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ON A GENERALIZATION OF THE TRANSFORMATION SEMIGROUP

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Introduction

In a series of papers ((1967), (1967a) and (1967b)) Magill has considered the semigroups $\mathscr{T}(X, Y; \theta)$ (definition below), a natural, but extensive, generalization of the usual transformation semigroup $\mathscr{T}(X)$. They have also been studied in Sullivan (to appear). Under the assumption that θ be onto Magill described their automorphisms and determined when one $\mathscr{T}(X, Y; \theta)$ is isomorphic to another.

It is the purpose of this paper to generalize these results to arbitrary θ.

The work is facilitated by the introduction in §1 of certain congruences. These congruences which bear some striking resemblances to Green's relations on an arbitrary semigroup shed light on the algebraic structure of $\mathscr{T}(X, Y; \theta)$ and appear to be a powerful tool for such considerations.

1. Some structure results for $\mathscr{T}(X, Y; \theta)$

Our notation is that of Clifford and Preston (1967) with some additions and departures. Let X and Y be sets and $\theta : Y \to X$. We shall write $\mathscr{T}(X, Y; \theta)$ (T for short) for the semigroup consisting of the set of all mappings from X into Y, $\mathscr{T}(X, Y)$, together with the operation

$$\alpha * \beta = \alpha\theta\beta \qquad (\alpha, \beta \in T)$$

If $Y = X$ we shall put $T = \mathscr{T}(X; \theta)$; and if $\theta = \iota$ we write $T = \mathscr{T}(X)$. Clearly $\mathscr{T}(X)$ is the usual full transformation semigroup. For $\alpha, \beta \in T$ we define $\alpha \, d \, \beta (\alpha \, l \, \beta, \alpha \, r \, \beta)$ to mean $\theta\alpha\theta = \theta\beta\theta$ ($\theta\alpha = \theta\beta, \alpha\theta = \beta\theta$). (Note: juxtaposition of functions will always mean composition in the usual sense.) Clearly $d \supseteq l, r$. We put $h = l \cap r$ and prove

LEMMA 1.1. *The relations d, l, and r are congruences on $\mathscr{T}(X, Y; \theta)$.*

PROOF. It is evident that l, r, and d are equivalences so we need only demonstrate the compatibility property. If $\alpha d \beta$ then $\theta\alpha\theta = \theta\beta\theta$. Hence for all $\gamma \in T$, $\gamma\theta\alpha\theta = \gamma\theta\beta\theta$. It follows that $\theta(\gamma * \alpha)\theta = \theta\gamma\theta\alpha\theta = \theta\gamma\theta\beta\theta = \theta(\gamma * \beta)\theta$ which means $\gamma * \alpha d \gamma * \beta$. Similarly $\alpha * \gamma d \beta * \gamma$, and this gives the result for d. The proofs for l and r are similar.

The relations l and r have a simple interpretation. Let $\alpha, \beta \in T$; then $\alpha l \beta$ means that α and β have the same action on $Y\theta$, while $\alpha r \beta$ is equivalent to the statement: "α and β map each element of X into the same partition class of θ". In the same vein $\alpha d \beta$ means that α and β map each element of $Y\theta$ into the same partition class of θ. It follows that each d-class is determined by a mapping $Y\theta \to Y/\theta \circ \theta^{-1}$. Since $|Y/\theta \circ \theta^{-1}| = |Y\theta|$, T has $|Y\theta|^{|Y\theta|}$ d-classes. In the same way the number of l-and r-classes of T are $|Y|^{|Y\theta|}$ and $|Y\theta|^{|X|}$ respectively.

THEOREM 1.2. *The congruences* 1 *and r commute and we have*

$$d = l \circ r = r \circ l = l \vee r.$$

PROOF. We show $d = l \circ r$. Let $\alpha, \beta \in T$ be such that $\alpha(l \circ r)\beta$. Then there exists $\gamma \in T$ such that $\theta\alpha = \theta\gamma$ and $\beta\theta = \gamma\theta$. Now $(\theta\gamma)\theta = \theta(\gamma\theta)$ so $(\theta\alpha)\theta = \theta(\beta\theta)$ and it follows that $\alpha d \beta$. Conversely if $\alpha d \beta$ then $\theta\alpha\theta = \theta\beta\theta$. Define $\gamma : X \to Y$ by

$$\begin{aligned} x\gamma &= x\beta \quad \text{when } x \in Y\theta \\ &= x\alpha \quad \text{otherwise.} \end{aligned}$$

Evidently $\theta\gamma = \theta\beta$: we complete the proof by showing that $\gamma\theta = \alpha\theta$. If $x \notin Y\theta$ then $x\gamma = x\alpha$ so that $x\gamma\theta = x\alpha\theta$. Otherwise $x = y\theta$ for some $y \in Y$. Hence

$$x\gamma\theta = x\beta\theta = y\theta\beta\theta = y\theta\alpha\theta = x\alpha\theta$$

Thus $\alpha(l \circ r)\beta$ and it follows that $d = l \circ r$.

Now $d = d^{-1} = (l \circ r)^{-1} = r^{-1} \circ l^{-1} = r \circ l$, and it only remains to show that $l \circ r = l \vee r$. But this is immediate, since l and r commute. (See Clifford and Preston (1967), Lemma 1.4).

Many of the simpler results concerning Green's relations follow from their congruence properties and the formula $\mathscr{D} = \mathscr{L} \circ \mathscr{R}$. Thus we may reinterpret such theorems in our context. (However the analogy is *only* formal as will be verified by examining the table set out in Example A, below.) Thus in the same way as Clifford and Preston (1967), page 48, we have

THEOREM 1.3. *If R is an r-class, L an l-class then R meets L if and only if they are contained in the same d-class.* Similarly (c.f. Clifford and Preston (1967), page 49),

THEOREM 1.4. *The set product (with respect to* $*$*) of any l-class and any r-class of* $\mathscr{T}(X, Y; \theta)$ *is always contained in a single d-class.*

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The former result shows that l, r, and d break up $\mathscr{T}(X, Y; \theta)$ in a manner analogous to Greens relations, $\mathscr{L}, \mathscr{R}$, and $\mathscr{D}$, on an arbitrary semigroup. In particular we have the familiar "egg-box" picture. We also have a partial analogue of Green's Lemma (Clifford and Preston (1967), page 49). For $\alpha \in T$, let L_α denote the l-class of T containing α and define, for $\beta \in T, G(\alpha, \beta) : L_\alpha \to L_\beta$ by demanding

$$\gamma\, G(\alpha,\beta)\,\big|\, Y\theta \;=\; \beta\,\big|\, Y\theta$$

$$\gamma\, G(\alpha,\beta)\,\big|\, X\backslash Y\theta \;=\; \gamma\,\big|\, X\backslash Y\theta \qquad (\gamma \in L_\alpha)$$

Since $L_\alpha = \{\gamma \in T;\ \theta\gamma = \theta\alpha\} = \{\gamma \in T;\ \gamma \,|\, Y\theta = \alpha \,|\, Y\theta\}$ it is clear that $G(\alpha, \beta)$ is a bijection. Moreover $G(\alpha, \beta)$ and $G(\beta, \alpha)$ are mutually inverse, and it follows that all the l-classes of T have the same cardinality.

Further, if $\alpha\, r\, \beta$ then $G(\alpha, \beta)$ is r-class preserving. For, let $\gamma \in L_\alpha$ and put $\gamma' = \gamma\, G(\alpha, \beta)$. Then if $x \in Y\theta$,

$$\begin{aligned} x\gamma'\theta &= x\beta\theta && \text{(by definition of } G(\alpha,\beta)) \\ &= x\alpha\theta && (\text{since } \alpha\, r\, \beta) \\ &= x\gamma\theta && (\text{since } x \in Y\theta \text{ and } \alpha\,|\,Y\theta = \gamma\,|\,Y\theta) \end{aligned}$$

— while if $x \notin Y\theta, x\gamma' = x\gamma$. Hence $\gamma'\theta = \gamma\theta$ so that $\gamma' r \gamma$. Dually, $G(\beta, \alpha)$ preserves r-classes, and it follows easily that $|H_\alpha| = |H_\beta|$. To summarize:

THEOREM 1.5. *The l-classes of $\mathscr{T}(X, Y; \theta)$ all have the same cardinality, and any two h-classes contained in a single r-class have the same cardinality.*

EXAMPLE A. We take $X = Y = \{1, 2, 3\}$ and write (i, j, k) for the mapping $1 \to i, 2 \to j, 3 \to k$. We shall calculate the l-, r-, and d-classes for $\mathscr{T}(X, X; \theta) = \mathscr{T}(X; \theta)$ with $\theta = (1, 2, 2)$. From the above discussion there are $|X\theta|^{|X\theta|} = 2^2 = 4$ d-classes of T. Each d-class is determined by a mapping

$$\Delta : \{1, 2\} \to \{\{1\}\{2, 3\}\}$$

where $\alpha d \beta$ if and only if $x\alpha$ and $x\beta$ belong to $x\Delta, x \in X\theta$.

Below is written out the class structure of T: the four large blocks represent the d-classes and within each d-class the rows represent the r-classes and the columns the l-classes. The cells formed by the conjunction of an l-class and an r-class are the h-classes of T. Under each d-class we have written the particular mapping which determined the class.

We have imitated the format of Clifford and Preston (1967), pages 55 and 56, to underline the formal duality between our relations and Green's. However the example points out a distinction: in the latter case all $\mathscr{H}$-classes within a single $\mathscr{D}$-class have the same cardinality. This is evidently not so here.

Mappings				Label	l-classes	r-classes
(122) (123) (121)	(132) (133) (131)			$1 \to \{1\}$ $2 \to \{2,3\}$	2	2
(212) (213) (211)	(312) (313) (311)			$1 \to \{2,3\}$ $2 \to \{1\}$	2	2
(112) (113) (111)				$1 \to \{1\}$ $2 \to \{1\}$	1	2
(222) (223) (221)	(232) (233) (231)	(322) (323) (321)	(332) (333) (331)	$1 \to \{2,3\}$ $2 \to \{2,3\}$	4	2
				Total	$9 = 3^2$	$8 = 2^3$

If S is a semigroup then we may hope to obtain a semigroup with reductive properties (see Clifford and Preston (1967)) by identifying those elements of S which multiply identically. More precisely, define $\lambda(\rho)$ on S by putting $a\lambda b(a\rho b)$ when $xa = x\,b(ax = bx)$ for all $x \in S$. If δ is the congruence generated by $\lambda \cup \rho$ then S/λ is left reductive, S/ρ is right reductive, and S/δ is both left and right reductive.

In the case of $\mathscr{T}(X, Y; \theta)$ it is elementary to show $\lambda = l$, $\rho = r$, and $\delta = d$. It follows that the l-(r-)classes of T consist of those mappings which multiply identically from the right (left), and the h-classes of those mappings which multiply identically from either side.

We have

THEOREM 1.6. *The following statements are equivalent*:

(1) *T is weakly reductive*

(2) *Each h-class contains one element*

(3) *θ is either injective or surjective*

PROOF. The foregoing discussion shows the equivalence of (1) and (2). If θ is one to one or onto X, then $\alpha\theta = \beta\theta$ and $\theta\alpha = \theta\beta$ together imply that $\alpha = \beta$. It follows that (3) implies (2). Finally if θ is neither one to one nor onto we may select $y_1, y_2 \in Y, y_1 \neq y_2$, with $y_1\theta = y_2\theta$; and an $x \in X$ such that $x \notin Y\theta$. Choose two mappings in T, α_1 and α_2, which agree on $X \backslash x$, with the additional property:

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$$x\alpha_n = y_n \qquad (n = 1, 2)$$

Since $x \notin Y\theta, \theta\alpha_1 = \theta\alpha_2$. However it is clear that $\alpha_1\theta = \alpha_2\theta$ so we have $\alpha_1 h \alpha_2$, with $\alpha_1 \neq \alpha_2$. Hence (2) implies (3), which completes the proof.

NOTE. One may also prove:

T is left (right) reductive if and only if the cardinality of each l-(r-)class is one, if and only if θ is onto (one to one). It follows that no $\mathscr{T}(X, Y; \theta)$ is "properly" weakly reductive: that is, weakly reductive and neither left nor right reductive.

The following theorem indicates the result of identifying the elements of T which multiply identically from any one side.

THEOREM 1.7. $\mathscr{T}(X, Y; \theta)/d \cong \mathscr{T}(Y\theta)$

PROOF. Let d_α be a d-class of T containing $\alpha \in T$.
Define $\bar{d}_\alpha : Y\theta \to Y\theta$ by

$$x\bar{d}_\alpha = x\alpha\theta \qquad (x \in Y\theta)$$

Now $\bar{d}_\alpha$ is well defined, for if $\alpha' \in d_\alpha$ then

$$x\theta\alpha'\theta = x\theta\alpha\theta \qquad (x \in Y)$$

from which follows $x\alpha'\theta = x\alpha\theta$ for all $x \in Y\theta$. To check that $d_\alpha \to \bar{d}_\alpha$ is one to one we merely reverse this argument. For surjectivity, let $\beta : Y\theta \to Y\theta$. Define $\alpha' : Y\theta \to Y$ by taking $x\alpha'$ as any element of $x\beta\theta^{-1}$, and extend α' to a mapping in T. Then for all $x \in Y\theta$,

$$x\bar{d}_{\alpha'} = x\alpha'\theta = x\beta$$

Finally we verify that $d_\alpha \to \bar{d}_\alpha$ is a homomorphism. If the semigroup operation on T/d is denoted by $\circ$ we have

$$d_\alpha \circ d_\beta = d_{\alpha * \beta} = d_{\alpha\theta\beta} \qquad (\alpha, \beta \in T)$$

Hence for all $x \in Y\theta$

$$x\overline{d_\alpha \circ d_\beta} = x\alpha\theta\beta\theta = x\bar{d}_\alpha\bar{d}_\beta$$

The following example will illustrate (1.7).

EXAMPLE B. Take $X = \mathbb{R}$ and define $x\theta$ to be x correct to four decimal places. Then $\theta : \mathbb{R} \to \mathbb{R}$. We consider $\mathscr{T}(X; \theta)$. Multiplication in $\mathscr{T}(X; \theta)$, $\alpha * \beta = \alpha\theta\beta$, may be thought of as a method of composing functions where one approximates at the intermediate stage. Call $\theta\alpha\theta$ the *table* of α. In fact, if α is the sine function, say, $x\theta\alpha\theta$ is what is actually computed if one evaluates $\sin x$ by standard mathematical tables. In practical problems it is convenient to regard functions as the same if their tables are the same. Thus (1.7) asserts that if we identify functions in

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this way, the resulting structure is $\mathscr{T}(\mathbb{R}_4)$, where $\mathbb{R}_4 = \mathbb{R}\theta$ is the set of numbers of the form $n \times 10^{-4}$, $n \in \mathbb{Z}$.

Observe that $\theta^2 = \theta$. In general, if this is the case, we shall call $\mathscr{T}(X;\theta)$ an *approximation system on* X.

It will be of value later — and is of independent interest — to know, given $\alpha \in T$, whether we can find $\alpha' \neq \alpha$ such that α and α' multiply identically in T. That is to say, when is $|H_\alpha| > 1$?*. The following theorem provides a useful criterion.

THEOREM 1.9. *If* $\alpha \in \mathscr{T}(X, Y;\theta)$ *with* $|X\alpha| > |Y\theta|$ *where* $|X\alpha|$ *is infinite then* $|H_\alpha| > 1$. *Moreover there exists* $\alpha' \in H_\alpha$, $\alpha' \neq \alpha$, *such that* $|X\alpha| = |X\alpha'|$.

PROOF. If $|H_\alpha| > 1$ then we may find $\alpha' \in T$ and $a \in X$ such that $\alpha | Y\theta = \alpha' | Y\theta$, $a\alpha \neq a\alpha'$, and $a\alpha\theta = a\alpha'\theta$. It follows that $|H_\alpha| > 1$ implies there exists $a \in X \backslash Y\theta$ with $|(a\alpha)\theta\theta^{-1}| > 1$. On the other hand, if we assume the latter statement then there exists $b \in (a\alpha)\theta\theta^{-1}$ with $b \neq a\alpha$. Define $\alpha' : X \to Y$ by

$$x\alpha' = x\alpha \qquad (x \neq a)$$

$$a\alpha' = b$$

Then $\theta\alpha = \theta\alpha'$ since $a \notin Y\theta$; and $\alpha\theta = \alpha'\theta$, since $a\alpha'\theta = b\theta = a\alpha\theta$. We have shown:

$$|H_\alpha| > 1 \text{ if and only if } |(a\alpha)\theta\theta^{-1}| > 1 \text{ for some } a \in X \backslash Y\theta.$$

Now assume $|X\alpha| > |Y\theta|$ where $|X\alpha|$ is infinite. We have

$$|X\alpha| \geqq |(X \backslash Y\theta)\alpha| \geqq |X\alpha \backslash Y\theta\alpha| = |X\alpha|,$$

where the last deduction follows from $|X\alpha| > |Y\theta| \geqq |Y\theta\alpha|$. Hence $|X\alpha| = |(X\backslash Y\theta)\alpha|$ and it is clearly impossible for $\theta | (X \backslash Y\theta)\alpha$ to be one to one. The result is now evident from the first part of the proof.

COROLLARY. If $|Y\theta| \geqq \aleph_0$ then every h-class of T containing an element whose rank (cardinality of range) is greater than rank θ, contains more than one element.

NOTE. In Example A $|H_{(321)}| = 1$. However $|X(321)| = 3 > |X\theta| = 2$. Hence the supposition that $|X\alpha|$ is infinite may not be removed.

2. Isomorphisms of $\mathscr{T}(X, Y; \theta)$

In this section we characterize $\mathscr{T}(X, Y;\theta)$ in terms of certain cardinals associated with θ. Magill (1967) has proved the same result, but with the restriction that θ maps Y *onto* X.

The next result indicates why the congruences we have introduced in section 1

* H_α, L_α, etc., will always be with respect to h, l, etc., and not Green's relations.

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are relevant to a discussion of isomorphisms and automorphisms of T. The proof is a simple consequence of some relationships established immediately before (1.6).

THEOREM 2.1. *If $\phi : \mathscr{T}(X, Y; \theta_1) \to \mathscr{T}(X, Y; \theta_2)$ is an isomorphism then ϕ maps each l-(r-, d-, h-)class of $\mathscr{T}(X, Y; \theta_1)$ onto a corresponding l-(r-, d-, h-) class of $\mathscr{T}(X, Y; \theta_2)$.*

We shall denote the operations in $\mathscr{T}(X, Y; \theta_1)$ and $\mathscr{T}(X, Y; \theta_2)$ by $*$ and $\times$ respectively. The following theorem is vital since it enables us to obtain information about isomorphisms by considering more tractable semigroups than $\mathscr{T}(X, Y; \theta)$.

THEOREM 2.2. *Let $\phi : \mathscr{T}(X, Y; \theta_1) \to \mathscr{T}(X, Y; \theta_2)$ be an isomorphism and define*

$$\rho : (\mathscr{T}(X, Y)\theta_1, \circ) \to (\mathscr{T}(X, Y)\theta_2, \circ) \quad \textit{and}$$

$$\lambda : (\theta_1 \mathscr{T}(X, Y), \circ) \to (\theta_2 \mathscr{T}(X, Y), \circ) \quad \textit{by}$$

$$(\alpha\theta_1)\rho = \alpha\phi\theta_2 \quad \textit{and}$$

$$(\theta_1\alpha)\lambda = \theta_2\alpha\phi$$

Then ρ and λ are isomorphisms.

NOTE. The symbol "$\circ$" means composition of functions and it is with respect to this operation that λ and ρ are isomorphisms. Also, to facilitate the notation, we shall assume henceforth that the symbol ϕ is evaluated immediatcly to its left. For example, $\alpha\phi\theta$ will mean $(\alpha\phi)\theta$ and $\theta\beta\phi$ will mean $\theta(\beta\phi)$.

PROOF. If $\alpha, \beta \in \mathscr{T}(X, Y)$ then, by (2.1), $\alpha\theta_1 = \beta\theta_1$ if and only if $\alpha\phi\theta_2 = \beta\phi\theta_2$. This shows at once that ρ is well defined and one to one, and the result for λ follows dually. Since ϕ itself is onto, λ and ρ are onto, and it only remains to demonstrate the morphism property. We prove the result for ρ:

$$\begin{aligned}(\alpha\theta_1\beta\theta_1)\rho &= [(\alpha\theta_1\beta)\theta_1]\rho = [(\alpha * \beta)\theta_1]\rho \\ &= [(\alpha * \beta)\phi]\theta_2 = (\alpha\phi \times \beta\phi)\theta_2 \\ &= \alpha\phi\theta_2 \cdot \beta\phi\theta_2.\end{aligned}$$

If an element of $\mathscr{T}(X, Y)$ has range $\{a\}$ we shall denote it by κ_a. If S is a transformation semigroup under composition, $S \leqq \mathscr{T}(X)$, then we write $K(S) = \{a \in X; \kappa_a \in S\}$. We have from Symons (to appear)

LEMMA 2.3. *If S and T are transformation semigroups, $S, T \subseteq \mathscr{T}(X)$, with $K(S), K(T) \neq \square$, and $\phi : S \to T$ is an isomorphism then*

$$x\alpha\phi = xg^{-1}\alpha g \qquad (x \in K(T))$$

where $g : K(S) \to K(T)$ is a bijection defined by the equality $\kappa_a\phi = \kappa_{ag}$.

We require one further result from Symons (to appear). It is an immediate corollary of Theorem (5.1) of that paper.

LEMMA 2.4. *If Y_1 and Y_2 are subsets of X_1 and X_2 respectively, and both contain more than two elements then $(\mathscr{T}(X_1, Y_1), \circ) \cong (\mathscr{T}(X_2, Y_2), \circ)$ if and only if $|X_1 \backslash Y_1| = |X_2 \backslash Y_2|$ and $|Y_1| = |Y_2|$.*

We now state the principal result of this section.

THEOREM 2.5. *$\mathscr{T}(X, Y; \theta_1) \cong \mathscr{T}(X, Y; \theta_2)$ if and only if $\theta_1 h = g\theta_2$, where $g \in \mathscr{G}(Y)$ and $h \in \mathscr{G}(X)$.*

PROOF. Assume $\theta_1 h = g\theta_2$ for $g \in \mathscr{G}(Y)$, $h \in \mathscr{G}(X)$. For $\alpha \in \mathscr{T}(X, Y)$, define $\alpha\phi = h^{-1}\alpha g$. Clearly ϕ is a permutation of $\mathscr{T}(X, Y)$. Moreover for all $\alpha, \beta \in \mathscr{T}(X, Y)$

$$\alpha\phi \times \beta\phi = h^{-1}\alpha g\theta_2 h^{-1}\beta g = h^{-1}\alpha(\theta_1 h)h^{-1}\beta g$$
$$= h^{-1}\alpha\theta_1\beta g = (\alpha * \beta)\phi$$

Conversely let $\phi : \mathscr{T}(X, Y; \theta_1) \to \mathscr{T}(X, Y; \theta_2)$ be an isomorphism. We recall (2.2) and apply (2.3) to λ and ρ. (This is permissible since $\theta\mathscr{T}(X, Y) \leqq \mathscr{T}(Y)$ and $\mathscr{T}(X, Y)\theta \leqq \mathscr{T}(X)$). Since both $\theta_1\mathscr{T}(X, Y)$ and $\theta_2\mathscr{T}(X, Y)$ contain all the constant functions of $\mathscr{T}(Y)$ we have

$$(\theta_1\alpha)\lambda = g^{-1}\theta_1\alpha g \qquad (\alpha \in \mathscr{T}(X, Y))$$

for some $g \in \mathscr{G}(Y)$. On the other hand $K(\mathscr{T}(X, Y)\theta_n) = Y\theta_n$ $(n = 1, 2)$ and it follows that

$$x(\alpha\theta_1)\rho = xh^{-1}\alpha\theta_1 h \qquad (x \in Y\theta_2, \alpha \in \mathscr{T}(X, Y))$$

where $h : Y\theta_1 \to Y\theta_2$ is a bijection. Now $\theta_2(\alpha\theta_1)\rho = \theta_2\alpha\phi\theta_2 = (\theta_1\alpha)\lambda\theta_2$. If we substitute the determinations of ρ and λ obtained above we have $\theta_2 h^{-1}\alpha\theta_1 h = g^{-1}\theta_1\alpha g\theta_2$. Putting $\alpha = \kappa_a$, where $\alpha \in Y$, this equation yields $a\theta_1 h = ag\theta_2$. Hence $\theta_1 h = g\theta_2$.

To complete the proof we show that h may be regarded as the restriction of a function in $\mathscr{G}(X)$, that is, $|X \backslash Y\theta_1| = |X \backslash Y\theta_2|$.

If X is finite, the result follows since $|Y\theta_1| = |Y\theta_2|$, while if $|X|$ is infinite and $|Y\theta_1| = |Y\theta_2|$ is 1 or 2 the result is clear. Hence we may assume that $|X|$ is infinite and $|Y\theta_n| > 2$. Now $\mathscr{T}(X, Y)\theta_n = \mathscr{T}(X, Y\theta_n)$ (this requires little proof) and from (2.2), $\mathscr{T}(X, Y\theta_1) = \mathscr{T}(X, Y\theta_2)$. An application of (2.4) completes the demonstration.

REMARKS (i). It is natural to ask for the conditions under which (A) the semigroups $T_1 = \mathscr{T}(X_1, Y_1; \theta_1)$ and $T_2 = \mathscr{T}(X_2, Y_2; \theta_2)$ are isomorphic. It is easy to see that if (B) $\theta_1 h = g\theta_2$ where h is now a bijection from X_1 to X_2 and g a bijection from Y_1 to Y_2, then (A) follows. In fact, $h^{-1}.g$ is an isomorphism from

T_1 to T_2. (Henceforth we shall call such isomorphims *formula-expressed*, and denote $h^{-1} \cdot g$ by $\psi(h, g)$). On the other hand, if (A) holds, it follows from a trivial extension of (2.2) and (2.3) that $|Y_1| = |Y_2|$ and $|Y\theta_1| = |Y\theta_2|$. If $|Y_n\theta_n| > 2$ then (2.4) gives $|X_1 \backslash Y_1\theta_1| = |X_2 \backslash Y_2\theta_2|$ in a manner analogous to that of the text. Hence $|X_1| = |X_2|$ and there is a formula-expressed isomorphism from T_2 to $\mathscr{T}(X_1, Y_1; \theta_3)$ for a certain θ_3. A little manipulation and (2.5) suffice to give (B). If $|Y_n\theta_n| = 2$ then we are denied (2.4) and the questions hinges on the following cardinal-theoretic proposition:

(C) $2^{\bar{\aleph}} = 2^{\aleph}$ implies $\aleph = \bar{\aleph}$, for $\aleph$ and $\bar{\aleph}$ infinite cardinals. (Note that (C) may be deduced from the Generalized Continuum Hypothesis. It is unknown whether it follows without such an assumption). If (C) is assumed then it can be shown that (B) follows, while if it is denied, one can exhibit a counterexample (see (5.1) of Symons (to appear). Finally, (A) together with $|Y_n\theta_n| = 1$ by no means implies (B): trivially $\mathscr{T}(X_1, Y_1; \theta_1)$ is isomorphic to $\mathscr{T}(X_2, Y_2; \theta_2)$ if $|Y_1| = |Y_2| = 1$. This affords a counterexample to (B) by choosing $|X_1| \neq |X_2|$.

(ii) Translated into our (more restricted) context, (2.3) of Magill (1967) yields the result: If $\mathscr{T}(X_1, Y_1; \theta_1) \cong \mathscr{T}(X_2, Y_2; \theta_2)$ then $g\theta_2 = \theta_1 h$ where $g \in \mathscr{G}(Y)$ and $h : Y\theta_1 \to Y\theta_2$ is a bijection.
Hence we could have deduced (2.5) from this result and the last paragraph of the proof of (2.5).

If we write $\theta_1 \sim \theta_2$ when $\mathscr{T}(X, Y; \theta_1) \cong \mathscr{T}(X, Y; \theta_2)$ then $\sim$ determines an equivalence on $\mathscr{T}(Y, X)$. One expects that $\theta_1 \sim \theta_2$ when θ_1 and θ_2 are the same "sort" of mapping. The following theorem makes precise what we mean by this.

THEOREM 2.6. *The following statements are equivalent*

(i) $\theta_1 \sim \theta_2$

(ii) *The defect of* θ_1 *(that is* $|X \backslash Y\theta_1|$*) equals the defect of* θ_2*, and there exists a bijection* $f : Y/\theta_1 \circ \theta_1^{-1} \to Y/\theta_2 \circ \theta_2^{-1}$ *such that* $|Af| = |A|$, $A \in Y/\theta_1 \circ \theta_1^{-1}$.

PROOF. If $\theta_1 \sim \theta_2$ then by (2.5) $\theta_1 h = g\theta_2$, where $g \in \mathscr{G}(Y)$ and $h \in \mathscr{G}(X)$. Hence $|X \backslash Y\theta_1| = |X \backslash Yg\theta_2 h^{-1}| = |X \backslash Y\theta_2 h^{-1}|$. But $X = Xh^{-1}$ and since h is one to one, $|Xh^{-1} \backslash Y\theta_2 h^{-1}| = |(X \backslash Y\theta_2)h^{-1}| = |X \backslash Y\theta_2|$. In the notation of Clifford and Preston (1967) put

$$\theta_1 = \begin{pmatrix} X_i \\ x_i \end{pmatrix} \text{ and } \theta_2 = \begin{pmatrix} Y_i \\ y_i \end{pmatrix}$$

Then

$$\begin{pmatrix} X_i \\ x_i \end{pmatrix} h = g \begin{pmatrix} Y_i \\ y_i \end{pmatrix}$$

so that

$$\begin{pmatrix} X_i \\ x_i h \end{pmatrix} = \begin{pmatrix} Y_j g^{-1} \\ y_j \end{pmatrix}$$

Defining $X_i f = Y_j$ when $X_i = Y_j g^{-1}$, we see that $f : Y/\theta_1 \circ \theta_1^{-1} \to Y/\theta_2 \circ \theta_2^{-1}$ satisfies the latter assertion of (ii).

Conversely if θ_1 and θ_2 are related as in (ii), with

$$\theta_1 = \begin{pmatrix} X_i \\ x_i \end{pmatrix}, \qquad \theta_2 = \begin{pmatrix} Y_i \\ y_i \end{pmatrix}$$

then we may take the indexing of θ_2 so that $X_i f = Y_i$. Choose $g \in \mathscr{G}(Y)$, demanding that $g \mid X_i : X_i \to Y_i$ is a bijection. Further, if we define $x_i h = y_i$ for all i then $h : Y\theta_1 \to Y\theta_2$ is a bijection. Since $|X \backslash Y\theta_1| = |X \backslash Y\theta_2|$ we may extend h to a permutation of X. We have

$$\theta_2 = \begin{pmatrix} Y_i \\ y_i \end{pmatrix} = \begin{pmatrix} X_i g \\ x_i h \end{pmatrix} = g^{-1} \begin{pmatrix} X_i \\ x_i \end{pmatrix} h = g^{-1}\theta_1 h,$$

as required.

Hence, up to isomorphism, $\mathscr{T}(X, Y; \theta)$ is determined by a (cardinal) number of cardinals, to wit, the defect of θ and the cardinalities of the partition classes of θ. If $|X|$ is finite then $\operatorname{Rank}\theta_1 = \operatorname{Rank}\theta_2$ implies $\operatorname{Defect}\theta_1 = \operatorname{Defect}\theta_2$. Hence given $\theta_1 = \begin{pmatrix} X_i \\ x_i \end{pmatrix}$, we may take θ_2 to be any mapping of the form $\begin{pmatrix} X_i \\ y_i \end{pmatrix}$ and have $\theta_1 \sim \theta_2$. In the particular case $X = Y$ we may demand that each $y_i \in X_i$ so that $\theta_2^2 = \theta_2$. Thus we have a

COROLLARY. *If $|X|$ is finite then $\mathscr{T}(X; \theta)$ is isomorphic to an approximation system on X.*

This argument fails in general. For example, take $X = N$, the natural numbers, and define $\theta : N \to N$ by $x\theta = [x/2]$ (= integral part of $x/2$). If $\theta \sim \varepsilon$ where $\varepsilon^2 = \varepsilon$ then $\square = \operatorname{Defect}\theta = \operatorname{Defect}\varepsilon$. Hence ε is onto and we have $\varepsilon = \iota$. However the partition classes of θ all have cardinality 2 and the bijection f of (2.6) cannot be constructed.

REMARKS (i). There is a natural extension of this result along the lines indicated by the remarks following (2.5) – provided we assume $|Y_n\theta_n| > 2$, $(n = 0, 1)$.

(ii) Theorem (3.1) of Magill (1967) gives (2.6) for θ_1 and θ_2 surjections.

3. Automorphisms of $\mathscr{T}(X, Y; \theta)$

In this section we shall describe the automorphism group of $\mathscr{T}(X,Y;\theta)$. We shall demand that $|Y\theta| > 2$ so that we may use the following result from Symons (to appear).

THEOREM 3.1. *Let $X \supseteq Y$ and assume $|Y| > 2$. Then any automorphism of $\mathscr{T}(X, Y)$ (the operation of composition is understood) is of the form $\alpha \to h^{-1}\alpha h$, $\alpha \in \mathscr{T}(X, Y)$, where $h \in \mathscr{G}(X)$ and $h \mid Y \in \mathscr{G}(Y)$.*

If ϕ is an automorphism of $T = \mathscr{T}(X, Y;\theta)$ then the mappings λ and ρ introduced in the last section are automorphisms of the semigroups $(\theta T, \circ)$ and $(T\theta, \circ)$. As in the proof of (2.5) we have, for some $g \in \mathscr{G}(Y)$

$$\theta\alpha\phi = (\theta\alpha)\lambda = g^{-1}\theta\alpha g \qquad (\alpha \in T) \tag{3.1.1}$$

while from (3.1) and the observation that $T\theta = \mathscr{T}(X, Y\theta)$ we have for some $h \in \mathscr{G}(X)$

$$\alpha\phi\theta = (\alpha\theta)\rho = h^{-1}\alpha\theta h \qquad (\alpha \in T) \tag{3.1.2}$$

Moreover in the same way as in the proof of (2.5) we can derive

$$\theta h = g\theta \tag{3.1.3}$$

If we apply this last formula to (3.1.1) and (3.1.2) we obtain

$$\alpha\phi\theta = h^{-1}\alpha g\theta = \alpha\psi(h,g)\theta \tag{3.1.4}$$

and

$$\theta\alpha\phi = \theta h^{-1}\alpha g = \theta\alpha\psi(h,g) \qquad (\alpha \in T)$$

If (3.1.3) is satisfied by bijections h and g then $\psi(h,g)$ is an automorphism of $\mathscr{T}(X, Y;\theta)$. Automorphisms of this nature will be called *formula-expressed automorphisms* and the group of all such by FEA(T).

Continuing with ϕ an arbitrary automorphism, let $\psi = \psi(h,g) \in$ FEA(T). Then $\theta \cdot \alpha\psi^{-1}\phi = \theta \cdot \alpha\psi^{-1}\psi = \theta\alpha$ so that $(\alpha\psi^{-1}\phi)l\alpha$. Since a similar result holds for r we have $(\alpha\psi^{-1}\phi)h\alpha$, and it follows that $\psi^{-1}\phi$ maps each h-class of T onto itself. Denoting the totality of such automorphisms by HCA(T) (h-class automorphisms) we summarize the foregoing in

THEOREM 3.2. *If* $T = \mathscr{T}(X, Y;\theta)$ *where* $|Y\theta| > 2$ *then*

$$\operatorname{Aut} T = \mathrm{FEA}(T) \cdot \mathrm{HCA}(T). \tag{3.2.1}$$

REMARKS. (i) Both FEA(T) and HCA(T) are subgroups of Aut T. Moreover HCA(T) is a normal subgroup. To see this let $\psi = \psi(h,g) \in$FEA(T). Then $\psi^{-1} = \psi(h^{-1}, g^{-1})$. If $\phi \in$ HCA(T) and H_α is an h-class containing α then it is easy to see that $\alpha \in H_\alpha\psi^{-1}\phi\psi$. Since h-classes are preserved by automorphisms we must have $H_\alpha\psi^{-1}\phi\psi = H_\alpha$. Thus $\psi^{-1}\phi\psi \in$ HCA(T). We also have that HCA(T) $\cap$ FEA(T) is trivial (this follows readily from (3.3) below) and hence (3.2.1) expresses Aut T as a *semi-direct product* of FEA(T) and HCA(T).

(ii) We are also able to exhibit in a similar fashion all isomorphisms between $T_1 = \mathscr{T}(X_1, Y_1;\theta_1)$ and $T_2 = \mathscr{T}(X_2, Y_2;\theta_2)$, $|Y\theta_n| > 2$. If ϕ is such a mapping then by the remarks following (2.5), there exists a formula-expressed isomorphism $\psi = \psi(l, m) : T_1 \to T_2$. If follows that $\phi^{-1}\psi \in \operatorname{Aut} T_2 = \mathrm{FEA}(T_2)\mathrm{HCA}(T_2)$. A simple calculation gives $\phi = \psi(l, m)\psi(h, g)\zeta$ where $\psi(h,g) \in \mathrm{FEA}(T_2)$ and $\zeta \in \mathrm{HCA}(T_2)$. Now $\psi(l,m)\psi(h,g) = \psi(lh, mg)$ and $\theta_1 lh = m\theta_2 h = mg\theta_2$. It

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follows that $\psi(lh, mg)$ is a formula-expressed isomorphism from T_1 to T_2. Hence we state:

If $\phi : T_1 \to T_2$ where $|Y_n\theta_n| > 2$ then ϕ is an isomorphism if and only if ϕ is the product of a formula-expressed isomorphism from T_1 to T_2 and an h-class automorphism of T_2.

If $\psi \in \mathrm{HCA}(T)$ and $\alpha, \beta \in T$ then $(\alpha\theta\beta)\psi = (\alpha * \beta)\psi = \alpha\psi\theta\beta\psi = \alpha\theta\beta\psi = \alpha\theta\beta$, where the last two deductions are derived from $\alpha\psi \cdot \theta = \alpha\theta$ and $\theta \cdot \beta\psi = \theta\beta$. It is easy to see that $\mathscr{T}(X, Y)\theta\mathscr{T}(X, Y)$ is the set $\{\gamma \in \mathscr{T}(X, Y); |X\gamma| \leqq |Y\theta|\}$, and the above shows that ψ fixes all elements of rank $\leqq$ rank θ. This property together with the invariance of h-classes characterizes $\mathrm{HCA}(T)$.

THEOREM 3.3. *If $T = \mathscr{T}(X, Y; \theta)$ where $|Y\theta| > 2$, and $\psi : T \to T$ then $\psi \in \mathrm{HCA}(T)$ if and only if*

(i) $\alpha\psi = \alpha$ *for all* $\alpha \in T$ *with* $|X\alpha| \leqq |Y\theta|$ *and*

(ii) $\psi | H$ *is a permutation of* H *for all h-classes,* H.

PROOF. Necessity was established prior to the statement of the theorem. On the other hand if $\psi : T \to T$ satisfies (i) and (ii) it is clear that ψ is a permutation. Moreover, we have $(\alpha\theta\beta)\psi = \alpha\theta\beta = (\alpha\theta)\beta = (\alpha\psi\theta)\beta = \alpha\psi(\theta\beta) = \alpha\psi\theta\beta\psi \cdot \|$

Let I denote the set of mappings in T whose rank does not exceed $|Y\theta|$ and if H is an h-class of T write $H^* = H \backslash I$. All we may say about the action of $\psi \in \mathrm{HCA}(T)$ upon H^* is that it is a permutation. Hence we have a

COROLLARY

$$\mathrm{HCA}(T) \cong \prod_{H \in T/h} \mathscr{G}(H^*)$$

(Note: we interpret $\mathscr{G}(\square)$ as the trivial group).

We now state our main result

THEOREM 3.4. *If $|Y\theta| > 2$ the automorphisms of $\mathscr{T}(X, Y; \theta)$ are of the form $\alpha \to \alpha\phi$, where, for some $g \in \mathscr{G}(Y)$ and $h \in \mathscr{G}(X)$ such that $g\theta = \theta h$,*

$$H_\alpha\phi = h^{-1}H_\alpha g = H_{h^{-1}\alpha g} \qquad (H_\alpha \in T/h)$$

and $\phi | H_\alpha$ is one to one; and for $|X\alpha| \leqq |Y\theta|$

$$\alpha\phi = h^{-1}\alpha g.$$

Moreover any mapping of this form is an automorphism.

PROOF. The first part of the theorem follows directly from (2.1), (3.2), (3.3), and the observation that formula-expressed automorphisms preserve rank. To see the converse it suffices to assume ϕ has the form set out in the statement of the theorem and to show $\alpha \to h\alpha\phi g^{-1}$ is an h-class automorphism. We omit the details.

Let $\mathcal{M}$ be a partition of Y. Following Magill (1967) we put $G(\mathcal{M}) = \{g \in \mathcal{G}(Y);$ $Mg \in \mathcal{M}$ for all $M \in \mathcal{M}\}$. Observe that $M \to Mg$ is a permutation of the elements of $\mathcal{M}$ and that $G(\mathcal{M})$ is a subgroup of $\mathcal{G}(Y)$. We have

THEOREM 3.5. $\mathrm{FEA}(T) \cong \mathcal{G}(X \backslash Y\theta) \times G(Y/\theta \circ \theta^{-1})$.

PROOF. Let $\psi = \psi(h, g) \in \mathrm{FEA}(T)$. The required isomorphism is $\psi(h, g) \to (h \,|\, X \backslash Y\theta, g)$. This map is well defined; for if $h^{-1}\alpha g = h_1^{-1}\alpha g_1$ for all $\alpha \in T$, letting α range through the constant functions, we have immediately $g = g_1$. We cancel g and it is sufficient to observe that we may separate any pair of elements of X with a mapping in T (since $|Y| \geqq |Y\theta| > 2$) to deduce $h^{-1} = h_1^{-1}$. Moreover our mapping is into the required group. To see this let $\theta = \begin{pmatrix} X_i \\ x_i \end{pmatrix}$. The formula $\theta = g^{-1}\theta h$ may be written $\begin{pmatrix} X_i \\ x_i \end{pmatrix} = \begin{pmatrix} X_i g \\ x_i h \end{pmatrix}$, and an examination of this equality gives the result. The homomorphism property is clear and if $g = \iota_Y$ and $h \,|\, X \backslash Y\theta = \iota_{X \backslash Y\theta}$ then $\theta h = g\theta$ becomes $\theta h = \theta$ so that $h = \iota_X$. It only remains to demonstrate that $\psi(h, g) \to (h | X \backslash Y\theta, g)$ is onto to complete the proof. Let $g \in G(Y/\theta \circ \theta^{-1})$ and $h \in G(X \backslash Y\theta)$. Extend h to $\mathcal{G}(X)$ by defining $x_i h = x_j$ when $X_i g = X_j$. Then

$$\theta = \begin{pmatrix} X_i \\ x_i \end{pmatrix} = \begin{pmatrix} X_i g \\ x_i h \end{pmatrix} = g^{-1} \begin{pmatrix} X_i \\ x_i \end{pmatrix} h = g^{-1}\theta h,$$

so that $\psi(h, g) \in \mathrm{FEA}(T)$.

REMARK. By the remarks following (3.2) we may regard Aut T as the cartesian product $\mathrm{FEA}(T) \times \mathrm{HCA}(T)$ with multiplication given by

$$(\psi_1, \phi_1) \cdot (\psi_2, \phi_2) = (\psi_1\psi_2, \psi_2^{-1}\phi_1\psi_2\phi_2)$$

Putting together (3.5) and the Corollary to (3.3) we have that up to isomorphism Aut T is $\mathcal{G}(X \backslash Y\theta) \times G(Y/\theta \circ \theta^{-1}) \times \Pi_{H \in T/h}\mathcal{G}(H^*)$ *as set*, with product given by

$$(h, g, [\nu_H]) \cdot (l, m, [\eta_H]) = (hl, gm, [\nu_H]R(l, m)[\eta_H])$$

where $R(l, m)$ is an automorphism of $\Pi\mathcal{G}(H^*)$. Explicitly, the Hth component of $[\nu_H]R(l, m)$ is $\phi^{-1}\nu_{H\phi^{-1}}\,\phi$ where $\phi = \phi(l, m)$ is the formula-expressed automorphism corresponding to

$$(l, m) \in \mathcal{G}(X \backslash Y\theta) \times G(Y/\theta \circ \theta^{-1})$$

Aut T has an exceedingly simple structure when Aut $T = \mathrm{FEA}(T)$, or equivalently, when $\mathrm{HCA}(T)$ is trivial. By the Corollary to (3.3) this is so if and only if $|H^*| \leqq 1$ for all h-classes, H. This fact will be used in the proof of the following

THEOREM 3.6. *If* $|Y\theta| > 2$ *then* Aut $T = \mathrm{FEA}(T)$ *if and only if at least one of the following hold*

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(i) $|Y| = |Y\theta|$
(ii) $|X| = |Y\theta|$
(iii) $|X| = |Y| = |Y\theta| + 1 < \aleph_0$.

PROOF. If either (i) or (ii) hold then every element of T has rank less than or equal to the rank of θ and the result follows from (3.4). In case (iii) assume that γ_1 and γ_2 belong to H^* for some h-class, H. Since $|X\gamma_i| > |Y\theta|$ we must have $X\gamma_i = Y$ for $i = 1, 2$. It follows that γ_1 and γ_2 are bijections. By the relation $\gamma_1 l \gamma_2$, they agree at all points of X except (possibly) one and it is immediate that $\gamma_1 = \gamma_2$. Thus $|H^*| \leqq 1$.

To complete the proof let Aut T = FEA(T) and assume that (i) and (ii) fail, that is, $|Y\theta| < |Y|$ and $|Y\theta| < |X|$. Let $a \in X \backslash Y\theta$ and $b_1, b_2 \in Y, b_1 \neq b_2$, with $b_1\theta = b_2\theta$. Select $\alpha : X\backslash a \to Y$ with the property

$$|(X\backslash a)\alpha| = \min(|X\backslash a|, |Y|) \geqq |Y\theta|$$

For $i = 1, 2$ define $x\alpha_i = x\alpha$, $x \neq a$, and $a\alpha_i = b_i$. As in the proof of (1.6), $\alpha_1 h \alpha_2$. By assumption, at least one of the $|X\alpha_i|$ does not exceed $|Y\theta|$, for otherwise $|H^*_\alpha| > 1$. However for both $i = 1$ and 2

$$|X\alpha_i| \geqq |(x\backslash a)\alpha| = \min(|X\backslash a|, |Y|) \geqq |Y\theta|$$

It follows that $\min(|X\backslash a|, |Y|) = |Y\theta|$ and since $|Y| > |Y\theta|$ we have

$$|Y| \geqq |X| = |Y\theta| + 1$$

where both $|X|$ and $|Y\theta|$ are finite. (For otherwise $|X| = |Y\theta|$.) Assume by way of contradiction that $|Y| > |X|$, and take $\beta_1 : X \to Y\backslash b_2$ with $|X\beta_1| = \min(|X|, |Y\backslash b_2|) = |X|$. Clearly we may require $a\beta_1 = b_1$. Define β_2 to be identical to β_1 except at a where $a\beta_1 = b_1$. Then $\beta_1 h \beta_2$, so for at least one i

$$|Y\theta| \geqq |X\beta_i| \geqq \min(|X|, |Y\backslash b_1|) = |X| > |Y\theta|.$$

REMARKS (i). Example A of Section 1 is an instance of the situation that arises in case (iii). An examination of the h-class structure shows that each h-class contains at most one mapping with rank 3. If an automorphism of T fixes all elements of rank < 3, then since such mappings merely permute h-classes, it must be trivial.

(ii) If θ is onto then $|Y\theta| = |X|$ so that by (3.6), Aut T = FEA(T) $\cong \mathscr{G}(X\backslash Y\theta) \times G(Y/\theta \circ \theta^{-1})$. Since $X\backslash Y\theta = \square$, FEA($T$) $\cong G(Y/\theta \circ \theta^{-1})$. This is Theorem (3.3) of Magill (1967).

To conclude we discuss some examples.

EXAMPLE C. Consider the semigroup $\mathscr{T}(X, Y)$, where $X \supseteq Y$, under the operation of composition. If ι_Y is the canonical imbedding of Y in X then $(\mathscr{T}(X, Y), \circ) = \mathscr{T}(X, Y; \iota_Y)$. Now $Y/\iota_Y \circ \iota_Y^{-1}$ is the identity partition of Y so $G(Y/\iota_Y \circ \iota_Y^{-1}) = \mathscr{G}(Y)$. Moreover $|Y\iota_Y| = |Y|$ so Aut T = $FEA(T) = \mathscr{G}(X\backslash Y) \times \mathscr{G}(Y)$. Since all automorphisms of T are formula-expressed they are of the form $\psi(h, g)$ where $h \in \mathscr{G}(X)$, $g \in \mathscr{G}(Y)$ and $\iota_Y h = g\iota_Y = g$. It follows that $h \mid Y = g$ and $\psi(h \cdot g)$ is the map $h^{-1} \cdot h$. Hence the automorphisms of T are the mappings of the form

$$\alpha \to h^{-1}\alpha h \qquad (\alpha \in T)$$

where $h \in \mathscr{G}(X, Y) = \{g \in \mathscr{G}(X); g \mid Y \in \mathscr{G}(Y)\}$. This is (3.1), derived from Symons (to appear). We have used this result extensively in developing our theory.

EXAMPLE D. In case (iii) of (3.6), $|X \backslash Y\theta| = 1$ so that Aut $T \cong G(Y/\theta \circ \theta^{-1})$. Clearly the partition classes of θ all contain one element except one class which contains 2. Hence $G(Y/\theta \circ \theta^{-1}) \cong Z_2 \times \mathscr{G}(|X| - 2)$, the direct product of the group of integers modulo two and the full symmetric group on $|X| - 2$ objects.

EXAMPLE E. We consider Example B of Section 1. Let $\mathscr{M}$ be the partition of $\mathbb{R}$ realized by all left-closed right-open intervals of length 10^{-4}. Clearly $\mathscr{M} = \mathbb{R}/\theta \circ \theta^{-1}$. There are $|\mathbb{R}\theta|^{|\mathbb{R}|} = \aleph_0^{\mathfrak{c}} = 2^{\mathfrak{c}}$ r-classes of T and since $|T| = \mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$, there must be $2^{\mathfrak{c}}$ h-classes. If H_α is an h-class of T then $\beta \in H_\alpha$ if and only if $\beta \mid \mathbb{R}_4 = \alpha \mid \mathbb{R}_4$ and α and β map each $x \in \mathbb{R} \backslash \mathbb{R}_4$ to the same set in $\mathscr{M}$. It is easy to see that H_α contains at least $2^{\mathfrak{c}}$ elements with rank $\mathfrak{c}$, so that $|H_\alpha^*| = 2^{\mathfrak{c}}$. Hence $\mathrm{HCA}(T) \cong \Pi_{H \in T/h}\mathscr{G}(H^*) = \mathscr{G}(2^{\mathfrak{c}})^{2^{\mathfrak{c}}}$. Moreover $|\mathbb{R} \backslash \mathbb{R}_4| = \mathfrak{c}$ and thus $\mathrm{FEA}(T) \cong \mathscr{G}(\mathfrak{c}) \times G(\mathscr{M})$. It follows that Aut T is isomorphic to a semidirect product of the group $\mathscr{G}(\mathfrak{c}) \times G(\mathscr{M})$ with the group $\mathscr{G}(2^{\mathfrak{c}})^{2^{\mathfrak{c}}}$.

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