16

Generalized Yang–Mills theory on a Riemann surface

16.1 Introduction

Pure gauge theory in two dimensions is locally trivial and has no propagating degrees of freedom. This was discussed in the first chapter of this part of the book and now in the last chapter we will describe the global properties of gauge theories in two dimensions. For the latter to be non-trivial we will either take the underlying manifold to be a compact Riemann surface or introduce Wilson loops external sources. We will show that in those cases the gauge theory has a rich structure and in fact is almost a topological field theory, which is a theory with no propagating degrees of freedom (see also Section 4.7). Moreover, it will be shown that the theory has an interpretation in terms of a string theory.

It is easy to realize that in two dimensions the pure YM theory is in fact the simplest member of a wide class of renormalizable theories that incorporate only gauge fields. These will be referred to as the generalized gauge theories gYM. In Chapter 15 we introduced an alternative formulation of the YM theory using the action[1]

$$S = -\int \mathrm{d}^2 z \mathrm{Tr}[iFB + g^2 B^2]. \tag{16.1}$$

Now it is easy to realize that the B^2 term can be generalized to an arbitrary invariant function $\Phi(B)$. This will constitute the family of gYM.[2] The partition function of the generalized theories on Riemann surfaces and the computation of Wilson loops for these theories will also be described in this chapter.

Pure YM_2 theory defined on an arbitrary Riemann surface is known to be exactly solvable. In one approach the theory was regularized on the lattice and, using a heat kernel action, explicit expressions for the partition function and loop averages were derived.[3] Identical results were derived also in a continuum path-integral approach.

In the following sections we briefly review the former derivation of the partition function and determine in a similar way the results for the partition function and

[1] In fact in (16.1) we have used a slightly different formulation which is however equivalent to the one used in Chapter 15.

[2] The notion of the generalized QCD theory in two dimensions was introduced and analyzed in [82].

[3] A lattice version of two-dimensional Yang–Mills theory was shown to be exactly solvable by Migdal in [161]. Correlators of Wilson lines on this formulation were computed in [139].

Wilson loops in the gYM_2 case. Since this chapter is based on non-trivial two-dimensional topology which has not been dealt with intensively in this book, to fully understand its content the reader will need to consult the references to this chapter. The reader who is not interested in the topological aspects of two-dimensional gauge dynamics may skip this chapter and proceed directly to the third part of the book.

This chapter is based mainly on [115], [118], [119] and [105].

16.2 The partition function of the YM_2 theory

The partition function for the ordinary YM_2 theory defined on a compact Riemann surface $\mathcal{M}$ of genus H and area A is,

$$\mathcal{Z}(N, H, \lambda A) = \int [DA^\mu] \exp\left[-\frac{1}{4g^2}\int_\Sigma \mathrm{d}^2x\sqrt{\det G_{\mu\nu}}\ \mathrm{tr} F_{\mu\nu}F^{\mu\nu}\right], \tag{16.2}$$

where the gauge group G is taken to be either $SU(N)$ or $U(N)$, g is the gauge coupling constant, $\lambda = g^2 N$, $G_{\mu\nu}$ is the metric on $\mathcal{M}$, and tr stands for the trace in the fundamental representation.[4]

As was emphasized in Chapter 8 the pure YM theory defined on a flat Minkowski space-time with trivial topology is empty since one can gauge away the gauge fields. However this does not hold if the underlying manifold $\mathcal{M}$ is topologically non-trivial. If $\mathcal{M}$ contains a non-trivial cycle γ such that $\mathrm{tr}[P\mathrm{e}^{\oint_\gamma A_\mu \mathrm{d}x^\mu}] \neq 1$, where P stands for path-ordering, one cannot gauge A_μ away along γ. Thus, the partition function depends on the topology of $\mathcal{M}$ and in fact only on the topology and its area. This follows from the fact that the theory is invariant under *area preserving diffeomorphism*. The field strength can be written in the form $F_{\mu\nu} = \epsilon_{\mu\nu}F$ and hence the action takes the form,

$$S = -\frac{1}{4g^2}\int_\Sigma \mathrm{d}^2x\sqrt{\det G_{\mu\nu}}F^2. \tag{16.3}$$

Apart from the area form $\mathrm{d}^2x\sqrt{\det G_{\mu\nu}}$ this action is independent of the metric and is therefore invariant under area preserving diffeomorphism.

The lattice partition function defined on an arbitrary triangulation of the surface, as described in Fig. 16.1, is given by,

$$\mathcal{Z}_{\mathcal{M}} = \int \prod_l \mathrm{d}U_l \prod_\triangle Z_\triangle[U_\triangle], \tag{16.4}$$

where $\prod_l$ denotes a product over all links, $U_\triangle$ is the holonomy around a plaquette $U_\triangle = \prod_{l\in\triangle} U_l$, and $Z_\triangle$ is a plaquette action. For the latter one uses a heat kernel

[4] The partition function on any Riemann surface of the discretized theory was written down in [184]. An identical result was found also in the continuum formulation [228].

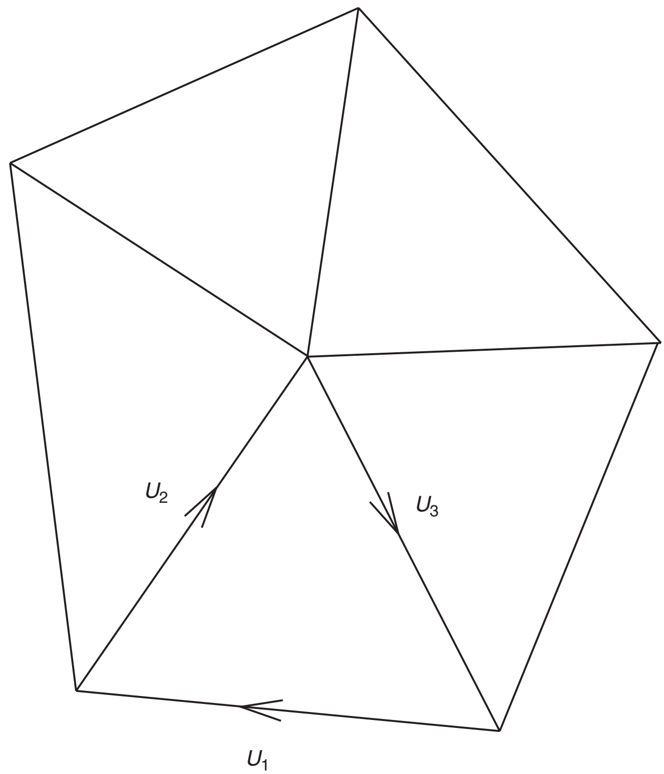

Fig. 16.1. A triangulation of the Riemann surface $\mathcal{M}$. A group matrix is placed on each link.

action rather than the Wilson action (which is $Z_\triangle[U_\triangle] = \mathrm{e}^{-\frac{1}{g^2}\mathrm{tr}(U+U^\dagger)}$), i.e.

$$Z_\triangle[U] = \sum_R d_R \chi_R(U)\mathrm{e}^{-tc_2(R)}, \tag{16.5}$$

where the summation is over the irreducible representations R of the group; d_R, $\chi_R(U)$ and $c_2(R)$ denote the dimension, character of U and the second Casimir operator of R, respectively, and $t = g^2a^2$ with a^2 being the plaquette area. The character, which was also discussed in Section 3.5 in relation to the ALA algebras, is defined here as $\chi_R(U) \equiv \mathrm{tr}_R[U]$. The holonomy U (its subscript $\triangle$ is omitted from here on) behaves as $U \approx 1 - iaF$ when a is small. Note that the region of validity of (16.5) is not only $a \to 0$ with F fixed, but actually also $a \to 0$ with F going to infinity as $a^{-1/2}$ because this is the region for which the exponential $-\frac{1}{4t}a^2\,\mathrm{Tr}\,F^2$ is of order unity.

We will briefly review the derivation that singles out (16.5) as a convenient choice among the different lattice theories which belong to the same universality class. Let us look for a function $\Psi(U,t)$ that will replace the continuum $\mathrm{e}^{-\frac{1}{4t}a^2\,\mathrm{Tr}\,F^2}$.

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The requirements which we impose on Ψ are:

1. As t goes to zero (and, therefore, for finite g also a goes to zero) we want the holonomy to be close to 1,

$$\Psi(U,0) = \delta(U-1). \tag{16.6}$$

2. For any $V \in G$ we have,

$$\Psi(V^{-1}UV,t) = \Psi(U,t). \tag{16.7}$$

   In other words Ψ is a *class function*.
3. Ψ satisfies the heat kernel equation,

$$\left\{ \frac{\partial}{\partial t} - \sum_{a,b} g_{ab}\partial^a\partial^b \right\} \Psi(U,t) = 0, \tag{16.8}$$

   where g_{ab} is the inverse of the Cartan metric,

$$g^{ab} = \mathrm{tr}(t^a t^b), \tag{16.9}$$

   which was defined and discussed in Section 3.2.1.

To see that (16.5) is an approximate solution to the heat-kernel equation we note that any class function is a linear combination of characters. The differentiation of a character in the direction of a Lie algebra element t^a is given by,

$$\partial^{a_1}\partial^{a_2}\cdots\partial^{a_k}\chi_R(U) = \frac{i^k}{k!}\chi_R(Ut^{(a_1}t^{a_2}\cdots t^{a_k)}) + O(U-1). \tag{16.10}$$

The notation $\chi_R(Ut^{a_1}t^{a_2}\cdots t^{a_k})$ stands for the trace of the multiplication of the matrices which represent U and $t^{a_1},\ldots,t^{a_k}$ in the representation R. The brackets $(\cdots)$ imply symmetrization with respect to the indices. The term $O(U-1)$ means that the corrections are of the order of $U-1 \sim aF \sim t^{1/2}$. Since,

$$\sum_{a,b} g_{ab}\partial^a\partial^b\chi_R(U) \approx -\frac{1}{2}\chi_R(U\sum_{a,b} g_{ab}t^{(a}t^{b)}) = -c_2(R)\chi_R(U), \tag{16.11}$$

we see that (16.5) is the correct answer up to terms of the order of $O(t^{3/2})$ which drop in the continuum limit. Using (16.5) as the starting point, we finally find the following form of the partition function,

$$\mathcal{Z}(N,H,\lambda A) = \sum_R d_R^{2-2H}\,\mathrm{e}^{-\frac{\lambda A c_2(R)}{2N}}. \tag{16.12}$$

To get from (16.5) to (16.12) we take the following steps. First we make use of the additivity property of the heat-kernel action. Consider two triangles glued along U_1 as depicted in Fig. 16.2.

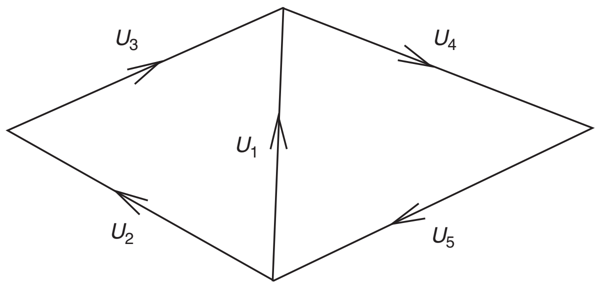

Fig. 16.2. Integrating over U_1 on a link which is common to the two triangles.

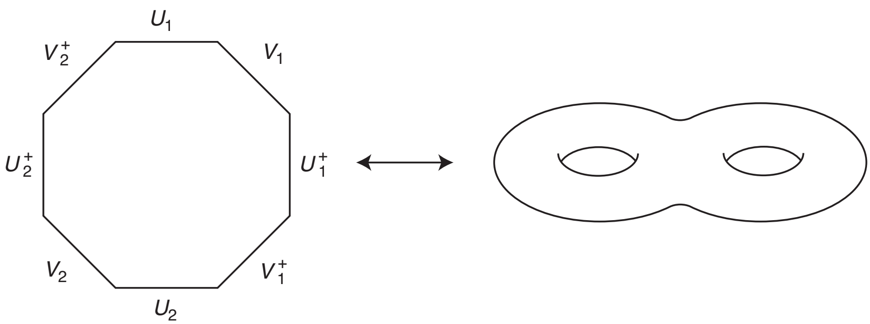

Fig. 16.3. Opening up of a genus-two surface.

Using the orthogonality of characters,

$$\int \mathrm{d}U \chi_{R_1}(VU)\chi_{R_2}(U^\dagger W) = \delta_{R_1,R_2} \frac{\chi_{R_1}(VW)}{\dim R_1}, \tag{16.13}$$

we find,

$$\int \mathrm{d}U_1 Z_{\triangle_1}(U_2U_3U_1)Z_{\triangle_2}(U_1^\dagger U_4U_5) = Z_{\triangle_1+\triangle_2}(U_2U_3U_4U_5). \tag{16.14}$$

This relation can be used to argue that the lattice representation is exact and independent of the triangulation since using this we can add as many triangles as desired, thus reaching the continuum limit. We can also use this relation to reduce the number of triangles to the minimum needed to capture the topology of $\mathcal{M}$. Describing a genus H manifold in term of a 4H-polygon with identified sides as described in Fig. 16.2 for a genus-two Reimann surface $a_1b_1a_1^{-1}b_1^{-1}\ldots a_Hb_Ha_H^{-1}b_H^{-1}$. The partition function on such a manifold can be written as,

$$\mathcal{Z}_\mathcal{M} = \sum_R d_R e^{-\frac{\lambda A c_2(R)}{2N}} \int \prod DU_l DV_l \chi_R[U_1V_1U_1^\dagger V_1^\dagger \ldots U_HV_HU_H^\dagger V_1^\dagger]. \tag{16.15}$$

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We can simplify this expression using again the orthogonality of the characters and the relation,

$$\int DU \chi_R[AUBU^\dagger] = \frac{1}{\dim_R}\chi_R[A]\chi_R[B], \tag{16.16}$$

to arrive at (16.12).

16.3 The partition function of gYM_2 theories

Pure YM_2 theory is in fact a special representative of a wide class of 2D gauge theories which are invariant under area preserving diffeomorphisms. These generalized YM_2 theories are described by the following generalized partition function,

$$\mathcal{Z}(G,H,A,\Phi) = \int [DA^\mu][DB] \exp[\int_\Sigma \mathrm{d}^2x \sqrt{\det G_{\mu\nu}}\ \mathrm{tr}(iBF - \Phi(B))], \tag{16.17}$$

where $F = F^{\mu\nu}\epsilon_{\mu\nu}$ with ϵ_{ij} being the antisymmetric tensor $\epsilon_{12} = -\epsilon_{21} = 1$. B is an auxiliary Lie-algebra-valued pseudo-scalar field.[5]

We wish to generalize the substitution (16.5) for the plaquette action (16.17),

$$Z_\triangle[U] = \int \mathcal{D}B \mathrm{e}^{\mathrm{tr}\{iaBF - t\Phi(B)\}} \overset{?}{\rightarrow} \Psi(U,t). \tag{16.18}$$

Here B is a Hermitian matrix and Φ is an invariant function (invariant under $B \rightarrow U^{-1}BU$ for $U \in G$). The quadratic case $\Phi(X) = g^2\mathrm{tr}(X^2)$ obviously corresponds to the YM_2 theory. We will take Φ to be of the form,

$$\Phi(X) = \sum_{\{k_i\}} a_{\{k_i\}} \prod_i \mathrm{tr}(X^i)^{k_i}, \tag{16.19}$$

(e.g. $\mathrm{tr}(X^3)^2 + \mathrm{tr}(X^6)$) For $SU(N)$ $(U(N))$, $\mathrm{tr}(X^i)$ can be expressed for $i \geq N$ $(i > N)$ in terms of $\mathrm{tr}(X^i)$ for smaller i. Thus the summands in (16.19) are not independent. This does not affect the following discussion. Moreover, in the large N limit that we will discuss in the following section, the terms do become independent.

We define the general structure constants $d_{abc\dots k}$ to be,

$$d_{abc\dots k} \overset{\text{def}}{=} g_{aa'}g_{bb'}g_{cc'}\cdots g_{kk'}tr(t^{a'}t^{b'}t^{c'}\cdots t^{k'}). \tag{16.20}$$

For every partition $r_1 + r_2 + \cdots + r_j$, we define the Casimir,

$$C_{\{r_1+r_2+\cdots+r_j\}} \overset{\text{def}}{=} \frac{1}{(r_1+r_2+\cdots+r_j)!} d_{a_1^{(1)}\dots a_{r_1}^{(1)}} d_{a_1^{(2)}\dots a_{r_2}^{(2)}} \cdots d_{a_1^{(j)}\dots a_{r_j}^{(j)}} \quad t^{(a_1^{(1)}} \cdots t^{a_{r_1}^{(1)}} t^{a_1^{(2)}} \cdots t^{a_{r_j}^{(j)})} \tag{16.21}$$

[5] In principle, we could perturb the ordinary YM_2 with operators of the form $\frac{1}{g^{2k-2}}\mathrm{tr}(F^k)$, without the need for an auxiliary field.

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Note that the index of $C_{\{\cdot\}}$ will always pertain to a partition. Thus $C_{\{p\}} \neq C_{\{r_1+r_2+\cdots+r_j\}}$ even if $p = r_1 + r_2 + \cdots + r_j$. The brackets in the t-s mean a total symmetrization ($(r_1 + r_2 + \cdots + r_j)!$ terms).

C_ρ can easily be seen to commute with all the group elements and so, by Schur's lemma, is a constant matrix in every irreducible representation.

We claim that the correct lattice generalization of (16.5) is,

$$\sum_R d_R \chi_R(U) \mathrm{e}^{-t\Lambda(R)}, \tag{16.22}$$

where,

$$\Lambda(R) = \sum_{\{k_i\}} a_{\{k_i\}} C_{\{k_1\cdot 1 + k_2 \cdot 2 + k_3 \cdot 3 + \cdots\}}(R). \tag{16.23}$$

This results from the requirements that $\Psi(U, t)$ must satisfy:

1. $\Psi(U, 0) = \delta(U - 1)$.
2. Ψ is a class-function.
3. Ψ satisfies the equation,

$$\left\{ \frac{\partial}{\partial t} - \sum_{\{k_j\}} a_{\{k_j\}} \prod_l \left((ia)^{-l} d_{a_1 a_2 \ldots a_l} \frac{\partial^l}{\partial F_{a_1} \cdots \partial F_{a_l}} \right)^{k_l} \right\} \Psi(U, t) + O(U - 1) = 0 \tag{16.24}$$

For the U s that are important in the weight for a single plaquette, $U - 1$ is of the order of magnitude of aF which, in turn, is of the order of magnitude of $O(t^{1/\nu})$ where ν is the maximal degree of Φ. Thus, the corrections to Ψ are $O(a^{-(1+1/\nu)})$ and drop out in the continuum limit.

The partition function for the generalized YM_2 theory is therefore,

$$\mathcal{Z}(G, \Sigma_H, \Phi) = \sum_R (\dim R)^{2-2H} \mathrm{e}^{-\frac{\lambda A}{2N}\Lambda(R)}, \tag{16.25}$$

where A is the area of the surface and $\Lambda(R)$ is defined in (16.23).

16.4 Loop averages in the generalized case

The full solution of the YM_2 theory includes, in addition to the partition function, closed expressions for the expectation values of products of any arbitrary number of Wilson loops,

$$W(R_1, \gamma_1, \ldots R_n \gamma_n) = <\prod_{i=1}^n \mathrm{Tr}_{R_i} \mathcal{P} e^{i \oint_{\gamma_i} A dx}>, \tag{16.26}$$

where the path-ordered product around the closed curve γ_i is taken in the representation R_i. Using loop equations, one can derive an algorithm to compute Wilson loops on the plane. This can be further generalized into a prescription for

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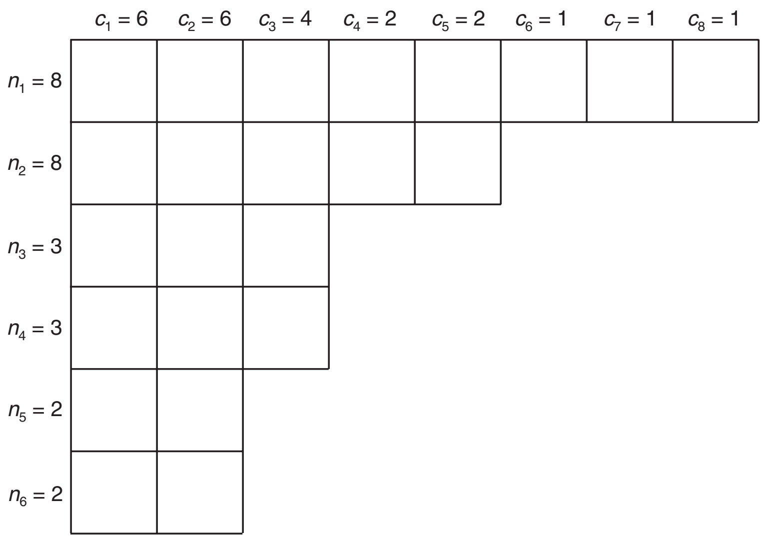

Fig. 16.4. Wilson loops on a torus.

computing those averages for non-intersecting loops on an arbitrary two manifold. Let us briefly summarize the latter. One cuts the 2D surface along the Wilson loop contours, see Fig. 16.4, forming several connected "windows". Each window contributes a sum over all irreducible representations of the form of (16.12). In addition, for each pair of neighbouring windows, a Wigner coefficient,

$$D_{R_1 R_2 f} = \int dU \chi_{R_1}(U)\chi_{R_2}(U^\dagger)\chi_f(U), \tag{16.27}$$

is attached. Altogether, one finds,

$$W(R_1, \gamma_1, \dots R_n \gamma_n) = \frac{1}{\mathcal{Z}} \frac{1}{N^n} \sum_{R_1} \dots \sum_{R_n} D_{R_1 \dots R_n} \prod_{i=1}^{N_w} d_{R_i}^{2-2G_i} \, \mathrm{e}^{\frac{-\lambda A_i C_2(R_i)}{2N}}, \tag{16.28}$$

where N_w is the number of windows, $2 - 2G_i$ is the Euler number associated with the window i and $D_{R_1 \dots R_n}$ is the product of the Wigner coefficients for neighboring windows. For the case of intersecting loops a set of differential equations provides a recursion relation by relating the average of a loop with n intersections to those of loops with $m < n$ intersections.

Generalizing these results to the gYM_2 is straightforward. The only alteration that has to be invoked is to replace the $\mathrm{e}^{\frac{-\lambda A C_2(R)}{2N}}$ factors that show up in those algorithms with similar factors where the second Casimir operator is replaced by the generalized Casimir operator (16.23). For instance the expectation value of a simple Wilson loop on the plane is given by,

$$\langle W(R, \gamma) \rangle = \mathrm{e}^{\frac{-\lambda A_\gamma \Lambda(R)}{2N}}, \tag{16.29}$$

where A_γ is the area enclosed by γ.

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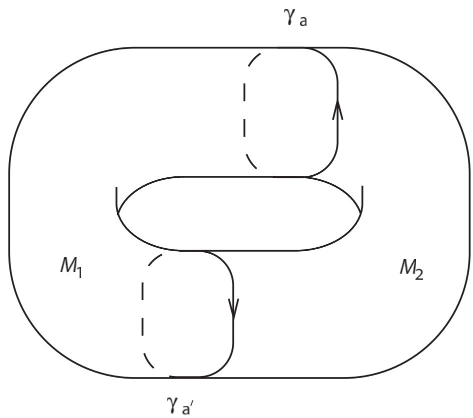

Fig. 16.5. Wilson loops on a torus.

It is interesting to note that for odd Casimir operators the expectation values of real representations (like the adjoint representation) equal unity due to the fact that the corresponding Casimirs vanish.

16.5 Stringy YM_2 theory

Before dwelling into the stringy description of the generalized YM_2 thoeries, we first review the proof of the stringy nature of pure YM_2 theory.[6] We then generalize this construction to the generalized YM_2 and present several examples to demonstrate the nature of the maps that contribute to the gYM_2 and their weights.

The partition function expressed as a sum over irreducible $SU(N)$ ($U(N)$) representations (16.12) can be expanded in terms of powers of $\frac{1}{N}$. This involves expanding the dimension and the second Casimir operator of the various representations. The representation of $U(N)$ or $SU(N)$ are described by Young tableau $Y(R)$, see for instance Fig. 16.5, composed of $r \leq N$ horizontal lines each with n_i boxes so that $n_1 \geq n_2 \geq \ldots \geq n_r$. The $U(N)$ and $SU(N)$ second Casimir operators of a representation R and its dimension are given by,

$$C_2^{U(N)}(R) = N\sum_{i=1}^{r} n_i + \sum_{i=1}^{r} n_i(n_i+1-2i) = Nn + 2\hat{P}_{\{2\}}(R)^{U(N)}(R),$$
$$C_2^{SU(N)}(R) = N\sum_{i=1}^{r} n_i + \sum_{i=1}^{r} n_i(n_i+1-2i) - \frac{(\sum_{i=1}^{r} n_i)^2}{N} = Nn + 2\hat{P}_{\{2\}}(R)^{SU(N)},$$
$$d_R = \frac{\prod_{i\leq j\leq N}(n_i - i - n_j + j)}{\prod_{i\leq j\leq N}(i-j)}, \tag{16.30}$$

[6] The stringy description of Yang–Mills theory in two-dimensional Riemann surfaces was introduced by D. Gross and W. Taylor in [115], [118] and [119]. The formulation of the two-dimensional Yang–Mills theory in terms of topological string theories was done in [126] and [69].

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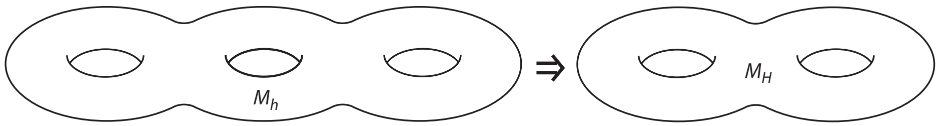

Fig. 16.6. A map from $\mathcal{M}_h$ to $\mathcal{M}_H$.

with $n = \sum_{i=1}^{r} n_i$. For every representation R there is a conjugate representation $\bar{R}$ whose Young tableau $Y(\bar{R})$ has its rows and columns interchanged. To determine the Casimir operator of the conjugate representation we use (16.30) with $2\hat{P}_{\{2\}}(R) = -2\hat{P}_{\{2\}}(R)^{U(N)}(R)$.

Using the Frobenius relations between representations of the symmetric group S_n and representations of $SU(N)$ $(U(N))$, the coefficients of this asymptotic expansion were written in terms of characters of S_n. The latter can be shown to correspond to permutations of the sheets covering the target space. The final result takes the form of,

$$Z(A,H,N) \sim \sum_{n^\pm, i^\pm=0}^{\infty} \sum_{p_1^\pm,\ldots,p_{i^\pm}^\pm \in T_2 \subset S_{n^\pm}} \sum_{s_1^\pm, t_1^\pm,\ldots,s_H^\pm, t_H^\pm \in S_{n^\pm}} \left(\frac{1}{N}\right)^{(n^+ + n^-)(2H-2)+(i^+ + i^-)}$$
$$\frac{(-1)^{(i^+ + i^-)}}{i^+! i^-! n^+! n^-!} (\lambda A)^{(i^+ + i^-)} e^{-\frac{1}{2}(n^+ + n^-)\lambda A} e^{\frac{1}{2}((n^+)^2 + (n^-)^2 - 2n^+ n^-)\lambda A/N^2}$$
$$\delta_{S_{n^+} \times S_{n^-}} \left(p_1^+ \cdots p_{i^+}^+ p_1^- \cdots p_{i^-}^- \Omega_{n^+,n^-}^{2-2H} \prod_{j=1}^{H} [s_j^+, t_j^+] \prod_{k=1}^{H} [s_k^-, t_k^-] \right), \tag{16.31}$$

where $[s,t] = sts^{-1}t^{-1}$. Here δ is the delta function on the group algebra of the product of symmetric groups $S_{n^+} \times S_{n^-}$, T_2 is the class of elements of $S_{n^\pm}$ consisting of transpositions, and Ω_{n^+,n^-}^{-1} are certain elements of the group algebra of the symmetric group $S_{n^+} \times S_{n^-}$.

The formula (16.31) nearly factorizes, splitting into a sum over $n^+, i^+, \cdots$ and $n^-, i^-, \cdots$. The contributions of the $(+)$ and $(-)$ sums were interpreted as arising from two “sectors” of a hypothetical worldsheet theory. These sectors correspond to orientation reversing and preserving maps, respectively. One views the $n^+ = 0$ and $n^- = 0$ terms as leading order terms in a $1/N$ expansion. At higher orders the two sectors are coupled via the $n^+ n^-$ term in the exponential and via terms in $\Omega_{n^+ n^-}$.

Thus, the conventional YM_2 theory has an interpretation in terms of sums of covering maps of the target space, see Fig. 16.5. Those maps are weighted by the factor of $N^{2-2h} e^{-\frac{1}{2} n \lambda A}$ where h is the genus of the world-sheet and A is the area of the target space. The power of N^{2-2h} is obtained from the Riemann–Hurwitz formula,

$$2h - 2 = (n^+ + n^-)(2H-2) + (i^+ + i^-), \tag{16.32}$$

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where $B = i^+ + i^-$ is the total branching number. The number of sheets above each point in target space (the degree of the map) is n and $\lambda = g^2 N$ is the string tension. Maps that have branch points are weighted by a factor of λA. The dependence on the area A results from the fact that the branch point can be at any point in the target space.

16.6 Toward the stringy generalized YM_2

Note that the stringy description of the YM_2 theory does not attribute any special weight for maps that have branch points of a degree higher than one, nor is there a special weight for two (or more) branch points that are at the same point in the target space. The latter maps are counted with weight zero, at least for a toroidal target space, since they constitute the boundary of map space.

The main idea behind a stringy behavior of the gYM_2 is to associate nonzero weights to those boundary maps, once one considers the general $\Phi(B)$ case rather than the B^2 theory. In other words, we anticipate that we will have to add for the $\mathrm{tr}(B^3)$ theory, for example, maps that have a branch point of degree 2 and count them, as well, with a weight proportional to A. From the technical point of view the emergence of the YM_2 description in terms of maps followed from a large N expansion of the dimensions and the second Casimir operators of (16.12). Obviously a similar expansion of the former applies also for the generalized models and therefore what remains to be done is to properly treat the Casimirs appearing in the exponents of (16.25).

In [119], the expansion of the second Casimir operator $C_2(R)$ of a representation R introduced the branch points and the string tension contributions to the partition function. The $C_2(R)$ was expressed in terms of the eigenvalue of the sum of all the $\frac{n(n-1)}{2}$ transpositions of n elements (permutations containing a single cycle of length 2), where n is the number of boxes in R. This is the outcome of the following formula,

$$C_2(R) = nN + 2\hat{P}_{\{2\}}(R), \tag{16.33}$$

where $\hat{P}_2(R)$ is the value of the scalar matrix representing the sum of transpositions $\sum_{i<j\leq n}(ij)$ in the representation R of S_n (the matrix commutes with all permutations and thus is scalar). In the partition function, $C_2(R)$ was multiplied by $\frac{\lambda A}{2N}$. The resulting term $\frac{1}{2}n\lambda A$ arises from the action and is proportional to the string tension. The term $\lambda A\hat{P}_{\{2\}}$ arises from the measure and is interpreted as the contribution of branch points to the weight of a map.

Our task is, therefore, to express the generalized Casimirs C_ρ of (16.21) in terms of $\hat{P}_{\rho'}$, the generalizations of $\hat{P}_{\{2\}}(R)$. This is expressed as,

$$C_\rho(R) = \sum_{\rho'} \alpha_\rho^{\rho'} N^{h_\rho^{\rho'}} \hat{P}_{\rho'}(R), \tag{16.34}$$

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where $\alpha_\rho^{\rho'}$ are coefficients that are independent of R and the power factors $h_\rho^{\rho'}$, are adjusted so that a string picture is achieved.

The $\hat{P}_{\{\rho'\}}$ factors are associated with ρ' which is an arbitrary partition of certain numbers, namely,

$$\rho' : \sum_i k_i \cdot i = \overbrace{1+1+\cdots+1}^{k_1} + \overbrace{2+2+\cdots+2}^{k_2} + \cdots, \tag{16.35}$$

$\hat{P}_{\{\rho'\}}(R)$ is the product of two factors. The first is the sum of all the permutations in S_n (n is the number of boxes of R) which are in the equivalence class that is characterized by having k_i cycles of length i for $i \geq 2$. Just like the case of $\hat{P}_{\{2\}}(R)$, the matrix $\hat{P}_{\{\rho'\}}(R)$ commutes with all permutations and thus is a scalar. The sum is taken in the representation R of S_n. The second factor is,

$$\binom{n - \sum_{i=2} ik_i}{k_1}, \tag{16.36}$$

which can be interpreted later as the number of ways to put k_1 marked points on the remaining sheets that do not participate in the branch points.

16.7 Examples

A complete diagrammatic expansion of the operators was determined in [105]. Using this expansion one can write down the stringy description of the partition function for any generalized YM theory. We end this chapter with a few examples of the Casimir factors for various choices of $\Phi(B)$ for both $U(N)$ and $SU(N)$ groups.

1. For $\frac{\lambda}{N}\mathrm{tr}(B^2)$ which is the conventional YM_2 theory we get (16.33),

$$\frac{2\lambda}{N}\hat{P}_{\{2\}} + \lambda\hat{P}_{\{1\}}. \tag{16.37}$$

   The first term means that we give a factor of $\frac{2\lambda A}{N}$ for each branch point, and the second term means that we have a factor of λ for each marked point (i.e. this is the string tension).

2. For $\alpha N^{-2}\mathrm{tr}(B^3)$ in $U(N)$ we get,

$$3\alpha N^{-2}\hat{P}_{\{3\}} + 3\alpha N^{-1}\hat{P}_{\{2\}} + 3\alpha N^{-2}\hat{P}_{\{1+1\}} + \frac{1}{2}\alpha\hat{P}_{\{1\}} + \frac{1}{2}\alpha N^{-2}\hat{P}_{\{1\}}. \tag{16.38}$$

   The first term is the contribution from branch points of degree 2 (the simple branch points are of degree 1). The next term is a modification to the weight of the usual branch points. The third is the weight of two marked points at the same point (but different sheets), which will translate into $n^+(n^+-1)$ $+ n^-(n^- - 1)$ in the weight of a map for which (n^+, n^-) are the numbers of sheets of each orientability. The last two terms are modifications to the cosmological constant (or, in our terminology, to the weight of the single marked point).

Note that, because of the original power of N^{-2} we do not get the usual N^{2-2h} stringy behaviour in the partition function. We can overcome this problem by interpreting the last term not as the usual marked point, but as a *microscopic handle* that is attached to the point. By interpreting certain marked points as actually being microscopic handles (or higher Riemann surfaces) we can always adjust the power of N to be N^{2-2h}. Similarly, we should interpret the term $3\alpha N^{-2}\hat{P}_{\{1+1\}}$ as a connecting tube. We will come back to this interpretation towards the end of this section and investigate it further in the next subsection.

3. For $N^{-2}\mathrm{tr}(B^3)$ in $SU(N)$ we obtain the following corrections to (16.38),

$$\begin{aligned} &-\frac{6}{N^3}\hat{P}_{\{2+1\}} - \frac{12}{N^3}\hat{P}_{\{2\}} + \frac{12}{N^4}\hat{P}_{\{1+1+1\}} \\ &-\left(\frac{6}{N^2} - \frac{12}{N^4}\right)\hat{P}_{\{1+1\}} - \left(\frac{3}{N^2} - \frac{2}{N^4}\right)\hat{P}_{\{1\}}. \end{aligned} \tag{16.39}$$

These terms and the terms in the previous example (16.38) do not mix chiralities (i.e. sheets of opposite orientations). In the full theory (chiral and anti-chiral sectors) there is the corresponding anti-chiral term:

$$\begin{aligned} &+\frac{6}{N^3}\hat{P}_{\{\bar{2}+\bar{1}\}} + \frac{12}{N^3}\hat{P}_{\{\bar{2}\}} - \frac{12}{N^4}\hat{P}_{\{\bar{1}+\bar{1}+\bar{1}\}} \\ &+\left(\frac{6}{N^2} - \frac{12}{N^4}\right)\hat{P}_{\{\bar{1}+\bar{1}\}} + \left(\frac{3}{N^2} - \frac{2}{N^4}\right)\hat{P}_{\{\bar{1}\}}. \end{aligned} \tag{16.40}$$

For $SU(N)$ there are additional terms that do mix chiralities. They are,

$$-\frac{6}{N^3}(\hat{P}_{\{\bar{2}+1\}} - \hat{P}_{\{2+\bar{1}\}}) + \frac{12}{N^4}(\hat{P}_{\{\bar{1}+\bar{1}+1\}} - \hat{P}_{\{\bar{1}+\bar{1}+1\}}). \tag{16.41}$$

The first term is the contribution of maps that have a branch point in one orientability and a marked point in the other (at the same target space point). The second term is the contribution of maps with three marked points – two for one orientability and one for the other.

To illustrate the content of these formulae in terms of representations, we will calculate the value of the third Casimir $\frac{1}{6}d_{abc}t^{(a}t^{b}t^{c)}$ for a totally antisymmetric representation of $SU(N)$ with k boxes. The term $\hat{P}_{\{3\}}$ is translated into the sum of all the permutations of the k indices of a totally antisymmetric tensor that are 3-cycles, this gives $\frac{1}{3}k(k-1)(k-2)$. The term $\hat{P}_{\{2\}}$ gives the sum of all the permutations that are 2-cycles, that is $-\frac{1}{2}k(k-1)$ (a minus

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sign comes from antisymmetry). All in all we get,

$$
\begin{aligned}
C_{\{3\}}(k) &\equiv \frac{1}{6} d_{abc} t^{(a} t^b t^{c)} \\
&= k(k-1)(k-2) - \frac{3}{2} k(k-1) + \frac{3}{2} N k(k-1) \\
&+ \frac{1}{2} k + \frac{1}{2} N^2 k + \frac{3}{N} k(k-1)(k-2) \\
&- 3k(k-1) + \frac{6}{N} k(k-1) - 3k + \frac{2}{N^2} k(k-1)(k-2) + \frac{6}{N^2} k(k-1) + \frac{2k}{N^2} \\
&= \frac{k}{2N^2} (N+1)(N+2)(N-k)(N-2k).
\end{aligned} \tag{16.42}
$$

4. For $N^{-3}\mathrm{tr}(B^4)$ in $U(N)$ we get,

$$
\begin{aligned}
&4N^{-3}\hat{P}_{\{4\}} + 6N^{-2}\hat{P}_{\{3\}} + 6N^{-3}\hat{P}_{\{2+1\}} + \frac{8}{3}N^{-2}\hat{P}_{\{1+1\}} \\
&\quad + \left(\frac{4}{3}N^{-1} + 6N^{-3}\right)\hat{P}_{\{2\}} + \left(\frac{1}{6} + \frac{5}{6}N^{-2}\right)\hat{P}_{\{1\}}.
\end{aligned}
$$

The terms that have an extra microscopic handle are,

$$
6N^{-3}\hat{P}_{\{2+1\}} + \frac{8}{3}N^{-2}\hat{P}_{\{1+1\}} + 6N^{-3}\hat{P}_{\{2\}} + \frac{5}{6}N^{-2}\hat{P}_{\{1\}}.
$$

5. For $N^{-4}(\mathrm{tr}(B^2))^2$ in $U(N)$ we get,

$$
\begin{aligned}
&24N^{-4}\hat{P}_{\{3\}} + 8N^{-4}\hat{P}_{\{2+2\}} + 4N^{-3}\hat{P}_{\{2+1\}} \\
&+ \frac{16}{3}N^{-3}\hat{P}_{\{2\}} + \left(2 + \frac{8}{3}N^{-4}\right)\hat{P}_{\{1+1\}} + \left(\frac{2}{3}N^{-2} + \frac{1}{3}N^{-4}\right)\hat{P}_{\{1\}}.
\end{aligned} \tag{16.43}
$$

and additional terms,

$$
8N^{-4}\hat{P}_{\{2+\bar{2}\}} + 4N^{-3}(\hat{P}_{\{\bar{2}+1\}} + \hat{P}_{\{2+\bar{1}\}}) + 2N^{-2}\hat{P}_{\{1+\bar{1}\}},
$$

that mix the two chiral sectors.

The meaning of the term $2N^{-2}\hat{P}_{\{1+\bar{1}\}}$ is a factor of $N^{-2}\mathrm{e}^{-2\alpha n^+ n^- A}$, where α is the coefficient of the $N^{-4}(\mathrm{tr}(B^2))^2$ term in the action. n^+ is the number of sheets of positive orientability and n^- is the number of sheets of negative orientability for a given map.

16.8 Summary

In this chapter we studied the generalized two-dimensional Yang–Mills theory on Riemann surfaces. We reviewed the exact formulae for the partition function and Wilson loop averages of the conventional YM theory. We then presented the generalization of these results in the context of the generalized YM theories. These expressions are based on a replacement of the second Casimir operator with more general Casimir operators depending on the particular model. There is another method to obtain these results [228], i.e. by regarding the general Yang–Mills actions as perturbations of the topological theory at zero area.

Using the relations between $SU(N)$ representations and representations of the symmetric groups S_n, we wrote down the generalizations that have to be made in the Gross–Taylor string rules for 2D Yang–Mills theory, so as to make the generalized Yang–Mills theory for $SU(N)$ or $U(N)$ a local string theory as well. The extra terms are special weights for certain maps with branch points of a degree higher than one.

An obvious extension of the results presented in this chapter is to consider other gauge groups. The conventional YM_2 theory with gauge groups $O(N)$ or $Sp(N)$ which were shown to be related to maps from non-orientable world-sheets.

One can further couple the gYM_2 theories to fermionic matter in analogy to 't Hooft's analysis presented in Chapter 10. This domain of research is far from being fully explored. A particularly interesting question is to find out certain $\Phi(B)$s that lead to a special behaviour of the coupled system. For example, in the $U(1)$ case, the representations R are labeled by an integer n and for $\Phi(B) = -\alpha \log(1 + \lambda B^2)$ we get,

$$\mathcal{Z}(U(1), A) = \sum_n (1 + \lambda n^2)^{-\alpha A}$$

which has a singularity for $2\alpha A = 1$.