs, reversed in direction, so that the proof in this case is obvious.

This corollary is also true for cases in which the point of intersection of the chords lies on the circumference or outside the circle in a position where such a series of chords may be drawn.

3. *Application of Corollary to Trigonometry.*

The above corollary may be employed to illustrate the fact that the sum of the sines or cosines of n angles in arithmetical progression vanishes when the common difference of the angles is $\frac{2\pi}{n}$ or a multiple of $\frac{2\pi}{n}$.

In Fig. 2 let U be the centre of the circle and VUW the diameter which passes through O. Denote angle VOS by α, angle SOT, etc., $= \frac{2\pi}{5}$ by β, and let OU be taken as unit of length, then

$$\cos\alpha = \tfrac{1}{2}(OS + OD),$$
$$\cos(\alpha + \beta) = \tfrac{1}{2}(OT + OE),$$
$$\cos(\alpha + 2\beta) = \tfrac{1}{2}(OP + OA),$$
etc.

$$\therefore\ \cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \cos(\alpha + 3\beta) + \cos(\alpha + 4\beta) = \tfrac{1}{2}\Sigma(OS + OD) = 0.$$

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If instead of taking the projections of OU on the given chords, the projections on another set drawn through O at right angles to the given set are taken, a similar result is obtained for the sum of the sines of such a series of angles.

If the common difference of the angles is a multiple of $\frac{2\pi}{n}$, but not of 2π, the same results are obtained.

ALEX. D. RUSSELL.

Direct Proofs of Theorems in Elementary Geometry.

(1) If the straight line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.

(2) If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

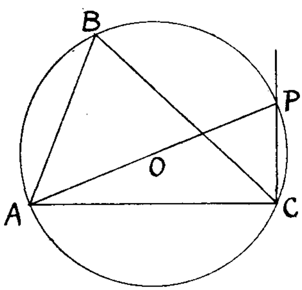

Fig. 1.

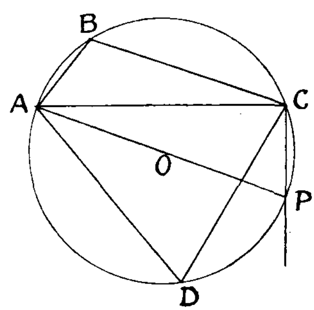

Fig. 2.

(1) Let A, C be the two points and B one of the other points. Let $\angle ABC$ be acute (Fig. 1).

Let O be the circumcentre of $\triangle ABC$; join AO and produce it to meet the perpendicular to AC through C in P.

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