AN EXTREMAL PROBLEM IN NUMBER THEORY

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Let n and k be integers with $n \geq k \geq 3$. Denote by $f(n, k)$ the largest positive integer for which there exists a set S of $f(n, k)$ integers satisfying (i) $S \subseteq \{1, 2, \ldots, n\}$ and (ii) no k members of S have pairwise the same greatest common divisor. The problem of determining $f(n, k)$ appears to be difficult. Erdős [2] proved that there is an absolute constant $c > 1$ such that for every $\epsilon > 0$ and every fixed k

$$c^{\frac{\log n}{\log \log n}} < f(n, 3) \leq f(n, k) \leq n^{3/4+\epsilon}, \tag{1}$$

provided $n \geq n_o(k, \epsilon)$. In [1] it is proved that for every $\epsilon > 0$ and every fixed k

$$f(n, k) \geq \{(k-1)^2 + [\frac{k-1}{2}]\}^{\frac{\log n}{(2+\epsilon) \log \log n}} \tag{2}$$

provided $n \geq n_o(k, \epsilon)$.

In this paper we investigate partially the case where $k \to \infty$ with n. In this connection it is known [2] that for $0 < \alpha < 1$

$$f(n, [n^\alpha]) \sim c_\alpha n \tag{3}$$

where c_α is a constant depending only on α. The main result that we prove is

THEOREM 1. Let $\epsilon > 0$ and $\alpha > 0$. Then

$$n^{\frac{\alpha}{1+\alpha} - \epsilon} < f(n, [\log^\alpha n]) < n^{\frac{2\alpha+3}{2\alpha+4} + \epsilon} \tag{4}$$

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provided $n \geq n_o(\alpha, \epsilon)$.

In addition we shall prove

THEOREM 2. Let $t \geq 2$ be a positive integer. Then

$$(5) \qquad f(n, [m^{1/t}]) > \frac{n(1-\epsilon)}{(\log n)^t} ,$$

for every $\epsilon > 0$ provided $n \geq n_o(t, \epsilon)$.

Note, however, that Theorem 2 is not as strong as (3). We remark also that in connection with Theorem 1, it would be of interest to know whether

$$f(n, [\log^\alpha n]) = n^{h(\alpha)+0(1)}$$

and, if so, to determine $h(\alpha)$.

To prove Theorems 1 and 2 we need the following lemma:

LEMMA. Let t and k be positive integers and let $P_1, P_2, \ldots, P_{tk}$ be the first tk primes. (P_r denotes the r^{th} prime.) Let S_t be the set of the k^t numbers $P_{i_1} P_{i_2} \ldots P_{i_t}$ where $(s-1)k + 1 \leq i_s \leq sk$ for $s = 1, 2, \ldots, t$. Then no $k+1$ members of S_t have pairwise the same greatest common divisor.

Proof. The lemma can be established by a straightforward induction argument on t .

Observe that the largest number in S_t is $N = P_k P_{2k} \ldots P_{tk}$. We thus have

$$(6) \qquad f(N, k+1) \geq k^t .$$

Now we prove Theorem 2. Let $t \geq 2$ be a fixed positive integer and set $k = [n^{1/t} / \log n]$. Then by the prime number theorem $(P_r \sim r \log r)$ we have

$$N = \prod_{m=1}^{t} P_{mk} \sim \prod_{m=1}^{t} mk \log mk \sim t! \ (k \log k)^t$$

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$$< t! \left(\frac{n^{1/t}}{t}\right)^t \le \frac{n}{2} .$$

Hence we have for all sufficiently large n

(7) $$N < n .$$

Also

(8) $$k^t = \left[\frac{n^{1/t}}{\log n}\right]^t > \left(\frac{n^{1/t}}{\log n} - 1\right)^t > \frac{(1-\epsilon)n}{(\log n)^t}$$

provided $n \ge n_o(t, \epsilon)$. Now (6), (7) and (8) imply

$$f(n, [n^{1/t}]) \ge f(N, [n^{1/t}]) \ge f(N, k+1)$$

$$\ge k^t > \frac{(1-\epsilon)n}{(\log n)^t} .$$

This establishes (5) and thus Theorem 2 is proved.

To obtain the lower bound in (4) choose $k = [\log^\alpha n] - 1$ and $t = \dfrac{\log n}{(1+\alpha)\log\log n}$. Then we have for all sufficiently large n

(9) $$N = \prod_{m=1}^{t} P_{mk} < (1+\epsilon)^t \, t! \, k^t \prod_{m=1}^{t} \log mk$$

$$< (1+\epsilon)^t \, t! \, k^t (\log tk)^t < n .$$

Also it is easy to verify that for n sufficiently large

(10) $$k^t > n^{\frac{\alpha}{1+\alpha} - \epsilon} .$$

Now (6), (9) and (10) yield

$$f(n, [\log^\alpha n]) \ge f(N, k+1) \ge k^t > n^{\frac{\alpha}{1+\alpha} - \epsilon} .$$

This establishes the lower bound in (4).

The argument used by Erdös to obtain the upper bound

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given by (1) can be used with only slight modifications to obtain the upper bound given in (4). Let $\{a_1, a_2, \ldots, a_\ell\}$ be an arbitrary subset of $\{1, 2, \ldots, n\}$, $\ell = n^{\frac{2\alpha+3}{2\alpha+4}+\epsilon}$. Split the a's into two classes. In one class put those a's which have at least $\frac{\log n}{2(2+\alpha)\log\log n}$ distinct prime factors. The remaining a's are placed in the second class. The Erdös argument can now be used to show that the second class contains at least $[\log^\alpha n]$ integers with pairwise the same greatest common divisor. We do not reproduce the details of the argument. This completes the proof of Theorem 1.

In [2] Erdös raised the following problem. Denote by $\vartheta(n)$ the largest positive integer for which there exists a set S of $\vartheta(n)$ integers satisfying $S \subseteq \{1, 2, \ldots, n\}$ and no three members of S have pairwise the same least common multiple. Is it true that $\vartheta(n) = 0(n)$? We do not settle this question here, but it may be worth noting that a very simple argument shows that for $n \geq n_0(\epsilon)$

$$\vartheta(n) > (1-\epsilon)\frac{n \log\log n}{\log n} . \tag{11}$$

To prove (11) let $\ell = [n^{1/4}]$ and consider the following set of numbers:

$$\begin{array}{llll} P_1P_{\ell+1}, & P_1P_{\ell+2}, & \ldots, & P_1P_{\ell+s_1} \\ P_2P_{\ell+1}, & P_2P_{\ell+2}, & \ldots, & P_2P_{\ell+s_2} \\ \vdots & \vdots & & \vdots \\ P_\ell P_{\ell+1}, & P_\ell P_{\ell+2}, & \ldots, & P_\ell P_{\ell+s_\ell} \end{array}$$

where s_i is defined by $P_{\ell+s_i} \leq \frac{n}{P_i} < P_{\ell+s_i+1}$.

Then it is clear that all of these numbers are distinct and do not exceed n and it is easy to verify that no three of the

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numbers have pairwise the same least common multiple. The number of numbers in the above array is

$$s_1 + s_2 + \ldots + s_\ell = \left(\frac{n}{P_1}\right) + \left(\frac{n}{P_2}\right) + \ldots + \left(\frac{n}{P_\ell}\right) - \ell^2$$

$$> \left(1 - \frac{\epsilon}{2}\right) \frac{n}{\log n} \sum_{i=1}^{\ell} \frac{1}{P_i} - \ell^2$$

$$> (1-\epsilon) \frac{n}{\log n} \log \log n .$$

This proves (11).

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REFERENCES

1. H. L. Abbott, Some remarks on a combinatorial theorem of Erdős and Rado. Can. Math. Bull. vol. 9, no. 2 (1966) pages 155-160.

2. P. Erdős, On a problem in elementary number theory and a combinatorial problem. Math. of Comp., vol. 18, no. 88 (1964), pages 644-646.

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