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PLURIHARMONIC SYMBOLS OF COMMUTING TOEPLITZ TYPE OPERATORS ON THE WEIGHTED BERGMAN SPACES

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ABSTRACT. A class of Toeplitz type operators acting on the weighted Bergman spaces of the unit ball in the n-dimensional complex space is considered and two pluriharmonic symbols of commuting Toeplitz type operators are completely characterized.

1. **Introduction and result.** Let B be the unit ball of the n-dimensional complex space $\mathbb{C}^n$ and V denote the Lebesgue volume measure on B normalized to have total mass 1. For $\alpha > -1$, define a measure dV_α on B by $dV_\alpha(z) = c_\alpha(1-|z|^2)^\alpha\, dV(z)$ where the constant c_α is a normalizing constant so that dV_α is a probability measure. The weighted Bergman space A^p_α $(1 \leq p < \infty)$ is the closed subspace of the usual Lebesgue space $L^p_\alpha = L^p(B, V_\alpha)$ consisting of holomorphic functions on B.

Corresponding to each $\sigma > -1$, we define an integral operator

$$P_\sigma(\psi)(z) = \lambda_\sigma \int_B \frac{(1-|w|^2)^\sigma}{(1-\langle z, w\rangle)^{n+1+\sigma}} \psi(w)\, dV(w) \quad (z \in B)$$

where $1/\lambda_\sigma = \int_B (1-|w|^2)^\sigma\, dV(w)$ and $\langle z, w\rangle$ is the ordinary Hermitian inner product for points $z, w \in \mathbb{C}^n$. It is known [2, Theorem 1] that for $p \geq 1$ and $\alpha, \sigma > -1$, P_σ is a bounded operator from L^p_α onto A^p_α if and only if $p(1+\sigma) > 1+\alpha$. Moreover, if $p(1+\sigma) > 1+\alpha$, then P_σ has the following reproducing properties:

$$P_\sigma f = f \quad \text{and} \quad P_\sigma \overline{f} = \overline{f}(0) \tag{1}$$

for every functions $f \in A^p_\alpha$. See [2] for more informations on P_σ and related facts.

Let $p \geq 1$ and $\alpha, \sigma > -1$ be such that $p(1+\sigma) > 1+\alpha$. For $u \in L^\infty$, the Toeplitz type operator T^σ_u with symbol u is the linear operator acting on A^p_α defined by

$$T^\sigma_u(f) = P_\sigma(uf)$$

for functions $f \in A^p_\alpha$. Then, T^σ_u is a bounded operator on A^p_α.

In this paper, we study the commuting problem of Toeplitz type operators, acting on the weighted Bergman spaces, with pluriharmonic symbols. A function $u \in C^2(B)$ is said

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to be *pluriharmonic* if its restriction to an arbitrary complex line that intersects the ball is harmonic as a function of single complex variable. As is well known [6, Theorem 4.4.9], every pluriharmonic function on B can be expressed, uniquely up to an additive constant, as the sum of a holomorphic function and an antiholomorphic function.

On the unweighted L^2-Bergman space, a characterization problem for pluriharmonic symbols of commuting Toeplitz operators induced by the Bergman projection was first solved by Axler and Čučković [1] on the disk. Later, on the setting of the ball, some partial results on this problem were obtained by Choe and Lee [3]. In [9], Zheng solved the problem completely by using the $\mathcal{M}$-harmonic function theory. Also, the same problem was considered for certain Toeplitz type operator. In a recent paper, the author [5] solved the problem for the certain Toeplitz type operator acting on the unweighted L^1-Bergman space of the ball.

In the present paper, we consider the same characterization problem for a class of Toeplitz type operators acting on the weighted Bergman spaces and completely characterize two pluriharmonic symbols for which the associate Toeplitz type operators commute. The following is the main theorem of the paper.

THEOREM 1. *Let $p \geq 1$ and $\alpha, \sigma > -1$ be such that $p(1+\sigma) > 1+\alpha$ and $p(n+1+\sigma) = 2(n+1+\alpha)$. Suppose that u and v are two bounded pluriharmonic symbols on B. Then $T_u^\sigma T_v^\sigma = T_v^\sigma T_u^\sigma$ on A_α^p if and only if one of the following properties holds:*

(a) u and v are both holomorphic on B.

(b) u and v are both antiholomorphic on B.

(c) There exist constants c and d, not both 0, such that $cu + dv$ is constant on B.

In the unweighted case of $p = 2$, $\alpha = 0$ and $\sigma = 0$, the above theorem was obtained in [1] on the disk and then, on the ball, in [3] partially and [9] completely. Recently, a certain weighted case was considered. In [5], the author obtained Theorem 1 in the weighted case $p = 1$, $\alpha = 0$ and $\sigma = n + 1$.

In Section 2, we collect some basic facts and preliminary results on $\mathcal{M}$-harmonic and pluriharmonic functions needed in the proof. In Section 3, we prove Theorem 1 and give an application (see Corollary 9).

2. **Preliminaries.**

For $z, w \in B, z \neq 0$, define

$$\varphi_z(w) = \frac{z - |z|^{-2}\langle w, z\rangle z - \sqrt{1-|z|^2}(w - |z|^{-2}\langle w, z\rangle z)}{1 - \langle w, z\rangle}$$

and $\varphi_0(w) = -w$. Then $\varphi_z \in \mathcal{A}$, the group of all automorphisms (= biholomorphic self-maps) of B and φ_z is an involution. That is, $\varphi_z \circ \varphi_z$ is the identity on B. Furthermore, each $\varphi \in \mathcal{A}$ has a unique representation $\varphi = \varphi_z \circ U$ for some $z \in B$ and $U \in \mathcal{U}$, the group of all unitary operators on $\mathbb{C}^n$. For $u \in C^2(B)$ and $z \in B$, we define

$$(\tilde{\Delta} u)(z) = \Delta(u \circ \varphi_z)(0)$$

where Δ denotes the ordinary Laplacian. The operator $\tilde{\Delta}$ is called the *invariant Laplacian* because it commutes with automorphisms of B in the sense that $\tilde{\Delta}(u \circ \varphi) = (\tilde{\Delta} u) \circ \varphi$ for

$\varphi \in \mathcal{A}$. We say that a function $u \in C^2(B)$ is $\mathcal{M}$-harmonic on B if it is annihilated on B by the invariant Laplacian $\tilde{\Delta}$. $\mathcal{M}$-harmonic functions are characterized by a certain mean value property. To be more precise, for a continuous function F on B, let $\mathcal{R}(F)$ denote the radialization of F defined by the formula

$$\mathcal{R}(F)(z) = \int_{\mathcal{U}} F(Uz)\, dU \quad (z \in B)$$

where dU denotes the Haar measure on $\mathcal{U}$.

It is not hard to see that if F is $\mathcal{M}$-harmonic on B and $\varphi \in \mathcal{A}$, then the radialization $\mathcal{R}(F \circ \varphi)$ is constant on B and hence extends to a continuous function on $\overline{B}$. The following proposition shows that this property, together with a certain mean value property, conversely characterizes the $\mathcal{M}$-harmonicity. The following characterization of $\mathcal{M}$-harmonicity given by the weighted area version of invariant mean value property will be useful in the proof of Theorem 1.

PROPOSITION 2. *Let $t > -1$ and suppose $F \in L^1_t$ is a continuous function on B. Then F is $\mathcal{M}$-harmonic on B if and only if*

$$(F \circ \varphi)(0) = \lambda_t \int_B (F \circ \varphi)(w)(1 - |w|^2)^t\, dV(w)$$

and $\mathcal{R}(F \circ \varphi)$ has a continuous extension on $\overline{B}$ for every $\varphi \in \mathcal{A}$.

PROOF. See Proposition 3 of [5].

To characterize the symbols, we also need a recent result of D. Zheng [9] on $\mathcal{M}$-harmonic products (the original statement in [9, Theorem 2] is in a slightly different form).

LEMMA 3. *Let $u = f + \overline{g}$ and $v = h + \overline{k}$ be two bounded pluriharmonic functions on B. If $f\overline{k} - h\overline{g}$ is $\mathcal{M}$-harmonic on B, then u and v are all holomorphic or antiholomorphic or there exist constants c and d, not both 0, such that $cu + dv$ is constant on B.*

Let $1 \leq s < \infty$. The Hardy space H^s is the space of all functions f holomorphic on B for which

$$\|f\|_s = \Big(\sup_{0<r<1} \int_S |f(r\zeta)|^s\, d\sigma(\zeta) \Big)^{1/s} < \infty.$$

Here the measure σ denotes the rotation invariant probability measure on S, the boundary of B. It is well known (see [6, Theorem 5.6.4]) that if $f \in H^s$, then $f^*(\zeta) = \lim_{r\to 1} f(r\zeta)$ exists for $[\sigma]$ a.e. $\zeta \in S$. The space BMOA consists of all $f \in H^2$ whose boundary functions f^* are functions of bounded mean oscillations with respect to the nonisotropic metric on S that corresponds to the Korányi approach regions. See [4] for details. Note that BMOA functions are closed under composition with automorphisms and BMOA $\subset H^s$ for all s. Also, it turns out (see, for example, [7]) that $f \in$ BMOA if and only if

$$\sup_{a \in B} \|f \circ \varphi_a - f(a)\|_2 < \infty.$$

Before turning to the proof, we prove several simple results which will be used in the proof of Theorem 1. The following proposition was obtained in [8, Proposition 3] on the disk.

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PROPOSITION 4. *Let $f+\bar{g}$ be a bounded pluriharmonic function on B. Then f and g are all in* BMOA.

PROOF. Put $u = f+\bar{g}$ for notational simplicity. By the mean value property for holomorphic functions, we see that

$$\int_S \big(f(r\zeta)-f(0)\big)\big(g(r\zeta)-g(0)\big)\,d\sigma(\zeta)=0$$

for all $0<r<1$. It follows from a simple manipulation that

$$\begin{aligned}\int_S |f(r\zeta)-f(0)|^2\,d\sigma(\zeta)+\int_S |g(r\zeta)-f(0)|^2\,d\sigma(\zeta) &= \int_S |u(r\zeta)-u(0)|^2\,d\sigma(\zeta)\\ &\le 4\|u\|_\infty^2 \quad (0<r<1)\end{aligned}$$

and hence $\|f-f(0)\|_2^2+\|g-g(0)\|_2^2 \le 4\|u\|_\infty^2$. Now, for $a\in B$, replace u by $u\circ\varphi_a$ to get

$$\|f\circ\varphi_a - f(a)\|_2^2 + \|g\circ\varphi_a - g(a)\|_2^2 \le 4\|u\|_\infty^2.$$

Therefore, taking the supremum over all $a\in B$, we have $f,g\in$ BMOA, as desired. The proof is complete. ∎

It is not hard to see that by an integration in polar coordinates, $H^s\subset A_t^s$ for all $s\ge 1$ and $t>-1$. Hence, the following corollary is an immediate consequence of Proposition 4 because BMOA $\subset H^s$ for all $s<\infty$.

COROLLARY 5. *Let $f+\bar{g}$ be a bounded pluriharmonic function on B. Then f and g are all in A_t^s for all $s\ge 1$ and $t>-1$.*

It is shown in [3] that if $F,G\in H^2$, then the radialization $\mathcal{R}(F\bar{G})$ has a continuous extension on $\bar{B}$. Since BMOA is invariant under composition with automorphisms, we have the following.

LEMMA 6. *Let $F,G\in$ BMOA. Then $\mathcal{R}(F\circ\varphi\overline{G\circ\varphi})$ extends to a continuous function on $\bar{B}$ for every $\varphi\in\mathcal{A}$.*

3. **Proof.** In this section we will prove Theorem 1 and give a simple application. We first need some elementary properties of canonical automorphisms. The real Jacobian $J_R\varphi_z$ of φ_z is given by

$$J_R\varphi_z(w)=\left(\frac{1-|z|^2}{|1-\langle w,z\rangle|^2}\right)^{n+1}\quad (w\in B) \tag{2}$$

and the identity

$$1-\langle\varphi_z(a),\varphi_z(b)\rangle=\frac{(1-|z|^2)(1-\langle a,b\rangle)}{(1-\langle a,z\rangle)(1-\langle z,b\rangle)} \tag{3}$$

holds for every $a,b\in B$. See [6, Chapter 2] for details. For $\alpha>-1$ and $a\in B$, we let

$$k_a^\alpha(z)=\left(\frac{\sqrt{1-|a|^2}}{1-\langle z,a\rangle}\right)^{n+1+\alpha}\quad (z\in B)$$

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for notational simplicity. Then, by (2) and (3), we have a useful change-of-variable formula:

$$\int_B F\,dV_\alpha = \int_B F\circ\varphi_a |k_a^\alpha|^2\,dV_\alpha \quad (a\in B) \tag{4}$$

for all measurable F on B whenever the integrals make sense.

Let $p\geq 1$ and $\alpha>-1$. For $a\in B$, let U_a denote the linear operator on L_α^p defined by

$$U_a f = (f\circ\varphi_a)(k_a^\alpha)^{2/p}.$$

Then, by the change-of-variable formula (4), one can easily see that

$$\int_B |U_a f|^p\,dV_\alpha = \int_B |f\circ\varphi_a|^p |k_a^\alpha|^2\,dV_\alpha = \int_B |f|^p\,dV_\alpha$$

for every $f\in L_\alpha^p$. Hence U_a is an isometry of L_α^p into L_α^p. Using (3), one can easily show that $k_a^\alpha(\varphi_a)k_a^\alpha = 1$ on B. It follows that U_aU_a is the identity operator on L_α^p and hence U_a takes A_α^p onto A_α^p.

Before proving Theorem 1, we have a couple of lemmas. The following lemma says that P_σ and U_a commute on L_α^p in certain cases.

LEMMA 7. *Let $p\geq 1$ and $\alpha,\sigma>-1$ be such that $p(1+\sigma)>1+\alpha$ and $p(n+1+\sigma)=2(n+1+\alpha)$. For $a\in B$, we have $P_\sigma U_a = U_a P_\sigma$ on L_α^p.*

PROOF. Let $f\in L_\alpha^p$ and $z\in B$. By the change-of-variable formula (4) and simple manipulations using (3), one can see

$$\begin{aligned}
P_\sigma U_a f(z) &= \lambda_\sigma \int_B \frac{(1-|w|^2)^\sigma f\big(\varphi_a(w)\big)\big(k_a^\alpha(w)\big)^{2/p}}{(1-\langle z,w\rangle)^{n+1+\sigma}}\,dV(w)\\
&= \lambda_\sigma \int_B \frac{(1-|\varphi_a(w)|^2)^\sigma f(w)\Big(k_a^\alpha\big(\varphi_a(w)\big)\Big)^{2/p}}{\big(1-\langle z,\varphi_a(w)\rangle\big)^{n+1+\sigma}}\left(\frac{1-|a|^2}{|1-\langle w,a\rangle|^2}\right)^{n+1} dV(w)\\
&= \frac{(1-|a|^2)^{(n+1+\sigma)-\frac{n+1+\alpha}{p}}}{(1-\langle z,a\rangle)^{n+1+\sigma}}\\
&\qquad\times \lambda_\sigma\int_B \frac{(1-|w|^2)^\sigma f(w)(1-\langle w,a\rangle)^{\frac{2(n+1+\alpha)}{p}-(n+1+\sigma)}}{\big(1-\langle\varphi_a(z),w\rangle\big)^{n+1+\sigma}}\,dV(w).
\end{aligned}$$

On the other hand, since $p(n+1+\sigma)=2(n+1+\alpha)$ by the assumption, the last expression of the above turns into

$$\left(\frac{\sqrt{1-|a|^2}}{1-\langle z,a\rangle}\right)^{\frac{2(n+1+\alpha)}{p}}\lambda_\sigma\int_B \frac{(1-|w|^2)^\sigma}{\big(1-\langle\varphi_a(z),w\rangle\big)^{n+1+\sigma}} f(w)\,dV(w) = \big(k_a^\alpha(z)\big)^{2/p} P_\sigma f\big(\varphi_a(z)\big),$$

which is exactly $U_aP_\sigma f(z)$. Hence $P_\sigma U_a = U_a P_\sigma$ on L_α^p, as desired. The proof is complete. ∎

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LEMMA 8. *Let $p \geq 1$ and $\alpha, \sigma > -1$ be such that $p(1+\sigma) > 1+\alpha$ and $p(n+1+\sigma) = 2(n+1+\alpha)$. For $a \in B$ and $u \in L^\infty$, we have $U_a T_u^\sigma U_a = T_{u\circ\varphi_a}^\sigma$ on A_α^p.*

PROOF. Let $f \in A_\alpha^p$. By Lemma 7, one obtains

$$\begin{aligned} T_{u\circ\varphi_a}^\sigma U_a f &= T_{u\circ\varphi_a}^\sigma [(f\circ\varphi_a)(k_a^\alpha)^{2/p}] \\ &= P_\sigma[(u\circ\varphi_a)(f\circ\varphi_a)(k_a^\alpha)^{2/p}] \\ &= P_\sigma U_a(uf) \\ &= U_a P_\sigma(uf) \\ &= U_a T_u^\sigma f. \end{aligned}$$

Thus $T_{u\circ\varphi_a}^\sigma U_a = U_a T_u^\sigma$ on A_α^p. Now, use the fact $U_a U_a$ is the identity operator on A_α^p to get the desired result. This completes the proof. ∎

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1. We begin with easy direction. First suppose that (a) holds, so that u and v are holomorphic on B. Then, by (1), we can see that T_u^σ and T_v^σ are, respectively, the operators on A_α^p of multiplication by u and v. Thus $T_u^\sigma T_v^\sigma = T_v^\sigma T_u^\sigma$ on A_α^p. Now assume (b), so that $\bar{u}$ and $\bar{v}$ are holomorphic on B. Using the explicit formula for P_σ, an application of Fubini's theorem and (1), one can see that $T_u^\sigma T_v^\sigma f = P_\sigma(uvf)$ for every bounded functions f in A_α^p. Note that the set of all bounded functions in A_α^p forms a dense subset of A_α^p. It follows from the continuity that T_u^σ and T_v^σ commute, as desired. Finally, suppose (c) holds and assume $c \neq 0$ (the other case is in a similar fashion). Then $u = c_1 v + c_2$ for some constants c_1 and c_2, which implies $T_u^\sigma = c_1 T_v^\sigma + c_2$, so that $T_u^\sigma T_v^\sigma = c_1 T_v^\sigma T_v^\sigma + c_2 T_v^\sigma = T_v^\sigma T_u^\sigma$ on A_α^p. Hence T_v^σ and T_u^σ commute on A_α^p.

Now we prove the converse implication. Write $u = f + \bar{g}$ and $v = h + \bar{k}$ for some holomorphic f, g, h and k. Then, by Corollary 5, the functions f, g, h and k are all in A_α^p. Moreover, fh, $h\bar{g}$ and $f\bar{k}$ are all in L_α^p by Corollary 5 again. Let 1 denote the constant function 1 on B. Then, by the reproducing properties (1), we have

$$\begin{aligned} T_u^\sigma T_v^\sigma 1 &= T_u^\sigma(P_\sigma v) \\ &= T_u^\sigma\big(h + \bar{k}(0)\big) \\ &= P_\sigma\big(fh + \bar{k}(0)f + h\bar{g} + \bar{g}\bar{k}(0)\big) \\ &= fh + \bar{k}(0)f + P_\sigma(h\bar{g}) + \bar{g}(0)\bar{k}(0). \end{aligned}$$

Note that $\int_B F\, dV_\alpha = F(0)$ for holomorphic functions $F \in L_\alpha^1$. It follows that

(5)
$$\begin{aligned} \int_B (T_u^\sigma T_v^\sigma 1)\, dV_\alpha &= (T_u^\sigma T_v^\sigma 1)(0) \\ &= f(0)h(0) + f(0)\bar{k}(0) + \bar{g}(0)\bar{k}(0) + P_\sigma(h\bar{g})(0) \\ &= f(0)h(0) + f(0)\bar{k}(0) + \bar{g}(0)\bar{k}(0) + \lambda_\sigma \int_B h(w)\bar{g}(w)(1-|w|^2)^\sigma\, dV(w). \end{aligned}$$

Similarly,

$$\int_B (T_v^\sigma T_u^\sigma 1)\, dV_\alpha = f(0)h(0) + h(0)\bar{g}(0) + \bar{g}(0)\bar{k}(0) + \lambda_\sigma \int_B f(w)\bar{k}(w)(1-|w|^2)^\sigma\, dV(w). \tag{6}$$

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Since $T_u^\sigma T_v^\sigma = T_v^\sigma T_u^\sigma$ by the assumption, letting $\Lambda = f\overline{k} - h\overline{g}$, we have by (5) and (6) that

$$\lambda_\sigma \int_B \Lambda(w)(1-|w|^2)^\sigma \, dV(w) = \Lambda(0). \tag{7}$$

Let $a \in B$. Multiplying both sides of the equation $T_u^\sigma T_v^\sigma = T_v^\sigma T_u^\sigma$ by U_a on the left and by U_a on the right, we obtain since $U_a U_a$ is the identity operator

$$U_a T_u^\sigma U_a U_a T_v^\sigma U_a = U_a T_v^\sigma U_a U_a T_u^\sigma U_a$$

and therefore by Lemma 8

$$T^\sigma_{u\circ\varphi_a} T^\sigma_{v\circ\varphi_a} = T^\sigma_{v\circ\varphi_a} T^\sigma_{u\circ\varphi_a}. \tag{8}$$

Equation (7) was derived under the assumption that $T_u^\sigma T_v^\sigma = T_v^\sigma T_u^\sigma$. Thus (8) says that (7) remains valid with $\Lambda \circ \varphi_a$ in place of Λ. That is,

$$\lambda_\sigma \int_B (\Lambda \circ \varphi_a)(w)(1-|w|^2)^\sigma \, dV(w) = \Lambda(a) \tag{9}$$

for any $a \in B$. Let $\varphi \in \mathcal{A}$ and suppose φ has the representation $\varphi = \varphi_a \circ U$ for some $a \in B$ and $U \in \mathcal{U}$. Then, by the rotation-invariance of the measure dV and (9), we have the weighted area version of the invariant mean value property for Λ:

$$\begin{aligned}\lambda_\sigma \int_B (\Lambda \circ \varphi)(w)(1-|w|^2)^\sigma \, dV(w) &= \lambda_\sigma \int_B (\Lambda \circ \varphi_a)(w)(1-|w|^2)^\sigma \, dV(w) \\ &= \Lambda(a) \\ &= (\Lambda \circ \varphi)(0)\end{aligned}$$

because $\varphi(0) = a$. On the other hand, since f, g, h and k are all in BMOA by Proposition 4, the function $\mathcal{R}(\Lambda \circ \varphi)$ has a continuous extension on $\overline{B}$ for any $\varphi \in \mathcal{A}$ by Lemma 6. Note from Corollary 5 that $\Lambda \in L^1_\sigma$. It follows from Proposition 2 with $t = \sigma$ that $\Lambda = f\overline{k} - h\overline{g}$ is $\mathcal{M}$-harmonic on B. Now, the characterization follows from Lemma 3. This completes the proof. ■

We conclude this paper with a simple application. We note that pluriharmonic functions are closed under complex conjugation.

COROLLARY 9. *Let $p \geq 1$ and $\alpha, \sigma > -1$ be such that $p(1+\sigma) > 1+\alpha$ and $p(n+1+\sigma) = 2(n+1+\alpha)$. Suppose u is a bounded pluriharmonic symbol on B. Then $T_u^\sigma T_{\overline{u}}^\sigma = T_{\overline{u}}^\sigma T_u^\sigma$ on A^p_α if and only if the image of B under u lies on some line in $\mathbb{C}$.*

PROOF. If $u(B)$ lies on some line in $\mathbb{C}$, a rotation and a translation show that there exist constants c ($|c| = 1$) and d such that $cu + d$ is real valued on B. It follows that

$$T_{\overline{u}}^\sigma = \frac{1}{\overline{c}}(T^\sigma_{\overline{cu+d}} - \overline{d}) = \frac{1}{\overline{c}}(T^\sigma_{cu+d} - \overline{d}).$$

Now, since $T_u^\sigma = \frac{1}{c}(T^\sigma_{cu+d} - d)$, we see that T_u^σ and $T_{\overline{u}}^\sigma$ commute. Conversely, assume $T_u^\sigma T_{\overline{u}}^\sigma = T_{\overline{u}}^\sigma T_u^\sigma$ on A^p_α. Then, by Theorem 1, u and $\overline{u}$ are holomorphic on B or a nontrivial linear combination of u and $\overline{u}$ is constant on B. The first case implies u is constant on B, so we are done. Also, a simple manipulation shows that the latter case implies $u(B)$ lies on some line in $\mathbb{C}$. This completes the proof. ■

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