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On Domination in Zero-Divisor Graphs

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Abstract. We first determine the domination number for the zero-divisor graph of the product of two commutative rings with 1. We then calculate the domination number for the zero-divisor graph of any commutative artinian ring. Finally, we extend some of the results to non-commutative rings in which an element is a left zero-divisor if and only if it is a right zero-divisor.

1 Introduction

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. We denote the degree of a vertex v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. The *open neighborhood* of a vertex $v \in V$ is denoted by $N(v) = \{u \in V \mid uv \in E\}$, while the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The *boundary* of S, denoted by $B(S)$, is $N(S) \setminus S$. The set S is a *dominating set* of G if $N[S] = V$, and a *total dominating set* of G (or just TDS) if $N(S) = V$. For sets $A, B \subseteq V$, we say that A *dominates* B if $B \subseteq N[A]$, while A *totally dominates* B if $B \subseteq N(A)$. The minimum cardinality of a dominating set of G is the *domination number*, denoted by $\gamma(G)$, and the minimum cardinality of a TDS of G is the *total domination number*, denoted by $\gamma_t(G)$. We call a TDS of cardinality $\gamma_t(G)$, a $\gamma_t(G)$-set. A subset S of vertices is a *connected dominating set* if S is a dominating set and $G[S]$ is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set. For references on domination and its varieties, we refer the reader to [4].

By the zero-divisor graph $\Gamma(R)$ of a commutative ring R, we mean the graph with vertices $Z(R) \setminus \{0\}$ such that there is an (undirected) edge between vertices a and b if and only if $a \neq b$ and $ab = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain. The concept of zero-divisor graphs has been studied extensively by many authors. For a list of references and the history of this topic, the reader is referred to [1–3].

In this paper, we study domination in zero-divisor graphs. We first study domination, total domination, and connected domination in the zero-divisor graph of the product of two commutative rings with 1. We then determine the domination number of zero-divisor graphs of commutative artinian rings, and in particular finite commutative rings. Finally, we extend some of the results to non-commutative rings in which an element is a left zero-divisor if and only if it is a right zero-divisor.

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We denote by K_n and C_n the complete graph and the cycle on n vertices, respectively. Also we denote by $K_{m,n}$ the complete bipartite graph, where one partite set has m vertices and the other partite set has n vertices.

Throughout this paper, all rings have a $1 \neq 0$. We also note that by $G \leq H$ for two graphs, we mean that G is a subgraph of H, while by $R \leq S$ for two rings, we mean that R is a subring of S.

All zero-divisor graphs of rings we handle in this paper have finite domination number.

2 Commutative Rings

In this section, we study domination in the zero-divisor graph of a commutative ring. We first determine the domination number, the total domination number, and the connected domination number for the zero-divisor graph of the product of two commutative rings with 1. We then determine the domination number for the zero-divisor graph of any commutative artinian ring. In particular, we determine the domination number in the zero-divisor graph of a finite commutative ring. We begin with the following lemma.

Lemma 2.1 *For two rings R_1 and R_2, $(a, b) \in Z(R_1 \times R_2)$ if and only if $a \in Z(R_1)$ or $b \in Z(R_2)$.*

Proposition 2.2 *If R is an integral domain, then $\gamma(\Gamma(\mathbb{Z}_2 \times R)) = 1$.*

Proof Notice that $\Gamma(\mathbb{Z}_2 \times R)$ is a star, and any star has domination number one. ■

As a consequence, if R is an integral domain, then

$$\gamma_c(\Gamma(\mathbb{Z}_2 \times R)) = 1 \quad \text{and} \quad \gamma_t(\Gamma(\mathbb{Z}_2 \times R)) = 2.$$

Observation 2.3 ([4]) If S is a TDS in a graph G, then $G[S]$ has no isolated vertex.

Definition 2.4 Let R be a commutative ring with 1 and $Z(R) \neq 0$. A semi-total dominating set in $\Gamma(R)$ is a subset $S \subseteq Z(R)$ such that S is a dominating set for $\Gamma(R)$ and for any $x \in S$ there is a vertex $y \in S$ (not necessarily distinct) such that $xy = 0$. The semi-total domination number $\gamma_{st}(\Gamma(R))$ of $\Gamma(R)$ is the minimum cardinality of a semi-total dominating set in $\Gamma(R)$. (Note that for all rings R, $\gamma(\Gamma(R)) \leq \gamma_{st}(\Gamma(R)) \leq 2\gamma(\Gamma(R))$).

Note that any TDS of $\Gamma(R)$ is also a semi-total dominating set. But the converse is not true in general. For example, $\{4\}$ is a semi-total dominating set in $\Gamma(\mathbb{Z}_8)$, but it is not a TDS by Observation 2.3. It is also easy to see that if $Z(R)$ has no nontrivial nilpotent elements, then $\gamma_{st}(\Gamma(R)) = \gamma_t(\Gamma(R))$. We refer to a semi-total dominating set of $\Gamma(R)$ of minimum cardinality as a $\gamma_{st}(\Gamma(R))$-set.

Proposition 2.5 *If R is not an integral domain, then $\gamma(\Gamma(\mathbb{Z}_2 \times R)) = \gamma_{st}(\Gamma(R)) + 1$.*

Proof Let S be a $\gamma_{st}(\Gamma(R))$-set. It follows that $\{(0,x) : x \in S\} \cup \{(1,0)\}$ is a dominating set for $\Gamma(\mathbb{Z}_2 \times R)$. So $\gamma(\Gamma(\mathbb{Z}_2 \times R)) \leq \gamma_{st}(\Gamma(R)) + 1$. Now let D be a $\gamma(\Gamma(\mathbb{Z}_2 \times R))$-set. Consider $D_1 = \{x : (0,x) \in D\}$. We show that D_1 is a dominating set for $\Gamma(R)$. Let $d \in Z(R)$. Then $(1,d) \in Z(\mathbb{Z}_2 \times R)$. So there exists $(a,b) \in D$ such that $(a,b)(1,d) = (0,0)$. This implies that $a = 0$ and $b \in D_1$. Thus D_1 is a dominating set for $\Gamma(R)$. On the other hand, for any $x \in D_1$, $(1,x) \in Z(R)$, and so there exists $(a,b) \in D$ such that $(a,b)(1,x) = (0,0)$. Then $a = xb = 0$. So $b \in D_1$ and $bx = 0$. Thus D_1 is a $\gamma_{st}(\Gamma(R))$-set.

We next show that $|D| > |D_1|$. Suppose to the contrary that $|D| = |D_1|$. Then $(0,1)$ is not dominated by D, a contradiction. Thus $|D| > |D_1|$. We conclude that $\gamma(\Gamma(\mathbb{Z}_2 \times R)) = |D| \geq |D_1| + 1 \geq \gamma_{st}(\Gamma(R)) + 1$. ■

We next assign a parameter $a(R)$ to a ring R. For a commutative ring R with 1, we let

$$a(R) = \begin{cases} 1 & \text{if } Z(R) = 0, \\ \gamma_{st}(\Gamma(R)) & \text{if } Z(R) \neq 0. \end{cases}$$

Theorem 2.6 *If R_1, R_2 are commutative rings with 1 and $\mathbb{Z}_2 \notin \{R_1, R_2\}$, then*

$$\gamma(\Gamma(R_1 \times R_2)) = a(R_1) + a(R_2).$$

Proof Let $R = R_1 \times R_2$. We consider the following cases.

Case 1: $Z(R_1) = Z(R_2) = 0$. It is easy to see that $\{(1,0),(0,1)\}$ is a dominating set for $\Gamma(R)$. So $\gamma(\Gamma(R)) \leq 2$. If $\gamma(\Gamma(R)) = 1$, then we let $S = \{(a,b)\}$ be a $\gamma(\Gamma(R))$-set. Since $|R_1| > 2$ and $|R_2| > 2$, there are $a_1 \in R_1 \setminus \{0,a\}$ and $b_1 \in R_2 \setminus \{0,b\}$. But $(a_1,0)$ and $(0,b_1)$ are dominated by S. So we obtain that $a = b = 0$. This is a contradiction. Thus, $\gamma(\Gamma(R)) = 2$.

Case 2: $Z(R_1) \neq 0, Z(R_2) = 0$. First, let S be a $\gamma_{st}(\Gamma(R_1))$-set. Let

$$S_1 = \{(x,0) : x \in S\} \cup \{(0,1)\}.$$

Since any vertex of $\Gamma(R)$ is of the form (x,y), where $x \in V(\Gamma(R_1))$, or $(0,b)$, where $b \neq 0$, we obtain that S_1 is a dominating set for $\Gamma(R)$. So $\gamma(\Gamma(R)) \leq a(R_1) + 1$. Let D be $\gamma(\Gamma(R))$-set, and let $A_1 = \{x : (x,0) \in D\}$. We show that A_1 is a semi-total dominating set for $\Gamma(R_1)$. For any $y \in V(\Gamma(R_1))$, $(y,1) \in V(\Gamma(R))$. So there is $(c,d) \in D$ such that $(c,d)(y,1) = (0,0)$. This implies that $cy = d = 0$. So $c \in A_1$, and y is dominated by an element of A_1. We deduce that A_1 is a dominating set for $\Gamma(R_1)$. On the other hand, for any $x \in A_1$, $(x,1) \in Z(R)$, and so is dominated by an element (a,b) of D. We obtain that $(a,b)(x,1) = (0,0)$. This implies that $ax = b = 0$. So $a \in A_1$ and $ax = 0$. Hence A_1 is a semi-total dominating set for $\Gamma(R_1)$. This implies that $|A_1| \geq a(R_1)$, and so $|D| \geq a(R_1)$. If $|D| = a(R_1)$, then $D = \{(x,0) : x \in A_1\}$. But then $(1,0)$ is not dominated by D, which is a contradiction. So $|D| \geq a(R_1) + 1$.

Case 3: $Z(R_1) \neq 0, Z(R_2) \neq 0$. Let D be a $\gamma(\Gamma(R))$-set. Let $A_1 = \{x : (x,0) \in D\}$, and $A_2 = \{y : (0,y) \in D\}$. Similar to Case 2, we obtain that A_1 is a semi-total dominating set for $\Gamma(R_1)$, and A_2 is a semi-total dominating set for $\Gamma(R_2)$. So $|D| \geq a(R_1) + a(R_2)$. On the other hand, let S_1, S_2 be a $\gamma_{st}(\Gamma(R_1))$-set and a $\gamma_{st}(\Gamma(R_2))$-set, respectively. Let $T_1 = \{(x, 0) : x \in S_1\}$ and $T_2 = \{(0, y) : y \in S_2\}$. Since any vertex of $\Gamma(R)$ is of the form (x, y), where $x \in Z(R_1)$ or $y \in Z(R_2)$, we obtain that $T_1 \cup T_2$ is a dominating set for $\Gamma(R)$. So $\gamma(\Gamma(R)) = a(R_1) + a(R_2)$. ∎

By the proof of Theorem 2.6, we obtain the following interesting corollary.

Corollary 2.7 *If R_1, R_2 are commutative rings with 1 and $\mathbb{Z}_2 \notin \{R_1, R_2\}$, then $\gamma(\Gamma(R_1 \times R_2)) = \gamma_{st}((R_1 \times R_2))$.*

The minimum dominating sets for $\Gamma(R)$ in the proof of Theorem 2.6 are connected. This leads to the following.

Corollary 2.8 *If R_1, R_2 are commutative rings with 1 and $\mathbb{Z}_2 \notin \{R_1, R_2\}$, then $\gamma(\Gamma(R_1 \times R_2)) = \gamma_t(\Gamma(R_1 \times R_2)) = \gamma_c(\Gamma(R_1 \times R_2))$.*

Recall that a local ring is a ring with exactly one maximal ideal. We use (R, M) for a local ring R with unique maximal ideal M. Also, $\operatorname{Spec}(R)$ is the set of all prime ideals of R, and $\operatorname{Ass}(R)$ denotes the set of associated prime ideals of R. Note that in any artinian local ring (R, M), $\operatorname{Nil}(R) = M$, where $\operatorname{Nil}(R)$ is the set of all nilpotent elements of R.

Lemma 2.9 *For any local commutative artinian ring (R, M) with identity, $a(R) = 1$.*

Proof The result is trivial if R is a field. So we assume that R is not a field. Since R is artinian, it is noetherian, and so $\operatorname{Ass}(R) \neq \varnothing$. Now there is an $x \in R$ such that $\operatorname{ann}(x) = M$, since $\operatorname{Spec}(R) = \{M\}$. But x is nilpotent. So there is an integer i such that $x^{2i} = 0$ and $x^i \neq 0$. It follows that $\operatorname{ann}(x) \subseteq \operatorname{ann}(x^i) \subset R$ and $\operatorname{ann}(x^i) = M$ $(= Z(R))$. We deduce that $\{x^i\}$ is a $\gamma_{st}(\Gamma(R))$-set, and so the result follows. ∎

Lemma 2.10 *For any integral domain D, $\gamma_{st}(\Gamma(\mathbb{Z}_2 \times D)) = 2$.*

The next corollary is a consequence of Theorem 2.6, Corollary 2.7, and Lemmas 2.9 and 2.10.

Corollary 2.11 *Let $R_1, R_2, \ldots, R_k$ be local commutative artinian rings with identity. If $R = R_1 \times R_2 \times \cdots \times R_k$, where $R \ncong F$ or $\mathbb{Z}_2 \times F$ for a field F, then $\gamma(\Gamma(R)) = k$.*

Corollary 2.12 *If $R = \mathbb{Z}_{p_1^{t_1}} \times \mathbb{Z}_{p_2^{t_2}} \times \cdots \times \mathbb{Z}_{p_k^{t_k}}$, where $R \ncong \mathbb{Z}_p$ or $\mathbb{Z}_2 \times \mathbb{Z}_p$ for a prime p, then $\gamma(\Gamma(R)) = k$.*

Remark Since any commutative artinian ring is a finite direct product of local commutative artinian rings, by Proposition 2.2 and Corollary 2.11, the domination number of the zero-divisor graph of any commutative artinian (and hence finite) ring has been calculated.

3 Non-Commutative Rings

A *directed graph* $D = (V, A)$ consists of a set V of vertices and a set A of directed edges, called *arcs*, where $A \subseteq V \times V$. The *outset* of a vertex u is the set $O(u) = \{v : (u, v) \in A\}$, and the *closed outset* of u is $O[u] = O(u) \cup \{u\}$. For a subset S of V, $O(S) = \bigcup_{u \in S} O(u)$ and $O[S] = \bigcup_{u \in S} O[u]$. A set $S \subseteq V$ is a dominating set of D if $O[S] = V$. The domination number $\gamma(D)$ of D is the minimum cardinality of a dominating set of D. We note that domination in a directed graph can be defined if we consider the *insets*, where the inset $I(v)$ of a vertex v is the set $\{w : (w, v) \in A\}$.

Zero-divisor graphs for non-commutative rings were first studied in [5] and further studied, for example, in [1]. The zero-divisor graph of a non-commutative ring R is the directed graph $\Gamma(R)$, where its vertices are all the non-zero zero-divisors of R and for any two distinct vertices x and y, $x \to y$ is an edge if and only if $xy = 0$.

Here we consider a non-commutative ring R with 1 such that for any element $x \in R$, x is a left zero-divisor if and only if it is a right zero-divisor. Then the proofs of Propositions 2.2, 2.5, and Theorem 2.6 hold for these rings. So we obtain the following.

Proposition 3.1 *If R is a domain, then $\gamma(\Gamma(\mathbb{Z}_2 \times R)) = 1$.*

Proposition 3.2 *If R is not a domain, then $\gamma(\Gamma(\mathbb{Z}_2 \times R)) = \gamma_{st}(\Gamma(R)) + 1$.*

Theorem 3.3 *If R_1, R_2 are non-commutative rings with 1, then $\gamma(\Gamma(R_1 \times R_2)) = a(R_1) + a(R_2)$.*

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