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REALIZATION OF CHERN CLASSES BY SUBVARIETIES WITH CERTAIN SINGULARITIES

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§ 0. Introduction

In this paper we are concerned with subvarieties which realize Chern classes of holomorphic vector bundles. The existence of these subvarieties is known in some cases (for instance, see A. Grothendieck [2] for projective algebraic varieties and M. Cornalba and P. Griffiths [1] for Stein manifolds). In the present paper we realize Chern classes by subvarieties with singularities of a certain type. Our main theorem is as follows (see Def. 1.1.3 for the definition of quasilinear subvarieties).

MAIN THEOREM. *Let M be a paracompact complex manifold of dimension n and $\xi = (E, \pi, M)$ a holomorphic vector bundle of rank q with the condition* 2.2.1. *Then, for any integer $1 \leq k \leq n$, there exists a subvariety V of M such that*

(a) *V realizes the k-th Chern class of ξ.*

(b) *V is quasilinear of degree $k-1$ and can be desingularized by means of σ-processes.*

(c) *In particular, V is non-singular for $[n/2] \leq k \leq n$.*

As an application of Main Theorem, we show in § 4.3 that Chern classes of arbitrary holomorphic vector bundles over Stein manifolds can be realized by quasilinear subvarieties.

The following is an outline of the proof of Main Theorem. Let Φ be a holomorphic map from M into the complex Grassmann manifold $G_{q,m}$ which induces the bundle. We regard Φ as a holomorphic map from M into $G_{q,p+m}$ through an embedding $G_{q,m} \subset G_{q,p+m}$. Given a holomorphic map f from the total space E into the complex euclidean space C^p, we associate to f a holomorphic map Φ_f from M into $G_{q,p+m}$. We deform Φ into Φ_f so that Φ_f is transversal to all the strata of the Schubert variety F_1 in $G_{q,p+m}$.

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Pulling back F_1 by Φ_f, we obtain a quasilinear realization of the $(q-p+1)$-th Chern class of ξ.

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§1. Quasilinear subvarieties

1.1. Definition of quasilinear subvarieties

Let $\mathfrak{M}(p, q)$ denote the set of all $p \times q$ complex matrices. For integers p, q and r such that $p \leq q$, we define

$$\mathfrak{M}_r(p, q) = \{A \in \mathfrak{M}(p, q); \operatorname{corank}(A) \geq r\},$$

where corank $= p -$ rank.

Let M be a complex manifold of dimension n, and let

$$V_1 \supset V_2 \supset \cdots \supset V_p \tag{1.1.1}$$

be a sequence of subvarieties of M.

Definition 1.1.2. The sequence (1.1.1) is said to be *quasilinear* of degree m if it satisfies the following:

(a) For any integer $2 \leq k \leq p$, V_k consists of all the singular points of V_{k-1}. And V_p is non-singular.

(b) V_1 has the regular stratification;

$$(V_1 - V_2) \cup (V_2 - V_3) \cup \cdots \cup V_p .$$

(c) For any integer $1 \leq k \leq p$ and any point $x_0 \in V_k - V_{k+1}$ (where we set $V_{p+1} = \phi$), there is a triple (φ, U, W) such that U is an open neighbourhood of x_0 in M, W is that of $(0; 0)$ in

$$\mathfrak{M}(k, m+k) \times C^s \qquad (s = n - k(m+k)),$$

and φ is a biholomorphic map from U onto W such that

$$\varphi(x_0) = (0; 0),$$
$$\varphi(U \cap V_r) = W \cap (\mathfrak{M}_r(k, m+k) \times C^s)$$

for any integer $1 \leq r \leq k$.

The triple (φ, U, W) is called a quasilinearity at x_0. Notice that the condition (c) yields both (a) and (b). If the sequence (1.1.1) is quasilinear of degree m and V_k is not empty for any $1 \leq k \leq p$, then V_k has the codimension $k(m + k)$ in M.

Definition 1.1.3. A subvariety V of M is said to be *quasilinear* of degree m if there is a quasilinear sequence (1.1.1) of degree m such that $V_1 = V$. In this case, the sequence is called the associated sequence with V.

In particular, any non-singular subvariety of codimension $m + 1$ is quasilinear of degree m. In view of this, M itself is also said to be quasilinear.

1.2. Examples of quasilinear subvarieties

In this section we show that Schubert varieties have a natural quasilinear structure and that transversality to Schubert varieties yields quasilinearity as in Proposition 1.2.6.

Let

$$A = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix}, \quad \mathscr{A} = (a^i_j), \quad \mathscr{B} = (b^i_j), \quad \mathscr{C} = (c^i_j), \quad \mathscr{D} = (d^i_j),$$

where $A \in \mathfrak{M}(p, q)$ $(p \leq q)$ and $\mathscr{A} \in \mathfrak{M}(p - k, p - k)$. We define a $k \times (q - p + k)$ complex matrix $\tilde{\mathscr{D}}_k(A) = (\tilde{d}^i_j(A))$ by

$$\tilde{d}^i_j(A) = \begin{vmatrix} & & & b^1_j \\ & \mathscr{A} & & b^2_j \\ & & & \vdots \\ & & & b^t_j \\ c^i_1 & c^i_2 & \cdots\ c^i_t & d^i_j \end{vmatrix}$$

where $t = p - k$. With this notation, we have

Lemma 1.2.1. *Suppose that* $\mathscr{A}$ *is non-singular. Then, for any integer* $1 \leq r \leq k$,

$$\operatorname{corank}(A) \geq r \Leftrightarrow \operatorname{corank}(\tilde{\mathscr{D}}_k(A)) \geq r\,.$$

For the proof of the above lemma, it suffices to consider only minor determinants which contain all the components of $\mathscr{A}$. Since

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$$|\mathscr{A}|^s \left|\begin{array}{c|cccc} \mathscr{A} & b^1_{j_1} b^1_{j_2} & \cdots & b^1_{j_s} \\ & b^2_{j_1} & \cdot & \\ & \vdots & & \cdot \\ & b^t_{j_1} & \cdots & b^t_{j_s} \\ \hline c^{i_1}_1 c^{i_1}_2 \cdots c^{i_1}_t & d^{i_1}_{j_1} d^{i_1}_{j_2} & \cdots & d^{i_1}_{j_s} \\ c^{i_2}_1 \quad \cdot & d^{i_2}_{j_1} & \cdot & \\ \vdots \qquad \cdot & \vdots & & \cdot \\ c^{i_s}_1 \cdots c^{i_s}_t & d^{i_s}_{j_1} & \cdots & d^{i_s}_{j_s} \end{array}\right| = |\mathscr{A}| \left|\begin{array}{cccc} \tilde{d}^{i_1}_{j_1} \tilde{d}^{i_1}_{j_2} & \cdots & \tilde{d}^{i_1}_{j_s} \\ \tilde{d}^{i_2}_{j_1} & \cdot & \\ \vdots & & \cdot \\ \tilde{d}^{i_s}_{j_1} & \cdots & \tilde{d}^{i_s}_{j_s} \end{array}\right| ,$$

where $s = k - r + 1$, it follows that all $(t + s)$-minor determinants of A vanish if and only if all s-minor determinants of $\tilde{\mathscr{D}}_k(A)$ vanish.

The above lemma gives us the following example of a quasilinear sequence in $\mathfrak{M}(p, q)$.

PROPOSITION 1.2.2. *For any pair of positive integers $p \leq q$, the sequence*

$$\mathfrak{M}_1(p, q) \supset \mathfrak{M}_2(p, q) \supset \cdots \supset \mathfrak{M}_{p+1}(p, q) = \{O_{p,q}\}$$

is quasilinear of degree $q - p$ in $\mathfrak{M}(p, q)$.

Proof. Fix any $1 \leq k \leq p$, and let $A_0 \in \mathfrak{M}_k(p, q) - \mathfrak{M}_{k+1}(p, q)$ (where $\mathfrak{M}_{p+1}(p, q) = \phi$). We denote A_0 and each $p \times q$ complex matrix A by

$$A_0 = \begin{pmatrix} \mathscr{A}_0 & \mathscr{B}_0 \\ \mathscr{C}_0 & \mathscr{D}_0 \end{pmatrix}, \qquad A = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix},$$

where $\mathscr{A}_0$ and $\mathscr{A}$ are $(p - k) \times (p - k)$ matrices. Since the corank of A_0 is k, we may assume that $\mathscr{A}_0$ is non-singular. For any $p \times q$ matrix A, we define $\varphi(A)$ by the composition

$$A = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix} \mapsto \begin{pmatrix} \mathscr{A} - \mathscr{A}_0 & \mathscr{B} - \mathscr{B}_0 \\ \mathscr{C} - \mathscr{C}_0 & \tilde{\mathscr{D}}_k(A) \end{pmatrix} \mapsto (\tilde{\mathscr{D}}_k(A); a^1_1 - a^1_{01}, a^1_2 - a^1_{02}, \cdots)$$
$$= \varphi(A) \in \mathfrak{M}(k, q - p + k) \times \boldsymbol{C}^s ,$$

where $s = pq - k(q - p + k)$. By Lemma 1.2.1, corank $(\tilde{\mathscr{D}}_k(A_0)) \geq k$, that is, $\tilde{\mathscr{D}}_k(A_0)$ is the $k \times (q - p + k)$ zero matrix. Hence

$$\varphi(A_0) = (O_{k,q-p+k}; 0) .$$

Since $\mathscr{A}_0$ is non-singular, a restriction of φ gives rise to a biholomorphic map

$$\varphi | U : U \to W \subset \mathfrak{M}(k, q - p + k) \times \boldsymbol{C}^s ,$$

where U is an open neighbourhood of A_0 such that $|\mathscr{A}| \neq 0$ for any $A \in U$. From Lemma 1.2.1, it follows that for any $1 \leq r \leq k$,

$$\varphi(U \cap \mathfrak{M}_r(p, q)) = W \cap (\mathfrak{M}_r(k, q-p+k) \times C^s) .$$

Consequently, the triple $(\varphi | U, U, W)$ makes a quasilinearity at A_0.

Q.E.D.

We now investigate Schubert varieties. Let $G_{q,p+m}$ be the complex Grassmann manifold of all q-planes in $C^p \times C^{q+m}$ through the origin. Suppose $p \leq q$. For each integer $1 \leq k \leq p$, we set

$$F_k = \{q\text{-plane } \tau \in G_{q,p+m};\ \text{codimension } (\pi_p(|\tau|)) \geq k\} ,$$

where $|\tau|$ is the carrier of τ in $C^p \times C^{q+m}$ and π_p is the projection of $C^p \times C^{q+m}$ onto C^p. Thus we have the sequence of Schubert varieties in $G_{q,p+m}$

$$(1.2.3) \qquad F_1 \supset F_2 \supset \cdots \supset F_p .$$

If we use the Schubert symbol, F_k is represented by

$$(\underbrace{p+m-1, \cdots, p+m-1}_{q-p+k}, \underbrace{p+m, \cdots, p+m}_{p-k}) ,$$

with respect to the increasing sequence of linear subspaces $\{0\} \times C^1 \subset \{0\} \times C^2 \subset \cdots \subset \{0\} \times C^{q+m} \subset C^1 \times C^{q+m} \subset C^2 \times C^{q+m} \subset \cdots \subset C^p \times C^{q+m}$. Under the Poincaré duality isomorphism, the fundamental homology class of F_1 in $H_*(G_{q,p+m})$ corresponds to the $(q-p+1)$-th Chern class of the universal vector bundle over $G_{q,p+m}$ (see *W.* T. Wu [5] for Schubert varieties).

In order to show that the sequence (1.2.3) is quasilinear, we shall use the following local charts of $G_{q,p+m}$. Put $C^p \times C^{q+m} = (y^1, \cdots, y^p; z^1, \cdots, z^{q+m})$. Let

$$\{s_1, s_2, \cdots, s_h\} \cup \{\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{p-h}\} = \{1, 2, \cdots, p\} ,$$
$$\{t_1, t_2, \cdots, t_{q-h}\} \cup \{\bar{t}_1, \bar{t}_2, \cdots, \bar{t}_{m+h}\} = \{1, 2, \cdots, q+m\}$$

be arbitrary partitions for some integer $0 \leq h \leq q$. We denote each $(p+m) \times q$ complex matrix A by

$$A = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \multicolumn{2}{c}{\mathscr{C}} \end{pmatrix}$$

where $\mathscr{A} \in \mathfrak{M}(p-h, h)$. We define a holomorphic map

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$$\varphi(s_1, \cdots, s_h; t_1, \cdots, t_{q-h}) : \mathfrak{M}(p+m, q) \to G_{q,p+m}$$

by letting each element A of $\mathfrak{M}$ $(p+m, q)$ correspond to the graph in $C^p \times C^{q+m}$ of the linear map

$$\begin{bmatrix} y^{\bar{s}_1} \\ y^{\bar{s}_2} \\ \vdots \\ y^{\bar{s}_{p-h}} \\ z^{\bar{t}_1} \\ z^{\bar{t}_2} \\ \vdots \\ z^{\bar{t}_{m+h}} \end{bmatrix} = \left(\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \multicolumn{2}{c}{\mathscr{C}} \end{array}\right) \begin{bmatrix} y^{s_1} \\ y^{s_2} \\ \vdots \\ y^{s_h} \\ z^{t_1} \\ z^{t_2} \\ \vdots \\ z^{t_{q-h}} \end{bmatrix} .$$

It is easily verified that this map makes a local chart of $G_{q,p+m}$ without any restriction. We denote this local chart by

$$U(s_1, s_2, \cdots, s_h; t_1, t_2, \cdots, t_{q-h}) \qquad (\cong \mathfrak{M}(p+m, q)) .$$

Notice that $G_{q,p+m}$ is covered with those charts.

From the definition of Schubert varieties, we have

LEMMA 1.2.4. *For any integer* $1 \leq k \leq p$,

$$\varphi^{-1}(s_1, \cdots, s_h; t_1, \cdots, t_{q-h})(F_k \cap U(s_1, \cdots, s_h; t_1, \cdots, t_{q-h})) \\ = \{A \in \mathfrak{M}(p+m, q); \text{corank}\,(\mathscr{B}) \geq k\} .$$

In view of Proposition 1.2.2, this lemma implies

PROPOSITION 1.2.5. *The sequence of Schbert varieties* $F_1 \supset F_2 \supset \cdots \supset F_p$ *is quasilinear of degree* $q-p$ *in* $G_{q,p+m}$.

From this proposition, we obtain quasilinear subvarieties in a complex manifold M through holomorphic maps from M into $G_{q,p+m}$ as follows.

PROPOSITION 1.2.6. *Let* M *be a complex manifold. If a holomorphic map* Φ *from* M *into* $G_{q,p+m}$ *is transversal to all the strata of* F_1, *then the sequence*

$$\Phi^{-1}(F_1) \supset \Phi^{-1}(F_2) \supset \cdots \supset \Phi^{-1}(F_p)$$

is quasilinear of degree $q-p$ *in* M.

1.3. Resolution of singularities of quasilinear subvarieties

Given subvarieties $V_1 \supset V_2$ of a complex manifold M such that V_2 is non-singular, we denote by

$$\sigma\colon \tilde{M} \to M$$

the σ-process centred in V_2 and by $\tilde{V}_1$ the closure of the set $\sigma^{-1}(V_1 - V_2)$.

For any pair of positive integers $p \leq q$, let

$$\tilde{\mathfrak{M}}_1(p, q) \supset \tilde{\mathfrak{M}}_2(p, q) \supset \cdots \supset \tilde{\mathfrak{M}}_{p-1}(p, q) \tag{1.3.1}$$

be the sequence obtained from the sequence

$$\mathfrak{M}_1(p, q) \supset \mathfrak{M}_2(p, q) \supset \cdots \supset \mathfrak{M}_{p-1}(p, q) \supset \{O_{p,q}\}$$

by the σ-process $\sigma\colon \tilde{\mathfrak{M}}(p, q) \to \mathfrak{M}(p, q)$ centred at the zero matrix $O_{p,q}$.

LEMMA 1.3.2. *The sequence* (1.3.1) *is quasilinear of degree* $q - p$ *in* $\tilde{\mathfrak{M}}(p, q)$. *In particular,* $\tilde{\mathfrak{M}}_{p-1}(p, q)$ *is non-singular.*

Proof. Since $\tilde{\mathfrak{M}}(p, q) - \sigma^{-1}(O_{p,q})$ is biholomorphic to $\mathfrak{M}(p, q) - O_{p,q}$ under σ, it suffices to show the quasilinearity at any point x_0 in

$$\sigma^{-1}(O_{p,q}) \cap (\tilde{\mathfrak{M}}_k(p, q) - \tilde{\mathfrak{M}}_{k+1}(p, q)) ,$$

for any integer $1 \leq k \leq p - 1$ ($\mathfrak{M}_{p+1}(p, q) = \phi$). By the definition of the σ-process, $\tilde{\mathfrak{M}}(p, q)$ may be regarded as a complex submanifold of $\mathfrak{M}(p, q) \times P^{pq-1}$ (where P^{pq-1} denotes the complex projective space of dimension $pq - 1$). Suppose that x_0 is represented as

$$x_0 = (O_{p,q}; \kappa(H_0)) \in \mathfrak{M}(p, q) \times P^{pq-1} ,$$

where $H_0 = (h^i_{0j}) \in \mathfrak{M}(p, q)$ and $\kappa(H_0)$ denotes the element of P^{pq-1} with homogeneous coordinates h^i_{0j}. Without loss of generality we may assume $h^1_{01} = 1$.

We denote by $A[z]$ the matrix obtained from $A = (a^i_j)$ by replacing a^1_1 with a complex number z. For any element A of $\mathfrak{M}(p, q)$, we set

$$\mu(A) = (a^1_1 \cdot A[1]; \kappa(A[1])) \in \mathfrak{M}(p, q) \times P^{pq-1} .$$

Then

$$\mu(H_0[0]) = (O_{p,q}; \kappa(H_0)) = x_0 . \tag{1}$$

Moreover, there is a neighbourhood W' of $H_0[0]$ in $\mathfrak{M}(p, q)$ such that the restriction of μ to W' is a local coordinate system of $\tilde{\mathfrak{M}}(p, q)$ at x_0. Set $U' = \mu(W')$. Then, we have

$$\mu(\{A \in W'; a \neq 0, A[1] \in \mathfrak{M}_r(p, q)\}) = \tilde{\mathfrak{M}}_r(p, q) \cap (U' - \sigma^{-1}(O_{p,q})) ,$$

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for any $1 \leq r \leq k$. Since $A[1]$ does not contain the a_1^1-variable, we have

$$(2) \qquad \mu(\{A \in W'; A[1] \in \mathfrak{M}_r(p, q)\}) = U' \cap \tilde{\mathfrak{M}}_r(p, q)$$

for any $1 \leq r \leq k$.

Denote $H_0[0]$ and each element A of W' by

$$H_0[0] = \begin{pmatrix} 0 & h_{02}^1 & \cdots \\ h_{01}^2 & h_{02}^2 & \\ \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} \mathscr{A}_0 & \mathscr{B}_0 \\ \mathscr{C}_0 & \mathscr{D}_0 \end{pmatrix}, \qquad A = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix},$$

where $\mathscr{A}_0$ and $\mathscr{A}$ are $(p-k) \times (p-k)$ matrices. From (2), we may assume that $\mathscr{A}_0[1]$ is non-singular. For any element A of W', we define $\varphi(A)$ by the following composition

$$\begin{aligned} A = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix} &\mapsto \begin{pmatrix} \mathscr{A} - \mathscr{A}_0 & \mathscr{B} - \mathscr{B}_0 \\ \mathscr{C} - \mathscr{C}_0 & \tilde{\mathscr{D}}_k(A[1]) \end{pmatrix} \\ &\mapsto (\tilde{\mathscr{D}}_k(A[1]); a_1^1, a_2^1 - a_{02}^1, a_3^1 - a_{03}^1, \cdots) \\ &= \varphi(A) \in \mathfrak{M}(k, q-p+k) \times \boldsymbol{C}^s, \end{aligned}$$

where $s = pq - k(q-p+k)$. From (1), (2) and Lemma 1.2.1, it is easily verified that a restriction of $\varphi \circ \mu^{-1}$ to a sufficiently small neighbourhood of x_0 gives a quasilinearity at x_0. Q.E.D.

Remark 1.3.3. Note that $\sigma^{-1}(O_{p,q})$ is locally determined by $a_1^1 = 0$ and $\tilde{\mathfrak{M}}_r(p, q)$ by the equations which do not contain the a_1^1-variable. Therefore, $\sigma^{-1}(O_{p,q})$ is non-singular and transversal to all the strata of $\tilde{\mathfrak{M}}_1(p, q)$.

From this lemma, we can describe the behaviour of quasilinear sequences under σ-processes as follows.

PROPOSITION 1.3.4. *Let $V_1 \supset V_2 \supset \cdots \supset V_p$ be a quasilinear sequence of degree m in a complex manifold M. Then, the sequence $\tilde{V}_1 \supset \tilde{V}_2 \supset \cdots \supset \tilde{V}_{p-1}$ obtained from the above sequence by the σ-process $\sigma: \tilde{M} \to M$ centred in V_p is also quasilinear of degree m in $\tilde{M}$. In particular, $\tilde{V}_{p-1}$ is non-singular.*

Proof. From the quasilinearity, it suffices to prove the result for the sequence

$$\mathfrak{M}_1(s, t) \times \boldsymbol{C}^m \supset \mathfrak{M}_2(s, t) \times \boldsymbol{C}^m \supset \cdots \supset \{O_{p,q}\} \times \boldsymbol{C}^m$$

for arbitrary integers s, t and m. In this case, the result is easily verified by Lemma 1.3.2. Q.E.D.

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By successive applications of this proposition, we obtain the following desingularization of quasilinear subvarieties.

THEOREM 1.3.5. *A quasilinear subvariety can be desingularized by means of σ-processes.*

Proof. Let $V \subset M$ be a quasilinear subvariety, and $V_1 \supset V_2 \supset \cdots \supset V_p$ the associated sequence with V. From Proposition 1.3.4, by induction on i, we can define a sequence

$$\{(M^i, \sigma_i; V_1^i, V_2^i, \cdots, V_{p-i}^i)\} \qquad (0 \leq i \leq p-1)$$

such that

(a) $M^0 = M$ and $V_k^0 = V_k$ for $1 \leq k \leq p$.

(b) The sequence $V_1^i \supset V_2^i \supset \cdots \supset V_{p-1}^i$ is obtained from the quasilinear sequence in M^{i-1}

$$V_1^{i-1} \supset V_2^{i-1} \supset \cdots \supset V_{p-i+1}^{i-1}$$

by the σ-process $\sigma_i : M^i \to M^{i-1}$ centred in V_{p-i+1}^{i-1}.

Then, the restriction

$$\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{p-1} | V_1^{p-1} : V_1^{p-1} \to V_1^0 = V$$

gives a desingularization of V. Q.E.D.

As one of applications of this theorem, we show the existence of "tubular neighbourhoods" around singularities of quasilinear subvarieties, which will be used in § 4.

LEMMA 1.3.6. *Let $V_1 \supset V_2 \supset \cdots \supset V_p$ be a quasilinear sequence in a compact complex manifold M. Then, there exists an open neighbourhood U of V_2 in V_1 such that both inclusions $V_2 \to U$ and $V_1 - U \to V_1 - V_2$ are homotopy equivalences.*

Proof. With the notation of the proof of Theorem 1.3.5, we set

$$N_1 = V_1^{p-1}, \quad N_2 = \sigma_{p-1}^{-1}(V_2^{p-2}), \quad N_3 = \sigma_{p-1}^{-1} \circ \sigma_{p-2}^{-1}(V_3^{p-3}), \cdots,$$
$$N_p = \sigma_{p-1}^{-1} \circ \sigma_{p-2}^{-1} \circ \cdots \circ \sigma_1^{-1}(V_p).$$

From Remark 1.3.3, it follows that $N_1, N_2 \cdots$ and N_p are non-singular and transversal to each other in M^{p-1}. Moreover, the map $\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{p-1}$ carries $N_1 - (N_2 \cup N_3 \cup \cdots \cup N_p)$ onto $V_1 - V_2$ biholomorphically, and $N_1 \cap (N_2 \cup N_3 \cup \cdots \cup N_p)$ onto V_2 (not biholomorphically). Therefore, it suffices to construct a neighbourhood of $N_1 \cap (N_2 \cup \cdots \cup N_p)$ in N_1 and to let it fall down into V_1 by the proper map $\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{p-1}$.

Q.E.D.

§ 2. Fibre-directional singularities of holomorphic mappings

2.1. Fibrewise singular sets

Let M be a complex manifold of dimension n and $\xi = (E, \pi, M)$ a holomorphic vector bundle of rank q over M. From now on, we shall deal with holomorphic maps from the total space E into the complex euclidean space C^p. M will be identified with the zero cross-section of the bundle.

Let $f: E \to C^p$ be a holomorphic map. At each point x of M, we consider the (C-linear) differential

$$d_x(f|F_x): T_x(F_x) \to T_{f(x)}(C^p),$$

where F_x denotes the fibre over x and $f|F_x$ the restriction of f to F_x. The map f is said to be fibrewise regular at a point x_0 of M if the differential $d_{x_0}(f|F_{x_0})$ has the maximum rank, that is, the differential is one to one in case $p \geq q$ or onto in case $p \leq q$. In this case, the point x_0 is called a fibrewise regular point of f.

Henceforth we shall suppose $p \leq q$. For each integer $1 \leq k \leq p$, we define the k-th *fibrewise singular set* of f by

$$FS_k(f) = \{x \in M; \operatorname{corank}(d_x(f|F_x)) \geq k\}.$$

These fibrewise singular sets are subvarieties of M such that

$$FS_1(f) \supset FS_2(f) \supset \cdots \supset FS_p(f).$$

We shall denote any holomorphic map f from E into C^p by

$$f = (f^1, f^2, \cdots, f^p): E \to C^p.$$

Take any local triviality of ξ at a point x_0 of M such that

$$\psi = (z; \zeta) = (z^1, \cdots, z^n; \zeta^1, \cdots, \zeta^q): \pi^{-1}(U) \to W \times C^q \subset C^n \times C^q,$$
$$\psi(x_0) = (0; 0).$$

We denote this local triviality by (U, W, ψ). With respect to (U, W, ψ), we define a holomorphic map $\bar{f}_\psi$ from W into $\mathfrak{M}(p, q)$ by

$$\bar{f}_\psi(z) = \left(\frac{\partial f^i \circ \psi^{-1}}{\partial \zeta^j}(z; 0)\right).$$

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For any integer $1 \leq k \leq p$, a holomorphic map $f: E \to C^p$ is said to be *in a general position* of order k at a point x_0 of M if there is a local triviality (U, W, ψ) of ξ at x_0 such that $\bar{f}_\psi$ is transversal on W to all the strata of $\mathfrak{M}_\ell(p, q)$;

$$\mathfrak{M}_\ell(p, q) - \mathfrak{M}_{\ell+1}(p, q), \mathfrak{M}_{\ell+1}(p, q) - \mathfrak{M}_{\ell+2}(p, q), \cdots, O_{p,q}$$

where $\ell = p - k + 1$ (see Proposition 1.2.2 for this stratification). This definition does not depend on the choice of local trivialities if we choose a sufficiently small U. For any subset K of M, f is said to be in a general position of order k on K if it is so at any point of K. In case $k = p$, f is said to be in a general position on K.

Remark 2.1.1. Regard the total space E as a complex manifold of dimension $n + q$. We can use the following local coordinate system in place of local trivialities for the verification of the general position requirement. Let (W, ψ) be a local coordinate neighbourhood of E at a point x_0 of M such that

$$\psi = (z; \eta) = (z^1, \cdots, z^n; \eta^1, \cdots, \eta^q): W \to W_1 \times W_2 \subset C^n \times C^q$$

and such that $\psi(W \cap M)$ is a graph in $W_1 \times W_2$ of a holomorphic map $\eta = \chi(z)$ and $\psi(W \cap F_x)$ is determined by $z = z(x)$ for any point x in $W \cap M$. Then, a holomorphic map f from E into C^p is in a general position of order k on $W \cap M$ if and only if a holomorphic map $\bar{f}_\psi$ from W_1 into $\mathfrak{M}(p, q)$ defined by

$$\bar{f}_\psi(z) = \left(\frac{\partial f^i \circ \psi^{-1}}{\partial \eta^j}(z; \chi(z))\right)$$

is transversal to all the strata of $\mathfrak{M}_{p-k+1}(p, q)$ on W_1.

2.2. Relation between Schubert varieties and $FS_k\ (f)$

Throughout this section, we shall suppose that a holomorphic vector bundle $\xi = (E, \pi, M)$ of rank q satisfies the following condition:

(2.2.1) There exists, for some integer m, a holomorphic map Φ from M into $G_{q,m}$ such that $\xi \cong \Phi^*(\gamma_{q,m})$,

where $\gamma_{q,m} = (E_{q,m}, \pi_{q,m}, G_{q,m})$ denotes the universal vector bundle over $G_{q,m}$ whose total space consists of all pairs (τ, v) of a q-plane τ in C^{q+m} through the origin and a vector v in $|\tau|$, and the equivalence means a holomorphic equivalence as holomorphic vector bundles.

Let $\tilde{\Phi}: E \to E_{q,m}$ be the lift of Φ and $\varphi_{q,m}$ the holomorphic map from $E_{q,m}$ into C^{q+m} which sends (τ, v) to v. We define

(2.2.2) $$\Psi = \varphi_{q,m} \circ \tilde{\Phi}: E \to C^{q+m} .$$

Note that Ψ is fibrewise regular at any point of M. Given a holomorphic map f from E into C^p, we define Ψ_f by

$$\Psi_f = (f, \Psi): E \to C^p \times C^{q+m} .$$

Since Ψ is fibrewise regular on M, so is Ψ_f. Hence, the image of $T_x(F_x)$ by $d_x(\Psi_f | F_x)$ makes a q-plane in $C^p \times C^{q+m}$ for any point x of M. Therefore, there is a holomorphic map

(2.2.3) $$\Phi_f: M \to G_{q,p+m}$$

which sends each point x of M to the q-plane in $C^p \times C^{q+m}$ through the origin which is parallel to the image of $T_x(F_x)$ by $d_x(\Psi_f | F_x)$. It follows from the way of constructing Φ_f that

(2.2.4) $$\xi \cong \Phi_f^*(\gamma_{q,p+m}) ,$$

where $\gamma_{q,p+m}$ is the universal bundle over $G_{q,p+m}$.

Under the map Φ_f, fibrewise singular sets and Schubert varieties have the following relation.

LEMMA 2.2.5. *Let $f: E \to C^p$ be a holomorphic map. Then,*

(a) *$FS_k(f) = \Phi_f^{-1}(F_k)$, for any integer $1 \leq k \leq p$.*

(b) *The map f is in a general position on M if and only if Φ_f is transversal to all the strata of the Schbert variety F_1.*

Proof. Let $x_0 \in M$. Take a local triviality of ξ at x_0 such that

$$\psi' = (z^1, \cdots, z^n; \zeta^1, \cdots, \zeta^q): \pi^{-1}(U') \to W' \times C^q \subset C^n \times C^q ,$$
$$\psi(x_0) = (0; 0) .$$

Put $C^p = (y^1, \cdots, y^p)$ and $C^{q+m} = (\eta^1, \cdots, \eta^{q+m})$. We denote $\eta^i \circ \Phi$ briefly by η^i for any integer $1 \leq i \leq q + m$. Since Ψ is fibrewise regular at x_0, there is a subset $\{t_1, t_2, \cdots, t_q\}$ of $\{1, 2, \cdots, q + m\}$ such that the map

$$(\eta^{t_1}, \eta^{t_2}, \cdots, \eta^{t_q}): E \to C^q$$

is fibrewise regular at x_0. Therefore, for some open subsets $U \subset U'$ and $W \subset W'$, the map

$$\psi = (z^1, \cdots, z^n; \eta^{t_1}, \eta^{t_2}, \cdots, \eta^{t_q}) : \pi^{-1}(U) \to W \times C^q$$

is a local triviality at x_0. We may assume $\{t_1, t_2, \cdots, t_q\} = \{1, 2, \cdots, q\}$. With respect to this local triviality, we have

$$(1) \qquad \bar{f}_\psi(z) = \left(\frac{\partial f^i \circ \psi^{-1}}{\partial \eta^j}(z; 0)\right).$$

With the notation in § 1.2, $\Phi_f(x_0) \in U(; 1, 2, \cdots, q)$ (consider the case $\{s_i\} = \phi$ and $\{t_j\} = \{1, 2, \cdots, q\}$. Moreover,

$$(2) \qquad \varphi^{-1}(; 1, 2, \cdots, q) \circ \bar{f}_\psi(z) = \begin{bmatrix} \dfrac{\partial f^i \circ \psi^{-1}}{\partial \eta^j}(z; 0) \\ \cdots\cdots\cdots \\ \dfrac{\partial \eta^{q+\ell} \circ \psi^{-1}}{\partial \eta^j}(z; 0) \end{bmatrix},$$

where $1 \leq i \leq p$, $1 \leq j \leq q$ and $1 \leq \ell \leq m$. From Lemma 1.2.4, F_k is determined in $U(; 1, \cdots, q)$ by

$$(3) \qquad \begin{aligned} &\varphi^{-1}(; 1, \cdots, q)(F_k \cap U(; 1, \cdots, q)) \\ &= \left\{\left(\frac{\mathscr{B}}{\mathscr{C}}\right) \in \mathfrak{M}(p + m, q); \operatorname{corank}(\mathscr{B}) \geq k\right\}. \end{aligned}$$

From (1), (2) and (3), the lemma follows. Q.E.D.

From this lemma and Proposition 1.2.6, we can describe the structure of singularities of subvarieties $FS_k(f)$ as follows.

THEOREM 2.2.6. *If a holomorphic map f from E into C^p is in a general position on M, then the sequence*

$$FS_1(f) \supset FS_2(f) \supset \cdots \supset FS_p(f)$$

is quasilinear of degree $q - p$ in M.

§ 3. Approximation Theorems

Throughout this section, M will be a paracompact complex manifold of dimension n and $\xi = (E, \pi, M)$ a holomorphic vector bundle of rank q with the condition 2.2.1, unless otherwise stated. M will be identified with the zero cross-section of ξ.

3.1. Pseudonorms

Regard the total space E as a complex manifold of dimension $n + q$ and let $\{\mathscr{W}_i\}_{i \in I}$ be a locally finite covering of E such that each $\mathscr{W}_i$ is a compact local coordinate neighbourhood of E with coordinates $w_i = (w_i^1, w_i^2, \cdots, w_i^{n+q})$. We shall fix this covering and define a pseudonorm $\| \ \|_L$ associated to an arbitrary compact subset L of E for holomorphic maps from E into C^p by

$$\|f\|_L = \sup_{i\in I(L)} \sup_{w_i\in\mathscr{W}_i} \left\{\sum_k |f^k(w_i)| + \sum_{k,\ell}\left|\frac{\partial f^k}{\partial w_i^\ell}(w_i)\right| + \sum_{k,\ell,m}\left|\frac{\partial^2 f^k}{\partial w_i^\ell \partial w_i^m}(w_i)\right|\right\},$$

where $f = (f^1, f^2, \cdots, f^p)$ and $I(L)$ is a finite subset of I defined by

$$I(L) = \{i \in I; L \cap \mathscr{W}_i \neq \phi\}.$$

Let K be a compact subset of M. Since K is also compact in L, a pseudonorm $\| \ \|_K$ is defined for holomorphic maps from E into C^p by the above definition. Note that $\| \ \|_K$ measures not only distances on K but also those on a compact neighbourhood of K in E.

Let $x \in M$. Since the covering is locally finite, there is a compact neighbourhood W of x in E such that $I(x) = I(W)$. If we set $U = W \cap M$, U is a compact neighbourhood of x in M. Thus, we have

LEMMA 3.3.1. *For any point x of M, there exists a compact neighbourhood U of x in M such that $\| \ \|_x = \| \ \|_U$.*

3.2. Preparatory lemmas for Approximation Theorems

We start with the following lemma.

LEMMA 3.2.2. *Let $f: E \to C^p$ $(p \leq q)$ be a holomorphic map and $x_0 \in FS_{p-k+1}(f) - FS_{p-k+2}(f)$ $(FS_{p+1}(f) = \phi)$. Suppose that f is in a general position of order k at x_0. Then,*

(a) *f is already in a general position at x_0.*

(b) *There is a compact neighbourhood U of x_0 in M and $\delta > 0$ such that if a holomorphic map $g: E \to C^p$ satisfies that $\|f - g\|_U < \delta$ then g is in a general position on U.*

Proof. Take a local triviality of ξ at x_0 such that

$$\psi = (z; \zeta): \pi^{-1}(U') \to W_1 \times C^q \subset C^n \times C^q, \qquad \psi(x_0) = (0; 0).$$

Put $\ell = p - k + 1$. Since $x_0 \in FS_\ell(f) - FS_{\ell+1}(f)$, we have

$$\bar{f}_\psi(0) \in \mathfrak{M}_\ell(p, q) - \mathfrak{M}_{\ell+1}(p, q).$$

Since $\mathfrak{M}_1(p, q)$ is quasilinear of degree $q - p$ from Proposition 1.2.2, there is a quasilinearity (φ, V, W_2) of $\mathfrak{M}_1(p, q)$ at $\bar{f}_\psi(0)$ such that

$$\varphi\colon V \to W_2 \subset \mathfrak{M}(\ell, q-p+\ell) \times C^s\,, \qquad \varphi(\bar{f}_\psi(0)) = (O; 0)\,,$$

where $s = pq - \ell(q-p+\ell)$. Contracting W_1 to W_3 so that $\bar{f}_\psi(W_3) \subset V$, we denote the composition $\varphi \circ \bar{f}_\psi$ on W_3 by

$$\varphi \circ \bar{f}_\psi = ((y^i_j[f]); (y^t[f]))\,,$$

where $(y^i_j[f])\colon W_3 \to \mathfrak{M}(\ell, q-p+\ell)$ and $(y^t[f])\colon W_3 \to C^s$.

Since f is in a general position of order k at x_0, $\bar{f}_\psi$ is transversal to $\mathfrak{M}_\ell(p,q) - \mathfrak{M}_{\ell+1}(p,q)$ at the origin in W_3, hence $\varphi \circ \bar{f}_\psi$ is transversal to $O \times C^s$ at the origin. Therefore, there is a compact neighbourhood $W \subset W_3$ of the origin such that, for any $z \in W$, the differential $d_z((y^i_j[f]))$ is onto. Noticing that the ontoness of the differential implies the transversality of $\varphi \circ \bar{f}_\psi$ to all the strata of

$$\mathfrak{M}_1(\ell, q-p+\ell) \times C^s\,,$$

we have (a). Set $U = \psi^{-1}(W)$. There is some $\delta > 0$ such that if $\|f-g\|_U < \delta$ then $\bar{g}_\psi(W) \subset V$ and $d_z((y^i_j[g]))$ is onto for any $z \in W$. This proves (b). Q.E.D.

From the above lemma, we have the following two lemmas.

LEMMA 3.2.3. *If a holomorphic map $f\colon E \to C^p$ is in a general position of order k on a subset K of M, then f is in a general position on $K \cap FS_{p-k+1}(f)$.*

This lemma is proved by applying (a) to points in each of

$$K \cap (FS_\ell(f) - FS_{\ell+1}(f)), \cdots, K \cap FS_p(f)\,,$$

where $\ell = p - k + 1$.

LEMMA 3.2.4. *If a holomorphic map $f\colon E \to C^p$ is in a general position of order k on a compact subset K of M, then there exists $\delta > 0$ such that if*

$$\|f - g\|_K < \delta$$

then g is also in a general position of order k on K.

Proof. For any $x \in K \cap FS_{p-k+1}(f)$, there exist, from (b) of Lemma 3.2.2 and Lemma 3.1.1, a compact neighbourhood $U(x)$ of x in M and $\delta(x) > 0$ such that if

$$\|f - g\|_x \; (= \|f-g\|_{U(x)}) < \delta(x)$$

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then g is in a general position on $U(x)$.

For any $y \in K - FS_{p-k+1}(f)$, there are $\delta(y) > 0$ and a compact neighbourhood $U(y)$ of y in M such that if

$$\|f - g\|_y \; (= \|f - g\|_{U(y)}) < \delta(y)$$

then $U(y) \cap FS_{p-k+1}(g) = \phi$ (hence g is in a general position of order k on $U(y)$).

Since K is compact, we have

$$K \subset \Big(\bigcup_{i=1}^{s} U(x_i)\Big) \cup \Big(\bigcup_{j=1}^{t} (U(y_j)\Big)$$

for some points $\{x_i\} \subset K \cap FS_{p-k+1}(f)$ and $\{y_j\} \subset K - FS_{p-k+1}(f)$. Define

$$\delta = \min_{i,j} \{\delta(x_i), \delta(y_j)\} .$$

Suppose that $\|f - g\|_K < \delta$. Then we have

$$\|f - g\|_x < \delta(x_i) , \qquad \|f - g\|_y < \delta(y_j)$$

for any $1 \leq i \leq s$ and any $1 \leq j \leq t$. Hence g is in a general position of order k on K. Q.E.D.

In the following lemma, we show that given a holomorphic map h from E into C^p and a point x_0 of M, we can approximate h with a holomorphic map g which is in a general position of some order at x_0. In this case, we can make the approximation extend over an arbitrarily given compact subset L of E.

LEMMA 3.2.5. *If a holomorphic map*

$$f = (f^1, f^2, \cdots, f^k) : E \to C^k \qquad (k \leq p - 1)$$

is fibrewise regular at a point x_0 of M, then there is a compact neighbourhood $U(x_0; f^1, f^2, \cdots, f^k)$ of x_0 in M satisfying the following: Given a holomorphic map h from E into C^p such that

$$h = (f^1, \cdots f^k, h^{k+1}, \cdots, h^p)$$

(i.e. *$h^i = f^i$ on E for any $1 \leq i \leq k$), a compact subset L of E and $\varepsilon > 0$, there is a holomorphic map g from E into C^p such that*

(a) $g = (f^1, \cdots, f^k, g^{k+1}, \cdots, g^p)$.

(b) *g is in a general position of order $k+1$ on $U(x_0; f^1, \cdots, f^k)$.*

(c) $\|h - g\|_{L \cup U(x_0;\, f^1, \cdots, f^p)} < \varepsilon$.

Remark 3.2.6. In case $k = 0$, the lemma asserts that for any point x_0 of M there is a compact neighbourhood $U(x_0)$ of x in M satisfying the following: Given a holomorphic map $h: E \to C^p$, a compact subset L of E and $\varepsilon > 0$, there is a holomorphic map g in a general position of order 1 on $U(x_0)$ such that

$$\|h - g\|_{L \cup U(x_0)} < \varepsilon .$$

Proof of Lemma 3.2.5. Denote the map Ψ of (2.2.2) by

$$\Psi = (\Psi^1, \Psi^2, \cdots, \Psi^{q+m}): E \to C^{q+m} .$$

Since $d_{x_0}(\Psi | F_{x_0})$ is one to one and $d_{x_0}(f | F_{x_0})$ is onto, there is a subset $\{t_1, t_2, \cdots, t_{q-k}\}$ of $\{1, 2, \cdots, q + m\}$ such that the map

$$(1) \qquad (f^1, \cdots, f^k, \Psi^{t_1}, \cdots, \Psi^{t_{q-k}}): E \to C^q$$

is fibrewise regular at x_0. Without loss of generality, we may assume

$$\{t_1, \cdots, t_{q-k}\} = \{1, 2, \cdots, q - k\} .$$

Take a local triviality of ξ at x_0 such that

$$\psi' = (z^1, \cdots, z^n; \zeta^1, \cdots, \zeta^q): \pi^{-1}(U') \to W' \times C^q \subset C^n \times C^q ,$$

and consider the following map

$$(z^1, \cdots, z^n; f^1, \cdots, f^k, \Psi^1, \cdots, \Psi^{q-k}): \pi^{-1}(U') \to W' \times C^q .$$

Because of the fibrewise-regularity of the map (1) at x_0, the above map makes a local coordinate system (not necessarily a local triviality) of E at x_0 within some neighbourhood W of x_0 in E. Denote this local coordinate system by (W, ψ). We may assume that (W, ψ) satisfies the requirements of Remark 2.1.1. Let W_1, W_2 and χ be those in the remark with respect to (W, ψ). Thus we have a biholomorphic map

$$\psi = (z^1, \cdots, z^n; f^1, \cdots, f^k, \Psi^1, \cdots, \Psi^{q-k}): W \to W_1 \times W_2 \subset C^n \times C^q .$$

We may assume that W, W_1 and W_2 are compact.

Define

$$U(x_0; f^1, \cdots, f^k) = W \cap M .$$

We show that this compact neighbourhood of x_0 in M is the required one. Let $h: E \to C^p$ be a holomorphic map such that

$$h = (f^1, \cdots, f^k, h^{k+1}, \cdots, h^p) .$$

For any $(p-k) \times (q-k)$ complex matrix $\mathscr{E} = (\varepsilon_j^i)$, we consider

$$g_{\mathscr{E}} = (f^1, \cdots, f^k, g_{\mathscr{E}}^{k+1}, \cdots, g_{\mathscr{E}}^p) ,$$

where $g_{\mathscr{E}}^{k+i}(i = 1, 2, \cdots, p-k)$ are defined by

$$g_{\mathscr{E}}^{k+i} = h^{k+i} + \sum_{j=1}^{q-k} \varepsilon_j^i \Psi^j .$$

Since Ψ^j is holomorphic on E for any j, $g_{\mathscr{E}}$ is a holomorphic map from E into C^p. For any compact subset L of E and any $\varepsilon > 0$, there is some $\delta > 0$ such that

$$\max_{i,j} |\varepsilon_j^i| < \delta \Rightarrow \|h - g_{\mathscr{E}}\|_{L \cup U(x_0;\, f^1, \cdots, f^k)} < \varepsilon .$$

There is a regular value $\mathscr{E}_0 = (\varepsilon_{0j}^i) \in \mathfrak{M}(p-k, q-k)$ of a map which sends each $z \in W_1$ to

$$-\left(\frac{\partial h^{k+i} \circ \psi^{-1}}{\partial \Psi^j}(z; \chi(z))\right) \in \mathfrak{M}(p-k, q-k)$$

such that $\max |\varepsilon_{0j}^i| < \delta$. Define $g = g_{\mathscr{E}_0}$. Then g satisfies (a), (b) and (c).

Q.E.D.

Notice that the determination of $U(x_0; f^1, \cdots, f^k)$ does not depend on functions $\{h^{k+1}, \cdots, h^p\}$ but only on $\{f^1, \cdots, f^k\}$, and that we need not deform these $\{f^1, \cdots, f^k\}$ when we turn h into g. Because of these, we can extend the region where an approximating map g is in a general position of some order to a certain extent as follows.

LEMMA 3.2.7. *Let $f = (f^1, \cdots, f^p): E \to C^p$ be a holomorphic map and K a compact subset of M. If there is a subset $\{s_1, s_2, \cdots, s_k\}$ of $\{1, 2, \cdots, p\}$ such that the map*

$$(f^{s_1}, f^{s_2}, \cdots, f^{s_k}): E \to C^k$$

is fibrewise regular at any point of K, then for any compact subset L of E and any $\varepsilon > 0$, there is a holomorphic map $g: E \to C^p$ such that

(a) *g is in a general position of order $k+1$ on K.*

(b) $\|f - g\|_{K \cup L} < \varepsilon$.

Proof. Without loss of generality, we may assume that $\{s_1, \cdots, s_k\} = \{1, 2, \cdots, k\}$. Then the map

$$(f^1, \cdots, f^k): E \to C^p$$

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satisfies the assumption of Lemma 3.2.5 at any point x of K. For each $x \in K$, we let $U(x; f^1, \cdots, f^k)$ be a compact neighbourhood of x in M given in that lemma. From Lemma 3.1.1, we may assume

$$(1) \qquad \| \quad \|_x = \| \quad \|_{U(x;\, f^1, \cdots, f^k)}$$

for any $x \in K$. Since K is compact, we have

$$K \subset \bigcup_{i=1}^{N} U(x_i; f^1, \cdots, f^k)$$

for some points $\{x_i\}$ in K.

We show the existence of a sequence $\{(g_i, \delta_i)\}$ $(i = 1, 2, \cdots, N)$ of pairs of holomorphic maps from E into C^p and positive numbers such that, for any $1 \leq i \leq N$,

(a, i) g_i is in a general position of order $k+1$ on $U(x_i; f^1, \cdots, f^k)$.

(b, i) $\|g_{i-1} - g_i\|_{K \cup L} < \min\{\varepsilon, \delta_1, \delta_2, \cdots, \delta_{i-1}\}/N$

where $g_0 = f$.

(c, i) $g_i = (f^1, \cdots, f^k, g_i^{k+1}, \cdots, g_i^p)$.

(d, i) If

$$\|g_i - h\|_{x_i} < \delta_i$$

then h is in a general position of order $k+1$ on $U(x_i; f^1, \cdots, f^k)$.

Suppose that this is verified. Then, $g = g_N$ satisfies (a) and (b) for the following reason. For any $1 \leq i \leq N-1$, from properties $(b, i+1)$, $\cdots$ and (b, N), we have

$$\|g_i - g_N\|_{x_i} < \delta_i .$$

From this and (d, i), g_N is in a general position of order $k+1$ on $U(x_i; f^1, \cdots, f^k)$ for any $1 \leq i \leq N-1$. Therefore, from (a, N), g_N is in a general position of order $k+1$ on K. On the other hand, from (b, i) $(i = 1, \cdots, N)$, we have

$$\|f - g_N\|_{K \cup L} < \varepsilon .$$

By induction on i, we prove the existence of the above sequence. Now suppose that there are (g_1, δ_1), (g_2, δ_2), $\cdots$ and (g_r, δ_r) with the above properties. Because of (c, r) and the property of $U(x_{r+1}; f^1, \cdots, f^k)$ asserted by Lemma 3.2.5, we can turn

$$g_r = (f^1, \cdots, f^k, g_r^{k+1}, \cdots, g_r^p)$$

into

$$g_{r+1} = (f^1, \cdots, f^k, g_{r+1}^{k+1}, \cdots, g_{r+1}^{k+1})$$

with $(a, r+1)$ and $(b, r+1)$ (obviously with $(c, r+1)$). From $(a, r+1)$ and Lemma 3.2.4, we obtain $\delta_{r+1} > 0$ with $(d, r+1)$. The existence of (g_1, δ_1) is similarly verified. Q.E.D.

3.3. Approximation Theorems

We are now in the position to prove two approximation theorems on maps in a general position. The first theorem gives a semi-global approximation for paracompact manifolds. This theorem yields the second one, which supplies a global approximation for compact manifolds. In either case, $\xi = (E, \pi, M)$ is supposed to satisfy the condition 2.2.1.

THEOREM 3.3.1. *Let $f: E \to C^p$ $(p \leq q)$ be a holomorphic map. Then, for any compact subset K of M, any compact subset L of E and any $\varepsilon > 0$, there is a holomorphic map $g: E \to C^p$ such that*

(a) *g is in a general position on K.*

(b) $\|f - g\|_{K \cup L} < \varepsilon$.

Proof. We show the existence of a sequence $\{g_r\}$ $(r = 1, 2, \cdots, p)$ of holomorphic maps from E into C^p such that for any $1 \leq r \leq p$,

(a, r) g_r is in a general position of order r on K.

(b, r) $\|f - g_r\|_{K \cup L} < r\varepsilon/p$.

If we set $g = g_p$, then g satisfies (a) and (b).

By induction on r, we prove the existence of the above sequence. Applying Lemma 3.2.7 to the case $k = 0$, we have g_1 with $(a, 1)$ and $(b, 1)$.

We now suppose that there are $g_1, g_2, \cdots$ and g_k with the above properties. The remainder part will be devoted to a construction of g_{k+1}.

From Lemma 3.2.3 and (a, k), g_k is already in a general position on

$$K \cap FS_{p-k+1}(g_k) ,$$

and hence on a compact neighbourhood A of $K \cap FS_{p-k+1}(g_k)$ in K (note that $K \cap FS_{p-k+1}(g_k)$ is compact). By Lemma 3.2.4, there is some $\delta(A) > 0$ such that

(1) If $\|g_k - h\|_A < \delta(A)$, then h is in a general position on A.

Let W be an open neighbourhood of $K \cap FS_{p-k+1}(g_k)$ in K such that $W \subset A$, and put $B = K - W$. Then B is compact and $K = A \cup B$. Let $\mathfrak{N}$ denote the set of all k-subsets of $\{1, 2, \cdots, p\}$. Line up $\mathfrak{N}$ in some order, and denote the ℓ-th element of $\mathfrak{N}$ by

$$\{s_1(\ell), s_2(\ell), \cdots, s_k(\ell)\} .$$

Put $N = \sharp(\mathfrak{N}) = {}_pC_k$. Let $V(\ell)$ be the set of all the regular points in B of a map

$$(g_k^{s_1(\ell)}, g_k^{s_2(\ell)}, \cdots, g_k^{s_k(\ell)}) \colon E \to C^k .$$

Since $B \cap FS_{p-k+1}(g_k) = \phi$, $\{V(\ell)\}$ is an open covering of B. Contracting each $V(\ell)$, we obtain a compact covering $\{U(\ell)\}$ of B such that $U(\ell) \subset V(\ell)$ for any $1 \leq \ell \leq N$. There is $\delta(\ell) > 0$, for any integer $1 \leq \ell \leq N$, such that

(2) If $\|g_k - h\|_{U(\ell)} < \delta(\ell)$ then

$$(h^{s_1(\ell)}, \cdots, h^{s_k(\ell)}) \colon E \to C^k$$

is fibrewise regular on $U(\ell)$.

Define $\delta(B) = \min\{\delta(1), \delta(2), \cdots, \delta(N)\}$.

We show that there is a sequence $\{(h_\ell, \delta_\ell)\}$ $(\ell = 1, 2, \cdots, N)$ of pairs of holomorphic maps $h_\ell \colon E \to C^p$ and positive numbers δ_ℓ such that for any $1 \leq \ell \leq N$,

(α, ℓ) h_ℓ is in a general position of order $k + 1$ on $U(\ell)$.

(β, ℓ) $\|h_{\ell-1} - h_\ell\|_{K \cup L} < \min\{\varepsilon/p, \delta(A), \delta(B), \delta_1, \delta_2, \cdots, \delta_{\ell-1}\}/N$, where $h_0 = g_k$.

(γ, ℓ) If $\|h_\ell - h\|_{U(\ell)} < \delta(\ell)$, then h is in a general position of order $k + 1$ on $U(\ell)$.

If this is verified, then $g_{k+1} = h_N$ satisfies $(a, k + 1)$ and $(b, k + 1)$ for the following reason. By $(\beta, *)$, we have

$$\|g_k - h_N\|_A < \delta(A) .$$

This implies from (1) that h_N is in a general position on A. For any $1 \leq \ell \leq N - 1$, from $(\beta, \ell + 1)$, $(\beta, \ell + 2)$, $\cdots$ and (β, N), we have

$$\|h_\ell - h_N\|_{U(\ell)} < \delta_\ell .$$

This implies by (γ, ℓ) that h_N is in a general position of order $k + 1$ on

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$U(\ell)$ for any $1 \leq \ell \leq N-1$. By (α, N), h_N is also in a general position of order $k+1$ on $U(N)$. Thus h_N is in a general position of order $k+1$ on K. Moreover, from (b, k) and $(\beta, *)$, we have

$$\|f - h_N\|_{K \cup L} < (k+1)\varepsilon/p\ .$$

We now prove the existence of the above sequence, by induction on ℓ. Suppose that there are (h_1, δ_1), (h_2, δ_2), $\cdots$ and (h_d, δ_d) with the above properties. From $(\alpha, *)$, we have

$$\|g_k - h_d\|_{U(d+1)} < \delta(B) \leq \delta(d+1)\ .$$

This implies by (2) that the map

$$(h_d^{s_1(d+1)}, \cdots, h_d^{s_k(d+1)}) \colon E \to C^k$$

is fibrewise regular on $U(d+1)$. From this and Lemma 3.2.7, it follows that there is h_{d+1} with $(\alpha, d+1)$ and $(\beta, d+1)$. By Lemma 3.2.4 and $(\alpha, d+1)$, we have $\delta_{d+1} > 0$ with $(\gamma, d+1)$. The existence of (h_1, δ_1) is similarly proved. Q.E.D.

In case M is compact, setting $K = L = M$ in the above theorem, we have

THEOREM 3.3.2. *Suppose that M is compact. Let $f \colon E \to C^p$ be a holomorphic map. Then, for any $\varepsilon > 0$, there exists a holomorphic map g from E into C^p in a general position on M such that*

$$\|f - g\|_M < \varepsilon\ .$$

3.4. An existence theorem

By successive applications of Theorem 3.3.1, we have the following existence theorem.

THEOREM 3.4.1. *For any integer $1 \leq p \leq q$, there exists at least one holomorphic map from E into C^p in a general position on M.*

Proof. Since E is paracompact, there is an increasing sequence $\{L_k\}$ $(k = 1, 2, \cdots)$ of compact subsets of E such that $L_k \cap M$ is a non-empty compact subset of M for any k. If we set $K_k = L_k \cap M$ for each k, then $\{K_k\}$ makes an increasing compact covering of M. Take any holomorphic map f_0 from E into C^p. By induction on k, we can define a sequence $\{(f_k, \delta_k)\}$ $(k = 1, 2, \cdots)$ such that for any k,

(a, k) f_k is in a general position on K_k.

(b, k) $\|f_{k-1} - f_k\|_{L_k} < \min\{1/2^k, \delta_1/2^k, \delta_2/2^{k-1}, \cdots, \delta_{k-1}/2^2\}$.

(c, k) If $\|f_k - g\|_{K_k} < \delta_k$, then g is in a general position on K_k.

The existence of the above sequence is proved by applying Theorem 3.3.1 to (f_{k-1}, K_k, L_k) and by applying Lemma 3.2.4 to (f_k, K_k).

For any pair of positive integers $k < \ell$, we have, from $(b, *)$,

(1) $$\|f_{\ell-1} - f_\ell\|_{L_k} < \|f_{\ell-1} - f_\ell\|_{L_\ell} < 1/2^\ell .$$

(2) $$\begin{aligned}\|f_k - f_\ell\|_{K_k} &\leq \|f_k - f_{k+1}\|_{L_{k+1}} + \cdots + \|f_{\ell-1} - f_\ell\|_{L_\ell} \\ &< \delta_k(1/2^2 + 1/2^3 + \cdots) = \delta_k/2 .\end{aligned}$$

Define

$$f = \lim_{\ell\to\infty} f_\ell \colon E \to C^p .$$

From (1), this map is well-defined and holomorphic on E. Moreover, from (2), we have

$$\|f_k - f\|_{K_k} < \delta_k .$$

This implies from (c, k) that f is in a general position on K_k for any k, hence on M. Q.E.D.

§ 4. Realization of Chern classes

4.1. Chern classes and fibrewise singular sets

H_* and H^* will be the singular homology and cohomology with coefficients Z.

Let M be a paracompact complex manifold of dimension n and V a subvariety of M of codimension k. We define the fundamental cohomology class $\{V\} \in H^{2k}(M)$ as follows (see [1], appendix A).

Let V' denote the set of all the singular points of V. There is a closed tubular neighbourhood U of $V - V'$ in $M - V'$. Then we have

$$\begin{aligned} H^0(V - V') &\cong H^{2k}(U, \partial U) \text{ (the Thom isomorphism)} \\ &\cong H^{2k}(M - V', M - V) \\ &\cong H^{2k}(M, M - V) \text{ (see [1])} \\ &\to H^{2k}(M) . \end{aligned}$$

We denote by

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$$\Psi_V^*: H^0(V - V') \to H^{2k}(M)$$

the homomorphism obtained by the composition of the above homomorphisms. Let $\{V_m\}$ be the connected components of $V - V'$ with inclusions $\ell_m: V_m \to V - V'$. Let

$$\omega(V - V') = \sum_m \omega_m (\in H^0(V - V')),$$

where $\omega_m \in H^0(V_m)$ is the canonical generator for each m. We define the fundamental cohomology class of V in M by

$$\{V\} = \Psi_V^*(\omega(V - V')).$$

DEFINITION 4.1.1. A cohomology class $c \in H^*(M)$ is said to be realized by a subvariety V if $c = \{V\}$.

The following lemma shows that in the case of quasilinear subvarieties in a compact complex manifold, the above definition coincides with the usual one (that is, V realizes $c \in H^{2k}(M)$ if and only if c is the Poincaré dual cohomology class of the fundamental homology class $[V] \in H_{2n-2k}(M)$).

LEMMA 4.1.2. *Let V be a quasilinear subvariety of a compact complex manifold M. Then $\{V\}$ corresponds to $[V]$ under the Poincaré duality isomorphism.*

Proof. Let $V_1 \supset V_2 \supset \cdots \supset V_p$ be the associated sequence with V, ℓ the dimension of V and k its codimension in M. Because $H_{2\ell}(V_2) \cong H_{2\ell-1}(V_2) \cong 0$, we have $H_{2\ell}(V_1) \cong H_{2\ell}(V_1, V_2)$. From Lemma 1.3.6, there is an open neighbourhood U of V_2 in V_1 such that inclusions $V_2 \to U$ and $V_1 - U \to V_1 - V_2$ are homotopy equivalences. Therefore we have

$$\begin{aligned} H_{2\ell}(V_1) &\cong H_{2\ell}(V_1, V_2) \cong H_{2\ell}(V_1, U) \\ &\cong H_{2\ell}(V_1 - V_2, U - V_2) \\ &\cong H_{2\ell}(V_1 - V_2, V_1 - V_2 - (V_1 - U)) \\ &\cong H^0(V_1 - U) \text{ (the Alexander duality)} \\ &\cong H^0(V_1 - V_2). \end{aligned}$$

The result comes from the following commutative diagram:

$$\begin{array}{ccc} H^0(V_1 - V_2) & \xrightarrow{\psi_V^*} & H^{2k}(M) \\ \uparrow \cong & & \uparrow \cong \\ H_{2\ell}(V_1) & \longrightarrow & H_{2\ell}(M). \end{array}$$

Q.E.D.

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The first fibrewise singular sets of maps in a general position have the following relation with Chern classes.

THEOREM 4.1.3. *Let M be a paracompact complex manifold and $\xi = (E, \pi, M)$ a holomorphic vector bundle of rank q with the condition* 2.2.1. *If a holomorphic map f from E into C^p is in a general position on M, then the first fibrewise singular set of f realizes the $(q-p+1)$-th Chern class of ξ, that is,*

$$\{FS_1(f)\} = C_{q-p+1}(\xi) .$$

Proof. Suppose that $FS_1(f)$ is not empty. Because $FS_1(f)$ is quasilinear of degree $q-p$ from Theorem 2.2.6, $FS_1(f)$ has the codimension $q-p+1$ in M. Put $k = q-p+1$. Let $\Phi_f: M \to G_{q,p+m}$ be the holomorphic map defined in 2.2.3. By Lemma 2.2.5, we have

$$FS_\ell(f) = \Phi_f^{-1}(F_\ell)$$

for any integer $1 \leq \ell \leq p$. Since f is in a general position on M, Φ_f is transversal to all the strata of F_1. Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccccc} H^0(FS_1(f) - FS_2(f)) & \longrightarrow & H^{2k}(M - FS_2(f), M - FS_1(f)) & \longrightarrow & H^{2k}(M) \\ \uparrow \Phi_f^* & & \uparrow \Phi_f^* & & \uparrow \Phi_f^* \\ H^0(F_1 - F_2) & \longrightarrow & H^{2k}(G_{q,p+m} - F_2, G_{q,p+m} - F_1) & \longrightarrow & H^{2k}(G_{q,p+m}) , \end{array}$$

where the commutativity of the left block comes from the transversality of Φ_f to $F_1 - F_2$. From this diagram, we have that $\{FS_1(f)\} = \Phi_f^*(\{F_1\})$.

On the other hand, we have that $C_k(\xi) = \Phi_f^*(C_k(\gamma_{q,p+m}))$ from 2.2.4. Because F_1 is quasilinear by Proposition 1.2.5, $\{F_1\}$ is the Poincaré dual class of $[F_1]$ by Lemma 4.1.2. Recall that $[F_1]$ is the Poincaré dual class of $C_k(\gamma_{q,p+m})$. Then, we have $\{F_1\} = C_k(\gamma_{q,p+m})$. Consequently, we obtain the equation $\{FS_1(f)\} = C_k(\xi)$.

In case that $FS_1(f)$ is empty, we see that $C_{q-p+1}(\xi)$ is equal to zero.

Q.E.D.

4.2. Proof of Main Theorem

Fix $1 \leq k \leq n$, and put $p = q - k + 1$. From Theorem 3.4.1, there is a holomorphic map f from the total space E into the complex euclidean space C^p which is in a general position on M. Set $V = FS_1(f)$. From Theorem 4.1.3, V realizes the $(q-p+1)$-th Chern class of ξ. This proves (a). From Theorem 2.2.6, V is quasilinear of degree $q-p$ in M. Moreover, from Theorem 1.3.5, V admits a desingularization by means of σ-processes. This proves (b). Because $FS_2(f)$ is empty or has the complex codimension $2(q-p+2) = 2(k+1)$, $FS_2(f)$ is empty for $[n/2] \leq k$. Since $FS_2(f)$ consists of all the singular points of $FS_1(f)$, it follows that V is non-singular for $[n/2] \leq k$. This proves (c) and completes the proof.

4.3. The case of Stein manifolds

Let us specialize Main Theorem to Stein manifolds. In case M is a Stein manifold, we can show that an arbitrary holomorphic vector bundle $\xi = (E, \pi, M)$ of rank q satisfies the condition 2.2.1. Since E is also a Stein manifold, there is a proper holomorphic embedding Ψ of E into $\boldsymbol{C}^{q+m}$ for some sufficiently large integer m. In particular, Ψ is fibrewise regular on M. Therefore, the image of $T_x(F_x)$ by $d_x(\Psi | F_x)$ makes a q-plane in $\boldsymbol{C}^{q+m}$ for any point x of M. By parallel transformations of these planes, we obtain a holomorphic map from M into $G_{q,m}$, which induces the bundle ξ. Thus, from Main Theorem, we have

THEOREM 4.3.1. *Let M be a Stein manifold of dimension n. Then, all the Chern classes of an arbitrary holomorphic vector bundle ξ over M can be realized by quasilinear subvarieties. In particular, the $[n/2]$-th Chern class of ξ can be realized by a non-singular subvariety.*

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