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INJECTIVE ENDOMORPHISMS OF $\mathcal{G}_X$-NORMAL SEMIGROUPS: FINITE DEFECTS

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Abstract

A semigroup of transformations of an infinite set X is called $\mathcal{G}_X$-*normal* if S is invariant under conjugations by permutations of X. In this paper we describe injective endomorphisms of $\mathcal{G}_X$-normal semigroups of total one-to-one transformations f such that the range of f has a finite non-empty complement in X.

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Let X be an infinite set and $\mathcal{G}_X$ be the symmetric group on X. A semigroup S of transformations of X is said to be $\mathcal{G}_X$-*normal* if for every $h \in \mathcal{G}_X$, $hSh^{-1} \subseteq S$. For a transformation f of X the *defect* of f, $\operatorname{def} f = |X - R(f)|$, where $R(f) = f(X)$ is the range of f, and the *shift* of f, $\operatorname{shift} f = |S(f)|$, where $S(f) = \{x \in X : f(x) \neq x\}$. Let $\mathcal{V}_X$ denote the semigroup of all one-to-one total transformations of X with finite non-zero defects. Note that $\mathcal{V}_X$ is a $\mathcal{G}_X$-normal semigroup, and if f is in $\mathcal{V}_X$ then $\operatorname{shift} f$ is always infinite (Lemma 2.2(iv)). Given an infinite cardinal α and a positive integer n, let $S(X, \alpha, n) = \{f \in \mathcal{V}_X : \operatorname{shift} f \leq \alpha, \operatorname{def} f = n\}$. It was proven in [4, Proposition 2.16] that if S is a $\mathcal{G}_X$-normal subsemigroup of $\mathcal{V}_X$ then for each $f \in S$, and every integer $k \geq 8$, S contains $S(X, \operatorname{shift} f, k \operatorname{def} f)$. We say that a $\mathcal{G}_X$-normal S is *closed* if whenever $f \in S$, then S also contains $S(X, \operatorname{shift} f, \operatorname{def} f)$. It follows that a given $\mathcal{G}_X$-normal subsemigroup S of $\mathcal{V}_X$ there exist closed subsemigroups H, K of $\mathcal{V}_X$ which are correspondingly the *largest* and the *smallest* with repect to the property $H \subseteq S \subseteq K$. We denote these semigroups by $S_{\min}$ and $S_{\max}$ respectively, so that $S_{\min} \subseteq S \subseteq S_{\max}$. Note that a semigroup is closed if and only if $S_{\max} = S = S_{\min}$. For example, $\mathcal{V}_X$ and the semigroup of all one-to-one transformations with even non-zero defects are closed (Lemma 2.2(v)). If a semigroup S is not closed, then

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the sets $S \setminus S_{\max}$ and $S_{\min} \setminus S$ are relatively 'small', as demonstrated in the remainder of this paragraph. Let σ-def $S = \{\operatorname{def} f : f \in S\}$, σ-shift $S = \{\operatorname{shift} f : f \in S\}$. Observe that σ-def $S \setminus \sigma$-def $S_{\max} = \sigma$-def $S_{\min} \setminus \sigma$-def S, and the difference is finite. Moreover, σ-shift $S_{\max} \subseteq \sigma$-shift $S = \sigma$-shift $S_{\min}$. Also if $|X| \in \sigma$-shift S then there is at most a finite number of integers k such that for some cardinal α,

$$S(X, \alpha, k) \cap S \neq S(X, \alpha, k) = S(X, \alpha, k) \cap S_{\min}. \tag{1}$$

If $|X| \notin \sigma$-shift S then for all but a finite number of integers k for which (1) holds we have that $S(X, \alpha, k) \cap S \supseteq S(X, \beta, k) = S(X, \alpha, k) \cap S_{\min}$, where $\beta < \alpha$.

This paper is concerned with a description of injective endomorphisms of a closed $\mathscr{G}_X$-normal subsemigroup S of $\mathscr{V}_X$ (Theorem 1.1). There are a number of ingredients that are involved in our description. Generally, an injective endomorphism ϕ of S determines a partition of X into sets W and U such that for an f in S, the behaviour of $\phi(f)$ on W is determined by a finite set of one-to-one functions h_i, $i = 1, \ldots, n$, from X to W (Theorem 1.1 (iii)), while $\phi(f)_{|U}$ is governed by a homomorphism $\xi : S \to \mathscr{V}_U \cup \mathscr{G}_U$. We also present a (more complicated) result describing injective endomorphisms of an arbitrary $\mathscr{G}_X$-normal semigroup of one-to-one transformations with finite non-zero defects (Proposition 1.4).

We note that our description of injective endomorphisms relates to Magill's description of α-monomorphisms of α-semigroups in [6]. A semigroup S of total transformations of X is an *α-semigroup* if S contains the identity transformation of X, and all the *constant* transformations of X that map every point of X onto a single fixed point in X. A monomorphism ϕ from a semigroup S into a semigroup T is called an *α-monomorphism* if $\phi(S)$ is a semigroup with identity e such that if $ez = z$ for any left zero z of T then z is in $\phi(S)$.

It was shown in [6] that a mapping ϕ from an α-semigroup S of transformations of X into an α-semigroup T of transformations of Y is an α-monomorphism if and only if there exist functions $h : X \to Y$ and $k : Y \to X$ such that $kh = i_X$ and $\phi(f) = hfk$, for all f in S. A generalization of the above result to transitive semigroups of (possibly partial) transformations that for every $x \in X$ contain a constant idempotent with range $\{x\}$ is given in [7].

We denote the semigroup of all injective endomorphisms of S by Iend S. We note that if S is $\mathscr{G}_X$-normal then Iend S contains an isomorphic copy of $\mathscr{G}_X$. Indeed, in this case every automorphism of S is inner [3] and so the group of all automorphisms of S is isomorphic to $\mathscr{G}_X$.

1. Main Theorem

Let S be a $\mathscr{G}_X$-normal subsemigroup of $\mathscr{V}_X$, the semigroup of all total one-to-one transformations of X with finite non-zero defects. We start by introducing the notation necessary for stating our main theorem.

For f and g in S, let $D(f, g) = \{x \in X : f(x) \neq g(x)\}$. Let $\Delta_{\aleph_0}$ be a relation on S such that $(f, g) \in \Delta_{\aleph_0}$ if and only if $|D(f, g)| < \aleph_0$. Then $\Delta_{\aleph_0}$ is a congruence. Indeed Lemma 2.7 implies that $\Delta_{\aleph_0}$ is compatible. To show that $\Delta_{\aleph_0}$ is transitive, take (f, g), $(g, t) \in \Delta_{\aleph_0}$. Then $D(f, t) \subseteq D(f, g) \cup D(g, t)$, so $|D(f, t)| \leq |D(f, g)| + |D(g, t)| < \Delta_{\aleph_0}$. Let $S/\Delta_{\aleph_0} = \{V_\alpha : \alpha \in \Lambda\}$, where Λ is an index set. The binary operation on the quotient semigroup $S/\Delta_{\aleph_0}$ induces a binary operation on Λ such that for $\alpha, \beta \in \Lambda, \alpha\beta = \gamma$ if $V_\alpha V_\beta \subseteq V_\gamma$, where $\gamma \in \Lambda$ and $V_\alpha, V_\beta, V_\gamma \in S/\Delta_{\aleph_0}$. With the semigroup S we associate a partial function λ from σ-def S to the set of all infinite cardinals that do not exceed $|X|$ such that for $n \in \sigma$-def S,

$$\lambda(n) = \{\alpha : S \supseteq S(X, \alpha, n)\}. \tag{2}$$

Now we are ready to present the main theorem of this paper that describes injective endomorphisms of an arbitrary closed $\mathscr{G}_X$-normal subsemigroup $S \subseteq \mathscr{V}_X$.

THEOREM 1.1. *Let ϕ be an injective endomorphism of a closed semigroup S. There exist*

(i) *a subset W of X with $|W| = |X|$;*
(ii) *a partition $\{X_i : i = 1, \ldots, n\}$ of W such that $|X_i| = |X|$, for each $i = 1, \ldots, n, n \in \mathbb{N}$;*
(iii) *a set of bijections $h_i : X \to X_i, i = 1, \ldots, n$;*
(iv) *an integer $r \geq n$ such that* $\operatorname{def} \phi(f) = r(\operatorname{def} f)$, *for each $f \in S$;*
(v) *a homomorphism $\xi : S \to \mathscr{G}_U \cup \mathscr{V}_U$, where $U = X - W$ such that*
    (a) *the congruence $\theta(\xi)$ induced by ξ contains $\Delta_{\aleph_0}$,*
    (b) $\operatorname{shift} f + \operatorname{shift} \xi(f) \in \lambda(r(\operatorname{def} f))$;
(vi) *a homomorphism $\tau : \Lambda \to \mathscr{G}_n$ such that if $\tau(\alpha) \neq 1_{\{1,\ldots,n\}}$ then $|X| \in \lambda(r(\operatorname{def} g))$ for each $g \in V_\alpha$;*

and for $f \in V_\alpha, x \in X$,

$$\phi(f)(x) = \begin{cases} h_{\tau(\alpha)(i)} f h_i^{-1}(x) & \text{if } x \in X_i, \\ \xi(f)(x) & \text{if } x \in U. \end{cases} \tag{3}$$

Conversely given (i)–(vi), *the mapping defined in* (3) *is an injective endomorphism of S.*

COROLLARY 1.2. *Let S be a closed semigroup in which every element has shift less than $|X|$. Given an injective endomorphism ϕ of S, there exist* (i)–(v) *(as in Theorem* 1.1) *such that for $f \in V_\alpha, x \in X$,*

$$\phi(f)(x) = \begin{cases} h_i f h_i^{-1}(x) & \text{if } x \in X_i, \\ \xi(f)(x) & \text{if } x \in U. \end{cases} \tag{3'}$$

Conversely, given (i)–(v) *the mapping defined in* (3′) *is an injective endomorphism of* S.

The next result provides us with additional information on homomorphism ξ.

PROPOSITION 1.3. *Let* ξ *be the homomorphism from* S *to* $\mathscr{G}_U \cup \mathscr{V}_U$ *associated with* ϕ.

(i) *Either* $\xi(S) \subseteq \mathscr{G}_U$ *or* $\xi(S) \subseteq \mathscr{V}_U$.
(ii) *If there exist* f, g *in* S *with* $\operatorname{def} f \neq \operatorname{def} g$ *and* $\xi(f) = \xi(g)$ *then* $\xi(S) \subseteq \mathscr{G}_U$.

We deduce Theorem 1.1 from the following result describing injective endomorphisms of an arbitrary $\mathscr{G}_X$-normal subsemigroup S of $\mathscr{V}_X$. We note that the restrictions (6) and (7) imposed in the 'converse' part of the theorem are intrinsically related to the structure of a $\mathscr{G}_X$-normal subsemigroup S of $\mathscr{V}_X$. Namely, if it is known that S contains f with $\operatorname{shift} f = \alpha$ and $\operatorname{def} f = m$, then

$$S \supseteq S(X, \alpha, km) \quad \text{for every } k \geq 8. \tag{4}$$

However, little can be said about $S \cap S(X, \alpha, km)$ for $1 \leq k \leq 8$ (see [4, p. 72–75]).

PROPOSITION 1.4. *Let* $\phi \in \operatorname{Iend} S$. *There exist*

(i) *a subset* W *of* X *with* $|W| = |X|$;
(ii) *a partition* $\{X_i : i = 1, \dots, n\}$ *of* W *such that* $|X_i| = |X|$, *for each* $i = 1, \dots, n, n \in \mathbb{N}$;
(iii) *a set of bijections* $h_i : X \to X_i, i = 1, \dots, n$;
(iv) *an integer* $r \geq n$ *such that* $\operatorname{def} \phi(f) = r \operatorname{def} f$, *for each* $f \in S$;
(v) *a homomorphism* $\xi : S \to \mathscr{V}_U \cup \mathscr{G}_U$, *where* $U = X - W$ *such that the congruence* $\theta(\xi)$ *on* S *induced by* ξ *contains* $\Delta_{\aleph_0}$,
(vi) *a homomorphism* $\tau : \Lambda \to \mathscr{G}_n$ *such that if* $\tau(\alpha) \neq 1_{\{1,\dots,n\}}$ *then* $S \cap S(X, |X|, r \operatorname{def} g) \neq \emptyset$, *for each* $g \in V_\alpha$;

and for $f \in V_\alpha, x \in X$,

$$\phi(f)(x) = \begin{cases} h_{\tau(\alpha)(i)} f h_i^{-1}(x) & \text{if } x \in X_i, \\ \xi(f)(x) & \text{if } x \in U. \end{cases} \tag{5}$$

Conversely, given (i)–(v) *such that for every* $f \in S$

$$\operatorname{shift} f + \operatorname{shift} \xi(f) \in \lambda(r \operatorname{def} f), \tag{6}$$

where λ *is as defined in* (2), *and if for* $\alpha \in \Lambda$, $\tau(\alpha) \neq 1_{\{1,\dots,n\}}$ *then*

$$|X| \in \lambda(r(\operatorname{def} g)), \tag{7}$$

where $g \in V_\alpha$, *the mapping defined in* (5) *is an injective endomorphism of* S.

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2. Definitions and Proofs

The foregoing discussion and the results of this section up to Proposition 2.9 are true for an arbitrary $\mathscr{G}_X$-normal semigroup S of total one-to-one transformations with non-zero defects. The following notion introduced by the author in [3] plays a very important role in our description of injective endomorphisms of S. Let T be a subsemigroup of S. For $x \in X$, let

$$\mathscr{R}(x, T) = \{r \in T : x \in X \setminus R(r)\}.$$

For $f, g \in T$, let

$$\mathscr{R}(f, g, T) = \{r \in T : fr = gr\}.$$

If $\mathscr{R}(x, T)$ and $\mathscr{R}(f, g, T)$ are non-empty they are right ideals of T termed *point right ideal* and *function right ideal* respectively. For briefness we denote $\mathscr{R}(x, S)$ and $\mathscr{R}(f, g, S)$ by $\mathscr{R}(x)$ and $\mathscr{R}(f, g)$ respectively. It was shown in [3, Corollary 2.10] that $\mathscr{R}(f, g)$ is a maximal function right ideal of S if and only if $\mathscr{R}(f, g) = \mathscr{R}(x)$, where $\{x\} = D(f, g)$. This characterization of maximal function right ideals $\mathscr{R}(f, g)$ as $\mathscr{R}(x)$ depends on S being $\mathscr{G}_X$-normal. We show that an injective endomorphism ϕ of S maps a maximal function right ideal of S onto a maximal function right ideal of $\phi(S)$ and the latter can be described in terms of certain point right ideals of $\phi(S)$ that are associated with subsets of M_x of X defined after Proposition 2.9.

A semigroup T of transformations of X is said to be *doubly transitive* if for all pairs x, y and u, v of distinct elements in X, there exists an f in T such that $f(x) = y$, $f(u) = v$.

LEMMA 2.1. (i) *Let $g \in S$ with* $\operatorname{shift} g = \alpha$, $\operatorname{def} g = \beta$. *For every integer $k \geq 8$, S contains $S(X, \alpha, k\beta)$;*

(ii) *S is doubly transitive;*

(iii) *let u, v, w be distinct points in X, then there exists $r \in S$ such that $r(u) = v$, $w \in X - R(r)$;*

(iv) *for all $f \in S$,* $\operatorname{shift} f$ *is infinite;*

(v) *for all $f, g \in S$,* $\operatorname{def} fg = \operatorname{def} f + \operatorname{def} g$.

PROOF. (i) This assertion was proved in [4, Theorem 2.7 and Proposition 2.16], and is stated here for future reference.

(ii) Take pairs x, y and u, v of distinct elements in X. Observe that there exist cardinals α, γ such that S contains $S(X, \alpha, \gamma)$, the set of all total one-to-one transformations t of X having $\operatorname{shift} t \leq \alpha$ and $\operatorname{def} t = \gamma$. Let $Y = X \setminus \{x, y, u, v\}$, $s \in S(Y, \alpha, \gamma)$, $h = (x, y)$, $p = (u, v)$ be transpositions interchanging x and y, and

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u and v. Let

$$f(a) = \begin{cases} s(a), & \text{if } a \in Y, \\ h(a), & \text{if } a \in \{x, y\}, \\ p(a), & \text{if } a \in \{u, v\}. \end{cases}$$

Then $f(x) = y$, $f(u) = v$, as required.

(iii) Assume that $w \in R(f)$, where f is as constructed as above. If $\operatorname{def} f > 1$, choose $z \in X \setminus R(f)$, $z \neq u$, and let $q = (w, z)$, $r = qfq^{-1}$. If $\operatorname{def} f = \gamma = 1$, then $S = \mathscr{V}_X$, and the result is clear.

(iv) Let $x \in X \setminus R(f)$. Then $S(f) \supseteq \{f^n(x) : n = 0, 1, 2, \ldots\}$, indeed, if for some non-negative integer k, $f(f^k(x)) = f^k(x)$, then $f^k(f(x)) = f^k(x)$, and since f is one-to-one, we have that $x = f(x) \in R(f)$, a contradiction.

(v) Observe that $X \setminus R(fg) = (X \setminus R(f)) \cup f(X \setminus R(g))$ so $\operatorname{def} fg = |X \setminus R(fg)| = |X \setminus R(f)| + |f(X \setminus R(g))| = \operatorname{def} f + \operatorname{def} g$.

LEMMA 2.2. *Let $f, g \in S$ with $(f, g) \in \Delta_{\aleph_0}$. Then*

(i) *$\mathscr{R}(f, g) \neq \emptyset$;*

(ii) *for every $x \in X \setminus D(f, g)$ there exists an $r \in \mathscr{R}(f, g)$ with $x \in R(r)$;*

(iii) *for every $x \in X$ there exists $t \in S$ such that $x \in R(t)$ and $(f, t) \in \Delta_{\aleph_0}$.*

PROOF. (i) Let $D = D(f, g)$. Then D is a finite set, and by Lemma 2.1(i) there exist cardinals α, γ such that $\gamma > |D|$ and $S \supseteq S(X, \alpha, \gamma)$. Then for any $s \in S(X, \alpha, \gamma) \subseteq S$ with $R(s) \subseteq X \setminus D$, we have that $s \in R(f, g)$.

(ii) Given $x \in X \setminus D(f, g)$, choose $r \in S(X, \alpha, \gamma)$ as above having $x \in R(r)$, $R(r) \subseteq X \setminus D$.

(iii) Fix an $x \in X$, and assume that $x \notin R(f)$. Choose a $y \in R(f)$, and let $h = (x, y)$ be a transposition interchanging x and y. Let $t = hfh^{-1} \in S$. Then $D(f, t) \subseteq \{x, y, f^{-1}(y)\}$ so $D(f, t)$ is finite, and $(f, t) \in \Delta_{\aleph_0}$.

The next proposition connects point right ideals and function right ideals of a subsemigroup T of S. It is an easy generalization of [3, Result 2.7].

PROPOSITION 2.3. *Let $f, g \in T$ with $\mathscr{R}(f, g, T) \neq \emptyset$. Then*

$$\mathscr{R}(f, g, T) = \cap\{\mathscr{R}(x, T) : x \in D(f, g)\}.$$

PROPOSITION 2.4. (i) *$\mathscr{R}(x, T) = \mathscr{R}(x) \cap T$ for every $x \cap X$;*
(ii) *$\mathscr{R}(f, g, T) = \mathscr{R}(f, g) \cap T$ for every $f, g \in T$.*

Fix an injective endomorphism ϕ of S.

PROPOSITION 2.5. *Let $f, g \in S$ with $\mathscr{R}(f, g) \neq \emptyset$.*

(i) $\phi(\mathscr{R}(f,g)) = \mathscr{R}(\phi(f), \phi(g), \phi(S))$;

(ii) $\mathscr{R}(f,g)$ *is a maximal function right ideal of* S *if and only if* $\mathscr{R}(\phi(f), \phi(g), \phi(S))$ *is a maximal function right ideal of* $\phi(S)$.

PROOF. (i)

$$\begin{aligned}\phi(\mathscr{R}(f,g)) &= \phi(\{r \in S : fr = gr\}) \\ &= \{\phi(r) \in \phi(S) : \phi(f)\phi(r) = \phi(g)\phi(r)\} \\ &= \mathscr{R}(\phi(f), \phi(g), \phi(S)).\end{aligned}$$

(ii) $\mathscr{R}(f,g)$ is a maximal function right ideal if and only if for all $p, q \in S$, $\mathscr{R}(f,g) \subseteq \mathscr{R}(p,q)$ implies $\mathscr{R}(f,g) = \mathscr{R}(p,q)$, and this statement is preserved under injective endomorphisms.

Let $f, g \in S$ with $D(f,g) = \{x\}$, for some $x \in X$. Then $\mathscr{R}(f,g)$ is a maximal function right ideal of S and so the above proposition ensures that $\mathscr{R}(\phi(f), \phi(g), \phi(S))$ is a maximal function right ideal of $\phi(S)$. Moreover, by Proposition 2.3 with $T = S$,

$$\begin{aligned}\phi(\mathscr{R}(x)) = \phi(\mathscr{R}(f,g)) &= \mathscr{R}(\phi(f), \phi(g), \phi(S)) \\ &= \cap\{\mathscr{R}(y, \phi(S)) : y \in D(\phi(f), \phi(g))\},\end{aligned}$$

by Proposition 2.3 again with $T = \phi(S)$. We show that this determines a function $x \to D(\phi(f), \phi(g))$, that does not depend on the choice of f and g. We start with the following lemma.

LEMMA 2.6. *Given distinct* f, g *and* p *in* S *with* $D(f,g) = \{x\} = D(g,p)$, *there exist* s *and* t *in* S *such that* $sf = tp$ *and* $sg = tg$.

PROOF. Observe firstly that $D(f,p) = \{x\}$, for $f \neq p$ and if $v \neq x$ then $f(v) = g(v) = p(v)$. Let $f(x) = y$, $g(x) = z$, $p(x) = u$. Since $D(f,g) = \{x\} = D(g,p)$, y, z and u are distinct.

Choose s in S with $s(u) = u$ and $y \notin R(s)$. To ensure the existence of such an s, choose distinct $v, x \in X \setminus \{u, y\}$ and $q \in S$ with $q(v) = u$, $q(x) = y$ (by Lemma 2.1(ii)). By Lemma 2.1(iii), choose $r \in S$ with $R(r) \subseteq X \setminus \{x\}$, $r(u) = v$, and let $s = qr$. Let $h = (u, y)$, where (u, y) denotes the transposition interchanging u and y. Let $t = hsh^{-1}$. We show that s and t are the required mappings. Firstly, it is easy to check that $D(s,t) = \{u, y\}$. Now, if $w \neq x$, then $f(w) = p(w) \neq p(x) = u$, and $f(w) \neq y = f(x)$, so $f(w) \in X \setminus D(s,t)$ and $sf(w) = tf(w) = tp(w)$. Also $sf(x) = s(y) = hs(y)$, for $s(y) \neq u = s(u)$ and $y \notin R(s)$ and $hs(y) = hsh^{-1}(u) = t(u) = tp(x)$. Thus, $sf = tp$. To show that $sg = tg$ it is sufficient to show that $u, y \notin R(g)$. Now, if $w \neq x$, then $g(w) = p(w) \neq u = p(x)$, and $g(w) = f(w) \neq f(x) = y$. Also, $g(x) = z \neq u, y$, as required.

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LEMMA 2.7. *If f, g and l are one-to-one transformations, then*

(i) $D(lf, lg) = D(f, g)$;

(ii) $D(f, g) \cap R(l) = l(D(fl, gl))$.

A semigroup S is called *right reversible* [1, p. 34] if any two principle left ideals of S have a non-empty intersection: $Sf \cap Sg \neq \emptyset$, for all $f, g \in S$.

PROPOSITION 2.8. *A $\mathcal{G}_X$-normal semigroup of total one-to-one transformations is right reversible.*

PROOF. Let $f, g \in S$ and $\mu = \max\{\operatorname{shift} f, \operatorname{shift} g\}$. We can assume that $\operatorname{def} f = \operatorname{def} g$ (else replace f and g with fg and gf respectively and note that $\operatorname{def} fg = \operatorname{def} f + \operatorname{def} g = \operatorname{def} gf$, by Lemma 2.1(v)). By Lemma 2.1(i) there exists a non-zero cardinal α such that $S \supseteq S(X, \mu, \alpha)$. Choose $p \in S(X, \mu, \alpha)$. Construct a one-to-one mapping q satisfying $pf = qg$ as follows. Let q_1 be a bijection from $R(g)$ onto $R(pf)$ defined by $q_1(g(x)) = pf(x)$, for all $x \in X$. Note that $\operatorname{def} pf = \operatorname{def} p + \operatorname{def} f$, so $|X \setminus R(pf)| > \operatorname{def} f = \operatorname{def} g$, and partition $X \setminus R(pf)$ into disjoint sets A and B with $|A| = \operatorname{def} g$. Let q_2 be a bijection from $X \setminus R(g)$ onto A, and let q be a transformation of X such that

$$q(x) = \begin{cases} q_1(x), & \text{if } x \in R(g), \\ q_2(x), & \text{if } x \in X \setminus R(g). \end{cases}$$

Then $\operatorname{def} q = |B| = \operatorname{def} pf - \operatorname{def} g = \operatorname{def} p + \operatorname{def} f - \operatorname{def} f = \operatorname{def} p$. Also

$$\operatorname{shift} q \leq |\{x \in X : g(x) \neq pf(x)\}| + \operatorname{def} g \leq \operatorname{shift} g + \operatorname{shift} pf + \operatorname{def} g \leq \mu,$$

since $\operatorname{shift} g$, $\operatorname{shift} p$, and $\operatorname{shift} f$ are at most μ, and $\operatorname{def} g \leq \operatorname{shift} g \leq \mu$ (for any $x \in X \setminus R(g)$, $g(x) \neq x$). Thus $q \in S(X, \mu, \alpha) \subseteq S$.

PROPOSITION 2.9. *Let $f, g, p, q \in S$ with $D(f, g) = \{x\} = D(p, q)$. Then $D(\phi(f), \phi(g)) = D(\phi(p), \phi(q))$.*

PROOF. Assume firstly that $q = g$. By Lemma 2.6. there exist $t, s \in S$ with $sf = tp$ and $sg = tg$. Thus

$$\begin{aligned} D(\phi(f), \phi(g)) &= D(\phi(s)\phi(f), \phi(s)\phi(g)), \quad \text{by Lemma 2.7} \\ &= D(\phi(t)\phi(p), \phi(t)\phi(g)) = D(\phi(p), \phi(g)). \end{aligned}$$

Now assume that $q \neq g$. Since S is right reversible (Proposition 2.8), there exist $k, l \in S$ such that $kf = lp$. Now, $D(f, g) = D(kf, kg) = D(lp, kg)$, $D(p, q) \setminus D(lp, lq)$, and the result follows from the previous argument and the fact that the above equalities are preserved under injective endomorphisms.

Note that for each $x \in X$, $\mathscr{R}(x) \neq \emptyset$ (Lemma 2.1(iii)), and there exist $f, g \in S$ with $D(f, g) = \{x\}$ [3, Result 2.8]. Given $x \in X$ define

$$M_x = D(\phi(f), \phi(g)),$$

where $f, g \in S$ with $D(f, g) = \{x\}$. The above result ensures that M_x does not depend on the choice of f and g (as long as $D(f, g) = \{x\}$).

Starting from now assume that S is a $\mathcal{G}_X$-normal semigroup of total one-to-one transformations with *finite* non-zero defects.

PROPOSITION 2.10. (i) $\phi(\mathscr{R}(x)) = \cap\{\mathscr{R}(y, \phi(S)) : y \in M_x\}$;
(ii) M_x *is finite for every* $x \in X$;
(iii) $|M_x| = |M_y|$ *for all* $x, y \in X$;
(iv) $M_x \cap M_y = \emptyset$ *for all distinct* $x, y \in X$;
(v) $f(x) = y$ *if and only if* $\phi(f)(M_x) = M_y$.

PROOF. Statement (i) follows from Proposition 2.3 and the definition of M_x. Sets M_x are finite because of (i) and an observation that $\phi(S)$ consists of transformations with finite defects. To show (iii) let $x, y \in X$ and choose $f, t, s \in S$ such that $f(x) = y$, $D(t, s) = \{y\}$ [3, Result 2.8]. Then $D(tf, sf) = \{x\}$ and by Lemma 2.7(ii)

$$\begin{aligned} \phi(f)(M_x) &= \phi(f)(D(\phi(t)\phi(f), \phi(s)\phi(f))) \\ &= D(\phi(t), \phi(s)) \cap R(\phi(f)) = M_y \cap R(\phi(f)). \end{aligned} \tag{8}$$

Since $\phi(f)$ is one-to-one, $|M_x| \leq |M_y|$. Because of arbitrariness of our choice of f, t, s we conclude that $|M_x| = |M_y|$. Note that this together with (ii) and (8) proves the 'only if' part of v), so that now

$$f(x) = y \text{ implies } \phi(f)(M_x) = M_y. \tag{9}$$

To show (iv) take distinct $x, y \in X$ and assume $z \in M_x \cap M_y$. Choose $g \in \mathscr{R}(y)$ with $g(v) = x$, for some $v \in X$. Then $\phi(g)(M_v) = M_x \ni z$, by (9), while (i) implies that $M_y \subseteq X \setminus R(\phi(g))$, a contradiction, since $z \in M_y$.

Finally, assume $\phi(f)(M_x) = M_y$, while $f(x) = z$, for some $z \in X$. Then by (9), $\phi(f)(M_x) = M_z$ and by iv) $z = y$.

Let $W = \cup\{M_x : x \in X\}$, $U = X \setminus W$. Note that U can be empty.

COROLLARY 2.11. *Given* $\phi \in$ Iend S *there exists a partition of* X *into sets* U *and* W *such that* W *is a disjoint union of sets* M_x, $x \in X$, *and for every* $F \in S$, $\phi(f)(W) \subseteq W$, $\phi(f)(U) \subseteq U$.

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PROOF. Proposition 2.10(v) implies that $\phi(f)(W) \subseteq W$. Also, if $u \in U$ and $\phi(f)(u) = v \in W$, then there exists $y \in X$ such that $v \in M_y$. If $y \in R(f)$, say $f(x) = y$, then $\phi(f)(M_x) = M_y$ (Proposition 2.10(v)), and since $\phi(f)$ is one-to-one, $u \in M_x \subseteq W$, a contradiction. Assume $y \notin R(f)$, so $f \in \mathscr{R}(y)$, then $\phi(f) \in \cap\{\mathscr{R}(z, \phi(S)) : z \in M_y\}$ (Proposition 2.10(i)), that is $R(\phi(f)) \subseteq X \setminus M_y$, a contradiction since $v \in R(\phi(f)) \cap M_y$.

It follows from the above result that ϕ induces a homomorphism $\xi : S \to \mathscr{V}_U \cup \mathscr{G}_U$ given by $\xi(f) = \phi(f)|_U$. The next Lemma shows that the natural congruence $\theta(\xi)$ on S induced by ξ contains $\Delta_{\aleph_0}$.

LEMMA 2.12. *Let* $(f, g) \in \Delta_{\aleph_0}$. *Then*

(i) $\phi(f)|_{M_x} = \phi(g)|_{M_x}$ *for all* $x \in X \setminus D(f, g)$*;*

(ii) $\phi(f)|_U = \phi(g)|_U$.

PROOF. (i) By Lemma 2.2, if $x \in X \setminus D(f, g)$, we can choose $s \in \mathscr{R}(f, g)$ with $s(y) = x$, for some $y \in X$. Then $\phi(s)(M_y) = M_x$ and

$$\phi(f)|_{M_x} = \phi(f)|_{\phi(s)(M_y)} = \phi(f)\phi(s)|_{M_y} = \phi(g)\phi(s)|_{M_y} = \phi(g)|_{M_x}.$$

(ii) Define a relation λ on S such that $(f, g) \in \lambda$ if and only if there exist $p_1, \ldots, p_n \in S$ such that $p_1 = f$, $p_n = g$ and $|D(p_i, p_{i+1})| \leq 1$, $i = 1, \ldots, n-1$. Clearly λ is an equivalence. Moreover, Lemma 2.7 implies that λ is a congruence.

Now, let $(f, g) \in \lambda$, $p_1, \ldots, p_n$ be as above and $u \in U$. Then for $i = 1, \ldots, n-1$, $\phi(p_i)(u) = \phi(p_{i+1})(u)$ (by the definition of M_x's and W), so that $\phi(f)(u) = \phi(p_1)(u) = \cdots = \phi(p_n)(u) = \phi(g)(u)$. Hence we have shown that

$$(f, g) \in \lambda \text{ implies } \phi(f)|_U = \phi(g)|_U. \tag{10}$$

Assume finally that $(f, g) \in \Delta_{\aleph_0}$. Then $\operatorname{def} f = \operatorname{def} g = n$, say. Let $\alpha = \max(\operatorname{shift} f, \operatorname{shift} g)$, and recall that S contains $S(X, \alpha, kn)$, for all $k \geq 8$ (Lemma 2.1). Let $T = \cup\{S(X, \alpha, kn) : k \geq 8\}$. Then $\Delta_{\aleph_0}|_{T\times T} \subseteq \lambda|_{T\times T}$. Indeed, let $(s, t) \in (T \times T) \cap \Delta_{\aleph_0}$. It was shown in [5, Lemma 8] that there exist one-to-one total transformations $s_1 = s, s_2, \ldots, s_m = t$ such that $|D(s_i, s_{i+1})| = 1$, for all $i = 1, \ldots, m-1$. But then $\operatorname{def} s_i = \operatorname{def} s$ and $\operatorname{shift} s_i = \operatorname{shift} s$, for all $i = 1, 2, \ldots, m$, so that $s_i \in T$. Therefore $(s, t) \in \lambda$. Now choose $q \in T$ with $\operatorname{shift} q = \alpha$, $\operatorname{def} q = 8n$, and note that $(qf, qg) \in \Delta_{\aleph_0}|_{T\times T}$. Hence $(qf, qg) \in \lambda$ so that $\phi(q)\phi(f)(u) = \phi(q)\phi(g)(u)$ by (10). But this implies $\phi(f)(u) = \phi(g)(u)$, as required.

Next we show that every homomorphism from S to $\mathscr{V}_Y$, $\Delta_{\aleph_0} \leq |Y| \leq |X|$, preserves the natural order relationship between the defects of transformations in S. This will enable us to describe $\operatorname{def} \phi(f)$ for $f \in S$.

LEMMA 2.13. *Let $\xi' : S \to \mathscr{V}_Y$ be a homomorphism $f, g \in S$. Then,*

(i) $\operatorname{def} f > \operatorname{def} g$ *implies* $\operatorname{def} \xi'(f) > \operatorname{def} \xi'(g)$, *and* def $=$ $\operatorname{def} g$ *implies* $\operatorname{def} \xi'(f) = \operatorname{def} \xi'(g)$,

(ii) $\operatorname{def} \phi(f) = r \operatorname{def} f$, *for a fixed integer* $r \geq |M_x|$.

PROOF. Our proof goes via the following four steps.

Step 1. $\operatorname{def} f > \operatorname{def} g$ if and only if there exist $m \in \mathbb{N}, t \in S$ with $f^m = tg^m$.

Note that it suffices to show that $\operatorname{def} f > \operatorname{def} g$ implies the right hand side of the above equivalence. Let $\operatorname{shift} f = \alpha$, $\operatorname{shift} g = \beta$, and

$$m = \begin{cases} 8(\operatorname{def} f), & \text{if } \alpha > \beta, \\ 8(\operatorname{def} g), & \text{if } \alpha < \beta. \end{cases}$$

Observe that if $p, q \in S$ with $\operatorname{def} p > \operatorname{def} q$ then $|D(p, q)|$ is infinite and a one-to-one t can be constructed so that $p = tq$, $\operatorname{def} t = \operatorname{def} p - \operatorname{def} q$ and $\operatorname{shift} t = |D(p, q)|$ which is at most $\max\{\operatorname{shift} p, \operatorname{shift} q\}$, since $D(p, q) \subseteq S(p) \cup S(q)$. In particular, there exists a one-to-one mapping t with $f^m = tg^m$, $\operatorname{shift} t \leq \max\{\alpha, \beta\}$, and

$$\operatorname{def} t = \operatorname{def} f^m \setminus \operatorname{def} g^m = \begin{cases} 8(\operatorname{def} f)(\operatorname{def} f - \operatorname{def} g), & \text{if } \alpha > \beta, \\ 8(\operatorname{def} f)(\operatorname{def} f - \operatorname{def} g), & \text{if } \alpha < \beta. \end{cases}$$

If $\alpha \geq \beta$, then $\operatorname{shift} t \leq \alpha$, $\operatorname{def} t = 8(\operatorname{def} f)(\operatorname{def} f - \operatorname{def} g)$ and so $t \in S$ by Lemma 2.1(i). Similarly, $t \in S$ if $\alpha < \beta$.

Step 2. $\operatorname{def} f > \operatorname{def} g$ implies $\operatorname{def} \xi'(f) > \operatorname{def} \xi'(g)$.

Follows from Step 1 and the fact that the equality $\xi'(f)^m = \xi'(t)\xi'(g)^m$, $m \geq 1$, implies that $\operatorname{def} \xi'(f) > \operatorname{def} \xi'(g)$ since $\xi'(f), \xi'(t), \xi'(g) \in \mathscr{V}_Y$.

Step 3. $\operatorname{def} f = \operatorname{def} g$ if and only if

$$\text{for all } k, l \in \mathbb{N},\ l \geq 9,\ \text{there exist } s, t \in S \text{ satisfying} \tag{11}$$
$$sf^k = g^{k+l},\ tg^k = f^{k+l},\ \operatorname{def} s < \operatorname{def} f^{l+1},\ \operatorname{def} t < \operatorname{def} g^{l+1}.$$

Let $\operatorname{def} f = \operatorname{def} g = a$. For all positive integers k, l, $l \geq 9$, there exist one-to-one transformations s, t satisfying the two equations in (11). Then $\operatorname{shift} s$, $\operatorname{shift} t \leq \max\{\operatorname{shift} f, \operatorname{shift} g\}$. Also, by Lemma 2.1, $\operatorname{def} s = \operatorname{def} g^{k+l} \setminus \operatorname{def} f^k = (k + l)a - ka = la < la + a = \operatorname{def} f^{l+1}$. Similarly, $\operatorname{def} t = la < \operatorname{def} g^{l+1}$. By Lemma 2.1(i), $s, t \in S$.

For the converse we show that (11) implies $\operatorname{def} f = \operatorname{def} g$ in any subsemigroup S of $\mathscr{V}_X$. Let $\operatorname{def} f = a$, $\operatorname{def} g = b$, $\operatorname{def} s = c$, $\operatorname{def} t = d$. Then (11) implies that

$$ak + c = (k + l)b; \tag{12}$$

$$bk + d = (k + l)a; \tag{13}$$

$$c < (l + 1)a; \tag{14}$$

$$d < (l + 1)b. \tag{15}$$

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We show that (12)–(15) imply $a = b$. Let $k + l = n$, then $n \geq 1 + 9 = 10$. Note that (12) and (14) together imply that

$$(n + 1)a > nb \tag{16}$$

while (13) and (15) imply

$$(n + 1)b > na. \tag{17}$$

It is easy to verify that if a and b satisfy (16) and (17) with $n \geq 10$, then $a = b$.

Step 4. $\operatorname{def} f = \operatorname{def} g$ implies $\operatorname{def} \xi'(f) = \operatorname{def} \xi'(g)$.

This result follows from Steps 2 and 3 (recall that the proof of '(11) implies $\operatorname{def} f = \operatorname{def} g$' in Step 3 is given for an arbitrary semigroup of total one-to-one transformations with finite defects).

Observe that Steps 1–4 above are applicable to ϕ, a particular homomorphism from S into $\mathscr{V}_X$. Therefore, we may define a mapping $\eta : \sigma\text{-def}\, S \to \sigma\text{-def}\, S$ such that for $a \in \sigma\text{-def}\, S$, $f \in S$ with $\operatorname{def} f = a$, $\eta(a) = \operatorname{def} \phi(f)$. It follows from Corollary 2.11 that $\operatorname{def}(\phi(f)) = |W \setminus R(\phi(f)|_W)| + |U \setminus R(\phi(f)|_U)|$. By Proposition 2.10, iii), iv), v), $|W - R(\phi(f)|_W)| = n \operatorname{def} f$, where $n = |M_x|$, for some $x \in X$. Recall that ϕ induces a homomorphism $\xi : S \to \mathscr{V}_U \cup \mathscr{G}_U$ (the remark following Corollary 2.11) given by $\xi(f) = \phi(f)|_U$ and $|U - R(\xi(f))| = \eta(\operatorname{def} f) \setminus n \operatorname{def} f$. Therefore ξ induces a mapping from σ-def S to $\mathbb{N} \cup \{0\}$ such that $\operatorname{def} f \mapsto \operatorname{def} \xi(f)$, for $f \in S$. Let $\sigma\text{-def}\, S = \langle m_1, m_2, \ldots, m_k \rangle$, where $m_i < m_{i+1}$, and $m_1, m_2, \ldots, m_k$ is a minimal set of generators of σ-def S (see [4]).

Step 5. $\eta(a) = (\eta(m_1)/m_1)a$, for an $a \in \sigma\text{-def}\, S$.

Observe that η is a homomorphism since for $a, b \in \sigma\text{-def}\, S$ and $f, g \in S$ such that $\operatorname{def} f = a$, $\operatorname{def} g = b$ we have that $\eta(a + b) = \operatorname{def} \varphi(fg) = \operatorname{def}(\varphi(f)\varphi(g)) = \operatorname{def} \varphi(f) + \operatorname{def} \varphi(g) = \eta(a) + \eta(b)$.

Now given $a \in \sigma\text{-def}\, S$, $a\eta(m_1) = \eta(am_1) = m_1\eta(a)$, since η is a homomorphism. Thus $\eta(a) = (\eta(m_1)/m_1)a$. Let $r = \eta(m_1)/m_1$; we show that r is an integer. Let $d = gcd(m_1, m_2, \ldots, m_k)$, and observe that there exists an integer t such that σ-def S contains all integers $s \geq t$ divisible by d [4, Theorem 2.17]. Choose a $\in \sigma$-def S with $a \geq t$ and $a = bd$, $gcd(b, m_1) = 1$. Then $\eta(a) = ra = (\eta(m_1)/m_1)a(\eta(m_1)/m_1)bd = (bd\eta(m_1))/m_1$. Since $\eta(a) \in \sigma\text{-def}\, S$, $\eta(a)$ is divisible by d, so $(b\eta(m_1)/m_1$ is an integer. Therefore $r = \eta(m_1)/m_1$ is an integer (since $gcd(b, m_1) = 1$). Next we shall define the partition $\{X_i : i = 1, \ldots, n\}$ of W (Theorem 1.1(ii) and Propositon 1.4(ii). We need the following preliminaries.

LEMMA 2.14. *For every $\alpha \in \Lambda$ and distinct $x, y \in X$, there exists $f \in V_\alpha$ such that $f(x) = y$. If α is such that v_α contains a transformation g with $g(u) = u$ for some $u \in X$ then for every $x \in X$ there exists $p \in V_\alpha$ for which $p(x) = x$.*

PROOF. For each $t \in V_\alpha$, $S(t)$ is infinite and so there exist distinct $a, b \in X \setminus \{x, y\}$ such that $t(a) = b$. Let $f = (a, x)(b, y)t(a, x)(b, y)$. Note that if $z \neq a, b, x, y$ and $t(z) \neq a, b, x, y$ then $f(z) = t(z)$ and hence $f \in V_\alpha$, $f(x) = y$. To prove the second statement let $g \in V_\alpha$ with $g(u) = u$. Then $p = g$ if $x = u$, and $p = (x, u)g(x, u)$ otherwise, is the required transformation.

We call a $\Delta_{\aleph_0}$-class V_α containing a transformation with a *fixed* point an f-*set*.

LEMMA 2.15. *Let* $\alpha, \beta \in \Lambda$. *Then* $V_{\alpha\beta}$ *is an* f-*set*.

PROOF. By the first part of Lemma 2.14, we can choose $f \in V_\alpha$, $g \in V_\beta$ such that for some distinct $u, v \in X$, $f(u) = v$, $g(v) = u$. Then $fg(v) = v$, $fg \in V_{\alpha\beta}$.

LEMMA 2.16. *Let* $\alpha \in \Lambda$. *Then*

(i) *for all* $f, g \in V_\alpha$, $\operatorname{def} f = \operatorname{def} g$ *and* $\operatorname{shift} f = \operatorname{shift} g$;
(ii) *if* $f \in V_\alpha$ *with* $\operatorname{def} f = m$, *then* $\{R(g) : g \in V_\alpha\} = \{B \subseteq X : |X \setminus B| = m\}$.

PROOF. While (i) follows easily from the definition of $\Delta_{\aleph_0}$, to show (ii) let $R(f) = A$ and $B \subseteq X$ with $|X \setminus B| = m$. Choose a permutation h of X such that $h(X \setminus A) = X \setminus B$, $h(X \setminus B) = X \setminus A$ and h is the identity of $A \cap B$. We show that $hfh^{-1} \in V_\alpha$. Indeed if $hfh^{-1}(x) \neq f(x)$ then either $f(x) \in A \cap (X \setminus B)$ or $f(x) \in A \cap B$. There are only finitely many x in the first cases since $X \setminus B$ is finite and f is one-to-one, and in the second case $fh^{-1}(x) \neq h^{-1}f(x) = f(x)$, and so $h(x) \neq x$. Hence, since h shifts only a finite number of points we conclude that $hfh^{-1} \in V_\alpha$ with $R(hfh^{-1}) = h(R(f)) = h(A) = B$. Finally, note that the reverse containment follows from (1).

LEMMA 2.17. *Let* α *and* β *be in* Λ. *Put* $\mu = \operatorname{shift} p + \operatorname{shift} q$ *and* $a = 8 \operatorname{def} p + 9 \operatorname{def} q$, *for some* $p \in V_\alpha$ *and* $q \in V_\beta$. *Then for all* $x \in X$, $m \geq 1$, *there exist* $\delta, \gamma \in \Lambda$ *such that for any* $k \in S(X, \mu, ma) \cap V_\delta$ *and for an* $f \in V_\alpha$ *with* $fk(x) \neq x$ *if* V_β *is not an* f-*set, we have* $fk = gl$, *for some* $g \in V_\beta$ *and* $l \in V_\gamma$ *such that* $l(x) = x$.

PROOF. Note that by Lemma 2.16(i), $\mu = \max\{\operatorname{shift} p, \operatorname{shift} q : p \in V_\alpha, q \in V_\beta\}$ and either p or q in the definition of a will have shift μ. Hence by Lemma 2.1(i), $S(X, \mu, ma) \subseteq S$ and so there is $k \in S(X, \mu, ma)$ such that $fk(x) = u$ with $u \neq x$ when V_β is not a f-set. Let $fk(x) = u$ and assume that there exists $g \in V_\beta$ with $g(x) = u$ and $R(g) \supseteq R(fk)$. Then there exists a one-to-one total transformation l such that $fk = gl$. It then follows that $S(l) = D(g, fk)$ and therefore $\operatorname{shift} l = |S(l)| \leq \mu$, while $\operatorname{def} l = \operatorname{def} f + \operatorname{def} k - \operatorname{def} g = (8m + 1) \operatorname{def} f + (9m - 1) \operatorname{def} g$, so by Lemma 2.1(i) $l \in S$ with $l(x) = g^{-1}(fk(x)) = g^{-1}(u) = x$.

To show the existence of $g \in V_\beta$ as above note that by Lemmas 2.16(i) and 2.1(v) for any $t \in V_\beta$, $\operatorname{def} t = \operatorname{def} g > \operatorname{def} g + \operatorname{def} l = \operatorname{def} gl = \operatorname{def} fk$. By Lemma 2.16(ii) we may choose t in V_β such that $R(t) \supseteq R(fk)$. If $t(x) = u$, let $g = t$. If $t(x) = v \neq u, x$ and $u \neq x$, let $g = (u, v)t(u, v)$ (note that $R(g) = R(t)$ since $v = t(x) \in R(t)$ and $u = fk(x) \in R(fk) \subseteq R(t)$). If $t(x) = x$ and $u \neq x$, then since $R(t)$ is infinite it contains some $z \neq u, x$ such that $t(z) = w \neq u, x, z$. Let $g = (x, z)(u, w)t(x, z)(u, w)$ then $D(g, t) \subseteq \{u, w, x, z, t^{-1}(u), t^{-1}(w), t^{-1}(x), t^{-1}(z)\}$, a finite set, and so $g \in V_\beta$. Finally let $v \neq x$ while $u = x$. Then V_β is an f-set, so that V_β contains a transformation s with a fixed point. Choose a permutation h of X such that $h(X \setminus R(s)) = X \setminus R(t), h(X \setminus R(t)) = X \setminus R(s)$ and h is the identity otherwise. Replace t with hth^{-1}. Then $t(y) = y$ for some $y \in X$, $R(t) \supseteq R(fk)$. If $y = x$ let $g = t$. Otherwise let $g = (x, y)t(x, y)$ (note that $R(g) = R(t)$ since $x = u \in R(fk) \subseteq R(t)$).

Now let V_δ and V_γ be the classes of $\Delta_{\aleph_0}$ containing k and l respectively. Then for any $f' \in V_\alpha$ and $k' \in V_\delta$ with $f'k'(x) \neq x$ if V_β is not an f-set, as above we can find $g' \in V_\beta, l' \in S$ with $l'(x) = x$ and $f'k' = g'l'$. We show that $l' \in V_\gamma$. Indeed, $(f, f') \in \Delta_{\aleph_0}$ and $(k'k') \in \Delta_{\aleph_0}$ imply $(fk, f'k') \in \Delta_{\aleph_0}$, so that $(gl, g'l') \in \Delta_{\aleph_0}$ or $|D(gl, g'l')| < \aleph_0$. We show that $D(l, l')$ is finite. Indeed, if a $\in D(l, l')$ then either $gl(a) \neq g'l'(a)$, so $a \in D(gl, g'l')$, a finite set, or $gl(a) = g'l'(a)$, and $l(a) \in D(g, g')$, again a finite set, since $g, g' \in V_\beta$. Therefore, $D(l, l') \subseteq D(gl, g'l') \cup l^{-1}(D(g, g'))$, hence $|D(l, l')| < \aleph_0$, so $(l, l') \in \Delta_{\aleph_0}$ and $l' \in V_\gamma$.

Fix an x in X and write $M_x = \{x_1, \dots, x_n\}$ (M_x is defined after Proposition 2.9). For every $i \in \{1, \dots, n\}$ and $\alpha \in \Lambda$ let

$$Y_{i,\alpha} = \{\phi(f)(x_i) : f \in V_\alpha\}.$$

Partitions $\mathscr{A}$ and $\mathscr{B}$ of a set Z are said to be *orthogonal* if for all $A \in \mathscr{A}$, $B \in \mathscr{B}$, $|A \cap B| = 1$. A subset C of Z is a *transversal* of the partition $\mathscr{A}$ if for all $A \in \mathscr{A}$, $|C \cap A| = 1$.

LEMMA 2.18.

(i) *If $i \neq j$ then $Y_{i,\alpha} \cap Y_{j,\alpha} = \emptyset$.*
(ii) *If V_α is an f-set then $\{Y_{i,\alpha} : i = 1, \dots, n\}$ forms a partition of W orthogonal to $\{M_y : y \in X\}$. Otherwise, $\{Y_{i,\alpha} : i = 1, \dots, n\}$ is a partition of $W \setminus M_x$ orthogonal to $\{M_y : y \in X \setminus \{x\}\}$.*

PROOF. (i) Assume that $z \in Y_{i,\alpha} \cap Y_{j,\alpha}$, so that there exist $f, g, \in V_\alpha$ such that $\varphi(f)(x_i) = z = \varphi(f)(x_j)$. Recall that $x_i \in M_x$ and let $f(x) = y$, $g(x) = a$. Then by Proposition 2.10(v), $\varphi(f)(M_x) = M_y$, $\varphi(g)(M_x) = M_a$, and by 2.10(iv),

$z \in M_y \cap M_a$ implies $y = a$. Therefore $x \notin D(f, g)$, and so 2.12(i) implies that $\varphi(f)(x_i) = \varphi(g)(x_i)$. Hence, $i = j$.

(ii) We start by showing that $|M_y \cap Y_{i,\alpha}| = 1$ for every $y \in X$ for which there exist $f \in V_\alpha$ with $f(x) = y$. We have that $\varphi(f)(M_x) = M_y$ so that $\varphi(f)(x_i) \in M_y \cap Y_{i,\alpha}$, that is, $|M \cap Y_{i,\alpha}| \geq 1$. Now assume $a, b \in M_y \cap Y_{i,\alpha}$, then there exist $t, s, \in V_\alpha$ with $\varphi(t)(x_i) = a$, $\varphi(s)(x_i) = b$ and so $t(x) = y = s(x)$. Hence $a = b$ and $|M_y \cap Y_{i,\alpha}| = 1$.

For every $y \in X$ such that there exists $f \in V_\alpha$ with $f(x) = y$, $M_y \subseteq \cup\{Y_{i,\alpha} : i = 1, \ldots, n\}$. This follows from (i) and the above. This and Lemma 2.14 together with an observation that if V_α does not contain mappings with fixed points then $M_x \cap Y_{i,\alpha} = \emptyset$, for every i completes the proof.

Our aim now is to associate with every $\alpha \in \Lambda$ a partition $\mathscr{X}_\alpha$ of W orthogonal to $\{M_y : y \in X\}$. If α is such that V_α contains a mapping with a fixed point let $\mathscr{X}_\alpha = \{Y_{i,\alpha} : i = 1, \ldots, n\}$. Otherwise we use the following construction. Choose $u \in X$, $u \neq x$ and let $M_u = \{u_1, \ldots, u_n\}$. Let

$$U_{i,\alpha} = \{\varphi(f)(u_i) : f \in V_\alpha\}.$$

LEMMA 2.19. *Given $i \in \{1, \ldots, n\}$ there exists unique $j \in \{1, \ldots, n\}$ such that $U_{i,\alpha} = Y_{j,\alpha}$ if V_α is an f-set and $U_{i,\alpha} \setminus M_x = Y_{j,\alpha} \setminus M_u$ otherwise.*

PROOF. Choose $y \in X$ depending on whether V_α is an f-set as follows. If V_α is an f-set let y be an arbitrary element of X. If V_α is not an f-set let y be an arbitrary element of $X \setminus \{x\}$. By Lemma 2.18(ii) applied to $U_{i,\alpha}$ there is $z \in U_{i,\alpha} \cap M_y$, with $z = \phi(f)(u_i)$, for some $f \in V_\alpha$. Then $\phi(f)(M_u) = M_y$ by Proposition 2.10(v) and (iv), and so by Proposition 2.10(v), $f(u) = y$. We use Lemma 2.17 with $\beta = \alpha$ and $a = 8 \operatorname{def} f + 9 \operatorname{def} f = 17 \operatorname{def} f$, and $\mu = \operatorname{shift} f$. Choose $k \in S(X, \mu, a) \subseteq S$ with $k(x) = u$ (Lemma 2.14). Then $fk(x) = f(u) = y$, and if V_α is not an f-set we have $y \neq x$. By Lemma 2.17 there exist l, g in S with $g \in V_\alpha$ and

$$\begin{aligned} z = \phi(f)(u_i) &= \phi(f)\phi(k)(x_t), \text{ for some } t \in \{1, \ldots, n\} \\ &= \phi(fk)(x_t) = \phi(gl)(x_t) = \phi(g)\phi(l)(x_t) \\ &= \phi(g)(x_j) \in Y_{j,\alpha}, \text{ for some } j \in \{1, \ldots, n\}. \end{aligned}$$

If v is also an element of $U_{i,\alpha}$, $v = \phi(f')(u_i)$ and $f' \in V_\alpha$, let k be chosen as before, so that $u_i = \phi(k)(x_t)$. By Lemma 2.17 again choose l', g' such that $l'(x) = x$, $f'k = g'l'$, $g' \in V_\alpha$. Then

$$\begin{aligned} v = \phi(f')(u_i) &= \phi(f')\phi(k)(x_t) = \phi(f'k)(x_t) \\ &= \phi(g'l')(x_t) = \phi(g')\phi(l')(x_t) = \phi(g')(x_j) \in Y_{j,\alpha}, \end{aligned}$$

since $(l, l') \in \Delta_{\aleph_0}$, and so by 2.12, $\phi(l)(x_t) = \phi(l')(x_t)$. Therefore we have shown that $U_{i,\alpha} \subseteq Y_{j,\alpha}$ if V_α is an f-set, and $U_{i,\alpha} \setminus M_x \subseteq Y_{j,\alpha}$, if V_α is not an f-set. If V_α is an f-set, $Y_{j,\alpha}$ and $U_{i,\alpha}$ are transversals of the partition $\{M_y : y \in X\}$ of W (Lemma 2.18(ii)), and so $Y_{j,\alpha} = U_{i,\alpha}$. If V_α is not an f-set $Y_{j,\alpha}$ is a transversal of $\{M_y : y \in X \setminus \{x\}\}$, $U_{i,\alpha}$ is a transversal of $\{M_y : y \in X \setminus \{u\}\}$ (Lemma 2.18(ii) again), so $U_{i,\alpha} \setminus M_x$ and $Y_{j,\alpha} \setminus M_u$ are transversals of $\{M_y : y \in X \setminus \{u, x\}\}$. Therefore the inclusion $U_{i,\alpha} \setminus M_y \subseteq Y_{j,\alpha}$ implies that $U_{i,\alpha} \setminus M_x = Y_{j,\alpha} \setminus M_u$.

If V_α is not an f-set, the above lemma determines a permutation ρ of $\{1, \dots, n\}$ such that $Y_{i,\alpha} \setminus M_u = U_{\rho(i),\alpha} \setminus M_x$. With every $\alpha \in \Lambda$ we associate a partition $\mathscr{X}_\alpha$ of W such that

$$\mathscr{X}_\alpha = \{Y_{i,\alpha} \cup U_{\rho(i),\alpha} : i = 1, \dots, n\}.$$

In view of the above lemma this is a natural extension of the definition of $\mathscr{X}_\alpha$ given after Lemma 2.18 for these α for which V_α is an f-set. Observe that for evey $\alpha \in \Lambda$, the partition $\mathscr{X}_\alpha$ of W is orthogonal to $\{M_y : y \in X\}$. Indeed, let $z \in X \setminus \{x, u\}$, and note that since $Y_{i,\alpha}$ and $U_{\rho(i),\alpha}$ are transversals of $\{M_y : y \in X \setminus \{x\}\}$ and $\{M_y : y \in X \setminus \{u\}\}$ respectively (Lemma 2.18), then $|Y_{i,\alpha} \cap M_z| = 1 = |U_{\rho(i),\alpha} \cap M_z|$. Since $Y_{i,\alpha} \setminus M_u = U_{\rho(i),\alpha} \setminus M_x$ we have that $Y_{i,\alpha} \cap M_z = U_{\rho(i),\alpha} \cap M_z = \{a\}$, for some $a \in M_z$. Therefore

$$[Y_{i,\alpha} \cup U_{\rho(i),\alpha}] \cap M_z = (Y_{i,\alpha} \cap M_z) \cup (U_{\rho(i),\alpha} \cap M_z) = \{a\} \cup \{a\} = \{a\}$$

so that $|[Y_{i,\alpha} \cup U_{\rho(i),\alpha}] \cap M_z| = 1$. The next result follows.

LEMMA 2.20. *For every $\alpha \in \Lambda$, $\mathscr{X}_\alpha$ is a partition of W orthogonal to $\{M_y : y \in X\}$ that does not depend on the choice of initial point (x) or points (x and y).*

We write $\mathscr{X}_a = \{X_{i,\alpha} : i = 1, \dots, n\}$, so that $X_{i,\alpha} = Y_{i,\alpha}$ if V_α is an f-set, and $X_{i,\alpha} = Y_{i,\alpha} \cup U_{\rho(i),\alpha}$ otherwise.

LEMMA 2.21. *There exists a function $\sigma : \Lambda \times \Lambda \to \mathscr{G}_n$ such that for $\alpha, \beta \in \Lambda$, $i, j \in \{1, \dots, n\}$, $\sigma(\alpha, \beta)(i) = j$ if $X_{i,\alpha} = X_{j,\beta}$.*

PROOF. Let $\alpha, \beta \in \Lambda$, $v \in X_{i,\alpha} \cap M_y$ for some $y \in X$ with $y \neq x$ if V_β is not an f-set, for which there exists $f \in V_\alpha$ with $f(x) = y$. Let k, g, l be as in Lemma 2.17 with k chosen such that $k(x) = x$. Then $x_i = \phi(k)(x_m)$ for some m and

$$\begin{aligned} v = \phi(f)(x_i) &= \phi(f)\phi(k)(x_m) = \phi(fk)(x_m) \\ &= \phi(gl)(x_m) = \phi(g)\phi(l)(x_m) = \phi(g)(x_j) \in X_{j,\beta}, \end{aligned}$$

for some $j \in \{1, \dots, n\}$. If $v' = \phi(f')(x_i)$, $f' \in V_\alpha$, choose k as above and l', g' such that $(l, l'), (g, g') \in \Delta_{\aleph_0}$, $l'(x) = x$, $f'k = g'l'$ (Lemma 2.17 again). Using the above argument we can show that $v' = \phi(g')\phi(l')(x_m) = \phi(g')(x_j) \in X_{j,\beta}$. If V_α and V_β are f-sets, then $X_{i,\alpha} = Y_{i,\alpha} \subseteq Y_{j,\beta} = X_{j,\beta}$, and since $X_{i,\alpha}, X_{j,\beta}$ are transversals of $\{M_y : y \in X\}$, we have $X_{i,\alpha} = X_{j,\beta}$. If either V_α or V_β are not f-sets, then $X_{i,\alpha} \setminus M_x = Y_{i,\alpha} \subseteq Y_{j,\beta}$. In this case we repeat the argument starting with u to deduce that $U_{\rho(i),\alpha} \subseteq U_{k,\beta}$ (note the same β as before). Hence, using 2.19, $U_{\rho(i),\alpha} \setminus M_x = Y_{i,\alpha} \cap U_{\rho(i),\alpha} \subseteq Y_{j,\beta} \cap U_{k,\beta}$. That is $X_{j,\beta} \cap X_{k,\beta}$ is non-empty and so, by 2.20, $j = k$. Hence $X_{i,\alpha} \subseteq X_{j,\beta}$, and the equality follows.

Using Lemma 2.20, for every $\alpha \in \Lambda$ define bijections

$$h_{i,\alpha} : X \to X_{i,\alpha} \text{ by } y \mapsto M_y \cap X_{i,\alpha}, \quad i = 1, \ldots, n.$$

LEMMA 2.22. *Given $f \in V_\alpha$, $y \in X_{i,\alpha}$ and $\beta \in \Lambda$ such that V_β is an f-set,*

$$\phi(f)(y) = h_{\sigma(\alpha,\beta)(i),\alpha\beta} f h_{i,\alpha}^{-1}(y).$$

PROOF. Let $\{y\} = M_z \cap X_{i,\alpha} = \{h_{i,\alpha}(z)\}$. Then $\phi(f)(y) \in \phi(f)(M_z) = M_{f(z)}$. Observe that Lemma 2.14 implies that V_β contains k with $k(x) = z$ (in particular if $x = z$ the required k exists by 2.14 since V_β is an f-set). Then $\phi(k)(M_x) = M_z$, so that $\phi(k)(x_j) = y$, $j \in \{1, \ldots, n\}$. Therefore $y \in X_{j,\beta} \cap X_{i,\alpha}$, and by Lemma 2.21, $X_{i,\alpha} = X_{k,\beta}$ for some $k \in \{1, \ldots, n\}$. But then $X_{j,\beta} \cap X_{k,\beta} \neq \emptyset$, so $X_{j,\beta} = X_{k,\beta} = X_{i,\alpha}$ by Lemma 2.18(i). Hence $X_{j,\beta} = X_{\sigma(\alpha,\beta)(i),\beta}$ (Lemma 2.21), so that $j = \sigma(\alpha, \beta)(i)$, and

$$\phi(f)(y) = \phi(f)\phi(k)(x_j) = \phi(fk)(x_j) \in X_{j,\alpha\beta} = X_{\sigma(\alpha,\beta)(i),\alpha\beta}.$$

Thus,

$$\phi(f)(y) = M_{f(z)} \cap X_{\sigma(\alpha,\beta)(i),\alpha\beta} = h_{\sigma(\alpha,\beta)(i),\alpha\beta} f(z) = h_{\sigma(\alpha,\beta)(i),\alpha\beta} f h_{i,\alpha}^{-1}(y).$$

LEMMA 2.23. *For all $\alpha, \beta, \gamma \in \Lambda$,*

(i) $\sigma(\alpha, \beta) = \sigma^{-1}(\beta, \alpha)$;
(ii) $\sigma(\alpha, \beta) = \sigma(\gamma, \beta)\, \sigma(\alpha, \gamma)$;
(iii) $\sigma(\alpha\beta, \alpha\gamma) = \sigma(\beta, \gamma)$, *for all β, γ for which V_β and V_γ are f-sets.*

PROOF. (i) $X_{i,\alpha} = X_{\sigma(\alpha,\beta)(i),\beta} = X_{\sigma(\beta,\alpha)\sigma(\alpha,\beta)(i),\alpha}$, so $\sigma(\beta, \alpha)\sigma(\alpha, \beta)(i) = i$ for every $i \in \{1, \ldots, n\}$.

(ii) $X_{i,\alpha} = X_{\sigma(\alpha,\beta)(i),\beta} = X_{\sigma(\beta,\gamma)\sigma(\alpha,\beta)(i),\gamma} = X_{\sigma(\gamma,\alpha)\sigma(\beta,\gamma)\sigma(\alpha,\beta)(i),\alpha}$, so $\sigma(\gamma, \alpha)\sigma(\beta, \gamma)\sigma(\alpha, \beta)(i) = i$ for every $i \in \{1, \ldots, n\}$. Hence $\sigma(\alpha, \beta) = [\sigma(\gamma, \alpha)\sigma(\beta, \gamma)]^{-1} = \sigma^{-1}(\beta, \gamma)\sigma^{-1}(\gamma, \alpha) = \sigma(\gamma, \beta)\sigma(\alpha, \gamma)$, by i).

(iii) By Lemma 2.22, we have $\phi(f)(y) = h_{\sigma(\alpha,\beta)(i),\alpha\beta} f h_{i,\alpha}^{-1}(y)$, for $f \in V_\alpha$, $y \in X_{i,\alpha}$. Also, $\phi(f)(y) = h_{\sigma(\alpha,\gamma)(i),\alpha\gamma} f h_{i,\alpha}^{-1}(y) = h_{\sigma(\alpha\gamma,\alpha\beta)\sigma(\alpha,\gamma)(i),\alpha\beta} f h_{i,\alpha}^{-1}(y)$, so that $\sigma(\alpha, \beta) = \sigma(\alpha\gamma, \alpha\beta)\sigma(\alpha, \gamma)$, or $\sigma(\alpha\gamma, \alpha\beta) = \sigma(\alpha, \beta)\sigma^{-1}(\alpha, \gamma) = \sigma(\alpha, \beta)\sigma(\gamma, \alpha) = \sigma(\gamma, \beta)$, by (ii).

Fix $\mu \in \Lambda$ with V_u being an f-set. Let $X_{i,\mu} = X_i$, $h_{i,\mu} = h_i$. Define $\tau : \Lambda \to \mathcal{G}_n$ via $\alpha \to \sigma(\alpha\mu, \mu)$.

LEMMA 2.24. *τ is a homomorphism.*

PROOF. Let $\alpha, \beta \in \Lambda$. Then, by Lemma 2.23(iii) and Lemma 2.15, $\tau(\alpha)\tau(\beta) = \sigma(\alpha\mu, \mu)\sigma(\beta\mu, \mu) = \sigma(\alpha\mu, \mu)\sigma(\alpha\beta\mu, \alpha\mu)$. Now, using Lemma 2.23(ii), $\sigma(\alpha\mu, \mu)\sigma(\alpha\beta\mu, \alpha\mu) = \sigma(\alpha\beta\mu, \mu) = \tau(\alpha\beta)$.

LEMMA 2.25. *Given $y \in X_i$, $f \in V_\alpha$, $\phi(f)(y) = h_{\tau(\alpha)(i)} f h_i^{-1}(y)$.*

PROOF. Observe that $X_i = X_{i,\mu} = X_{\sigma(\mu,\alpha)(i),\alpha}$ and by Lemma 2.22 with $\beta = \mu$,

$$\begin{aligned}\phi(f)(y) &= h_{\sigma(\alpha,\mu)\sigma(\mu,\alpha)(i),\alpha\mu} f h^{-1}_{\sigma(\mu,\alpha)(i),\alpha}(y) = h_{i,\alpha\mu} f h^{-1}_{\sigma(\alpha,\mu)\sigma(\mu,\alpha)(i),\mu}(y) \\ &= h_{\sigma(\alpha\mu,\mu)(i),\mu} f h^{-1}_{i,\mu}(y) = h_{\tau(\alpha)(i)} f h_i^{-1}(y).\end{aligned}$$

Recall that there exists an integer $r \geq |M_x|$ such that $\operatorname{def}\phi(f) = r \operatorname{def} f$, for $f \in S$ (Lemma 2.13(ii)).

COROLLARY 2.26. (i) *For every $i \in \{1, \ldots, n\}$, $\phi(f)(X_i) \subseteq X_{\tau(\alpha)(i)}$.*

(ii) *If $\tau(\alpha) \neq 1_{\{1,\ldots,n\}}$, then $S \cap S(X, |X|, r(\operatorname{def} f)) \neq \emptyset$ where $f \in V_\alpha$.*

(iii) *If* shift $f < |X|$ *for all f in S then $\tau(\alpha)$ is the identity on $\{1, \ldots, n\}$ for every $\alpha \in \Lambda$.*

PROOF. (i) The statement follows directly from Lemma 2.25 and the fact that the image of $h_{\tau(\alpha)(i)}$ is $X_{\tau(\alpha)(i)}$.

(ii) Let $\alpha \in \Lambda$ be such that $\tau(\alpha)(i) = j \neq i$, for some $i, j \in \{1, \ldots, n\}$. Then, by part i), $\phi(f)(X_i) \subseteq X_j \subseteq X \setminus X_i$. Therefore $\operatorname{shift}\phi(f) \geq |X_i| = |X|$. The result now follows from the fact that $\operatorname{def}\phi(f) = r \operatorname{def} f$ (Lemma 2.13(ii)), and $\phi(f) \in S \cap S(X, |X|, r \operatorname{def} f)$.

(iii) This statement is an immediate consequence of (ii).

PROOF OF PROPOSITION 1.4. The set W is defined prior to Corollary 2.11. The existence of the partition $\{X_i : i = 1, \ldots, n\}$ in (ii) is established in Lemma 2.20. Bijections h_i in (iii) are defined prior to Lemma 2.24, while (iv) and (v) are shown in Lemmas 2.13 and 2.12 respectively. The homomorphism τ in (vi) is established in Lemma 2.24 and Corollary 2.26(ii). Finally, (5) is proven in Lemma 2.25.

Conversely, assume (i)–(vi) are given and satisfy (6) and (7). Let $f \in S$. Clearly $\phi(f)$ is one-to-one, and if $f \in V_\alpha$ with $\tau(\alpha) = 1_{\{1,\ldots,n\}}$ then $\operatorname{shift}\phi(f) = n \operatorname{shift} f + \operatorname{shift}\xi(f) = \operatorname{shift} f + \operatorname{shift}\xi(f)$, and (6) implies that $\phi(f) \in S$. If $f \in V_\alpha$ with $\tau(\alpha) \neq 1_{\{1,\ldots,n\}}$ then $\phi(f) \in S$ by (7).

To show that ϕ is a morphism take $f \in V_\alpha$, $g \in V_\beta$ and $y \in X_i$, for $\alpha, \beta \in \Lambda$, $i \in \{1, \dots, n\}$. Then $fg \in V_{\alpha\beta}$ and

$$\phi(fg)(y) = h_{\tau(\alpha\beta)(i)} f g h_i^{-1}(y),$$

while

$$\begin{aligned}\phi(f)\phi(g)(y) = \phi(f)h_{\tau(\beta)(i)}gh_i^{-1}(y) &= h_{\tau(\alpha)\tau(\beta)(i)} f h^{-1}_{\tau(\beta)(i)} h_{\tau(\beta)(i)} g h_i^{-1}(y), \\ &\qquad \text{since } h_{\tau(\beta)(i)} g h_i^{-1}(y) \in X_{\tau(\beta)(i)} \\ = h_{\tau(\alpha)\tau(\beta)(i)} f g h_i^{-1}(y) &= h_{\tau(\alpha\beta)(i)} f g h_i^{-1}(y),\end{aligned}$$

since τ is a homomorphism. To show that ϕ is one-to-one, let $f \in V_\alpha$, $g \in V_\beta$ with $\phi(f) = \phi(g)$. Then for every $y \in X_i$, $i \in \{1, \dots, n\}$, $\phi(f)(y) = h_{\tau(\alpha)(i)} f h_i^{-1}(y) = h_{\tau(\beta)(i)} g h_i^{-1}(y) = \phi(g)(y)$, so that $h_{\tau(\alpha)(i)} f(u) = h_{\tau(\beta)(i)} g(u)$, for every $u \in h_i^{-1}(X_i) = X$. Moreover, $\tau(\alpha)(i) = \tau(b)(i)$, and so $f = g$.

Proof of Theorem 1.1. Suppose S is closed. Then Proposition 1.4 gives (i)–(iv), v(a) and (3) in Theorem 1.1. To show v(b), let $f \in S$, then $\operatorname{shift}\phi(f) \geq \operatorname{shift} f + \operatorname{shift}\xi(f)$, $\operatorname{def}\phi(f) = r \operatorname{def} f$, so $S \supseteq S(X, \operatorname{shift}\phi(f), r(\operatorname{def} f)) \supseteq S(X, \operatorname{shift} f + \operatorname{shift}\xi(f), r(\operatorname{def} f))$, and the statement v(b) holds. If $g \in V_\alpha$ such that $\tau(\alpha) \neq 1_{\{1,\dots,n\}}$ then $\operatorname{shift}\phi(g) = |X|$, and so $S \supseteq S(X, |X|, r(\operatorname{def} g))$, and (vi) holds. Conversely, given (i)-(vi), the mapping defined in (3) is an injective endomorphism provided it satisfies (6) and (7) of Proposition 1.4. This follows from (v) and (vi) of the statement of Theorem 1.1

Proof of Corollary 1.2. Follows from Corollary 2.26(iii).

Proof of Proposition 1.3. (i) Let $f \in S$ with $\xi(f) \in \mathcal{G}_U$ and let $g \in S$. We show that $\xi(g) \in \mathcal{G}_U$. Let $\operatorname{def} f = m$, $\operatorname{def} g = l$. Since $9lm^2 > lm$ there exists a one-to-one mapping p such that $f^{9lm} = pg^m$. Then $\operatorname{def} p = \operatorname{def} f^{9lm} - \operatorname{def} g^m = 9lm^2 - lm = lm(9m - 1) \geq 9lm$, $\operatorname{shift} p \leq \max\{\operatorname{shift} f, \operatorname{shift} g\}$, and so $p \in S$. But then $\xi(p)\xi(g)^m = \xi(f)^{9lm} \in \mathcal{G}_U$, so that $\xi(g) \in \mathcal{G}_U$.

(ii) It suffices to show that if $f, g \in S$ with $\operatorname{def} f \neq \operatorname{def} g$ and $\xi(f) = \xi(g)$, then $\xi(S) \cap \mathcal{G}_U \neq \emptyset$. Assume firstly that there exists $t \in S$ such that $tf = g$. Then $\xi(t)\xi(f) = \xi(g) = \xi(f)$, $\xi(t)$ is the identity on the range of $\xi(f)$ and since $\operatorname{def}\xi(f)$ is finite, $\xi(t) \in \mathcal{G}_U$. Therefore, it suffices to show that there exist $f', g', t' \in S$ with $\operatorname{def} f' \neq \operatorname{def} g'$, $t'f' = g'$ and $\xi(f') = \xi(g')$. Let $\operatorname{def} f = n$, $\operatorname{def} g = m$, where $m > n$, $f' = f^{8nm}$, $g' = g^{8nm}$. Let t' be a one-to-one mapping such that $t'f' = g'$. Then $\operatorname{def} t' = 8nm(m - n)$, $\operatorname{shift} t' \leq \max\{\operatorname{shift} f', \operatorname{shift} g'\}$, and so $t' \in S$, as required.

We conclude by presenting an example of an injective endomorphism with non-trivial ξ and τ.

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EXAMPLE. Let S be such that σ-def $S = \{2k : k \geq 1\}$ and $S \supseteq \{S(X, |X|, 2k) : k \geq 2\}$. Partition X into sets W and U with $|W| = |U| = |X|$. Let $n = 2$ and partition W into X_1 and X_2, $|X_1| = |X_2| = |X|$. Choose arbitrary bijections $h_i : X \to X_i$, $i = 1, 2$. Choose an infinite cycle h of U and let

$$\xi : S \to \mathcal{G}_U \text{ be such that } f \to h^m, \text{ if def } f = 2m.$$

Let $\tau : \Lambda \to \mathcal{G}_2$ be given by

$$\tau(\alpha) = \begin{cases} 1_{\{1,2\}}, & \text{if def } f = 0 \pmod 4, \ f \in V_\alpha, \\ (12) & \text{otherwise.} \end{cases}$$

Let $\phi : S \to S$ be defined by

$$\phi(f)(y) = \begin{cases} h_i f h_i^{-1}(y), & \text{if def } f = 0 \pmod 4, \ y \in X_i, \\ h_{i+1} f h_i^{-1}(y), & \text{if def } f \neq 0 \pmod 4, \ y \in X_i, \\ \xi(f)(y), & \text{if } y \in U, \end{cases}$$

where $f \in S$ and the addition of the indices is done modulo 2.

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