AFFINE SEMIGROUPS OVER AN ARBITRARY FIELD†

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Let $\mathscr{L}V$ denote the algebra of all linear transformations on an n-dimensional vector space V over a field Φ. A subsemigroup S of the multiplicative semigroup of $\mathscr{L}V$ will be said to be an *affine semigroup over* Φ if S is a linear variety, i.e., a translate of a linear subspace of $\mathscr{L}V$.

This concept in a somewhat different form was introduced and studied by Haskell Cohen and H. S. Collins [1]. In an appendix we give their definition and outline a method of describing possibly infinite dimensional affine semigroups in terms of algebras and supplemented algebras.

Except in the appendix, V and hence also $\mathscr{L}V$ will be supposed finite-dimensional.

In §1 we show that some power of every element of an affine semigroup S lies in a subgroup of S and that S always contains a completely simple minimal ideal K. (For definitions see below.) We then obtain a decomposition of S into a group, an algebra, and two vector spaces.

The minimal ideal K of an affine semigroup is not in general a linear variety, as was observed by Cohen and Collins [1]. When K is not a linear variety, one turns naturally to $M(K)$, the smallest linear variety containing K. We show in §2 that $M(K)^n = K$ for any integer n exceeding the dimension of $M(K)$.

If $K \neq M(K)$ and every element of K is idempotent, then we are able to find a subsemigroup of $M(K)$ isomorphic to the example of Cohen and Collins (see 2.1). In this case we also show that $M(K)^2 = K$. The requirement that every element of K be idempotent is of some interest since this is always true for affine semigroups on locally convex linear topological spaces which are generated by compact convex subsemigroups (see Theorem 3 of [1] and 2.10 below).

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0. Preliminaries.

(0.1) Let V and V' be vector spaces over an arbitrary field Φ. A mapping $\phi : V \to V'$ is said to be *affine* if

$$(x+\alpha y-\alpha z)\phi = x\phi+\alpha(y\phi)-\alpha(z\phi)$$

for all x, y, z in V and all α in Φ. The more usual definition of an affine mapping is obtained by taking $x = z$ in that above. For fields of characteristic different from two the concepts coincide. Note, however, that if Φ is the field of order two, then any mapping on V satisfies the condition

$$(\alpha x+(1-\alpha)y)\phi = \alpha(x\phi)+(1-\alpha)(y\phi),$$

since α can only be 0 or 1. On the other hand, a mapping on V will not in general satisfy

$$(x+y+z)\phi = x\phi+y\phi+z\phi.$$

(0.2) A subset M of a vector space V over the field Φ is said to be a *linear variety over* Φ if $M = x+N$, where $x \in V$ and N is a linear subspace of V. A linear variety M of V may also be

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characterized as a subset M of V such that $x, y, z \in M$ implies $x+\alpha y-\alpha z \in M$ for all $\alpha \in \Phi$.

If M is a linear variety, then clearly $M-x = M-y$ for all $x, y \in M$. One is therefore justified in defining the *dimension* of M to be the dimension, i.e., the cardinality of a basis, of the subspace $M-y$ for $y \in M$.

(0.3) The intersection of any number of linear varieties, if not empty, is a linear variety. Let A be a subset of the vector space V. We denote by $M(A)$ the intersection of all linear varieties containing A and call this the *linear variety generated by* A. It may be easily verified that $M(A)$ consists of all finite sums $\sum \alpha_i a_i$, where $\sum \alpha_i = 1$ and $\{a_i\} \subset A$.

(0.4) $\mathscr{L}V$ will denote the algebra of all linear transformations on an n-dimensional vector space V over a field Φ.

We shall say that a subset S of $\mathscr{L}V$ is an *affine semigroup* if S is both a subsemigroup of the multiplicative semigroup of $\mathscr{L}V$ and a linear variety in the vector space $\mathscr{L}V$.

One easily verifies that the left and right translations $x \rightarrow ax$ and $x \rightarrow xa$ of an affine semigroup S are affine mappings for each a in S.

(0.5) Let S be a semigroup. A non-empty subset I of S is an *ideal* of S if $SI \cup IS$ is contained in S. K is said to be a *minimal ideal* or *kernel* of S if K is an ideal of S and no other ideal of S is properly contained in K. An element z [u] of S is a *zero* [an *identity*] of S if $zx = xz = z$ [$ux = xu = x$] for all x in S. S is *simple* if S contains no proper ideals. An idempotent e of S is *primitive* if e is the only idempotent in eSe. A simple semigroup which contains a primitive idempotent is called *completely simple.* The structure of such semigroups is well known (see [**4**, Chapter 2]).

(0.6) If S is an affine semigroup, by *ideal, zero*, etc., of S is meant, of course, ideal, zero, etc., of the multiplicative semigroup of S. On the other hand, if A is an algebra, and is being considered as such, then by *ideal* of A is meant, as usual, a subalgebra of A which is a ring ideal.

(0.7) By *isomorphism* [*homomorphism*] of an affine semigroup we mean a semigroup isomorphism [homomorphism] which is simultaneously an affine mapping.

(0.8) A linear combination $\sum \alpha_i x_i$ will be called an *affine combination* if $\sum \alpha_i = 1$.

(0.9) LEMMA. *Let M and M' be linear varieties over the same field Φ and let $x \rightarrow x'$ be an affine mapping from M into M'. Then*

$$\left(\sum \alpha_i x_i\right)' = \sum \alpha_i x_i'$$

for all affine combinations $\sum \alpha_i x_i$ of elements of M.

Proof. We observe that

$$\sum_{i=1}^{n} \alpha_i x_i = u + \alpha_n x_n - \alpha_n x_{n-1},$$

where $u = \alpha_1 x_1 + \dots + \alpha_{n-2} x_{n-2} + (\alpha_{n-1} + \alpha_n) x_{n-1}$.

Since M is a linear variety, u is an element of M. The proof now follows by induction on n.

(0.10) COROLLARY. *If $\sum \alpha_i x_i$ is an affine combination of elements of an affine semigroup S, then*

$$(\sum \alpha_i x_i) y = \sum \alpha_i x_i y$$

and

$$y \sum \alpha_i x_i = \sum \alpha_i y x_i$$

for all y in S.

1. Let a be an element of $\mathscr{L}V$. By the rank $\rho(a)$ of a is meant the dimension of the range of a.

If e is an idempotent of the semigroup S, we shall let $H(e, S) = H(e)$ denote the maximal subgroup of S containing e.

(1.1) THEOREM. *If a is an element of the affine semigroup S such that $\rho(a) = \rho(a^2)$, then a lies in a subgroup of S.*

Proof. If $\rho(a) = \rho(a^2)$, then it is well known (see e.g. [6, p. 273]) that a lies in some subgroup $H(e, \mathscr{L}V)$ of $\mathscr{L}V$ for some idempotent e in $\mathscr{L}V$. We shall show that e is in fact in S. Let $M = M(H(e, \mathscr{L}V))$ (see 0.3) and let $S' = S \cap M$. Since M and S are affine semigroups, S' is also. S' is not empty since it contains a. Since all elements of S' are linear combinations of elements of $H(e, \mathscr{L}V)$ and a belongs to $H(e, \mathscr{L}V)$, the mapping $x \to xa$ of S' into $S'a$ is 1–1. The fact that S' is a semigroup implies that $S'a \subseteq S'$ and, since 1–1 linear transformations preserve dimension, we must have $S'a = S'$. Similarly, $aS' = S'$. Now, since $ax = a$ has the unique solution $x = e$ in M, we have $e \in S' \subseteq S$. Likewise, the inverse of a in $H(e, \mathscr{L}V)$ is contained in S and hence $a \in H(e, S)$. This completes the proof.

(1.2) COROLLARY. *Every affine semigroup contains an idempotent.*

(1.3) COROLLARY. *Some power of every element of an affine semigroup S lies in a subgroup of S.*

Proof. Clearly, for each a in S, $\rho(a^n) = \rho(a^{n+1})$ for some positive integer n. Then $\rho(a^{2n}) = \rho(a^n)$ and therefore, by 1.1, a^n lies in a subgroup of S.

After Drazin [7] we shall say that a semigroup S is *pseudo-invertible* if for each x in S there is an element $\bar{x}$ in S possessing the following properties:

(i) $\bar{x}x = x\bar{x}$,

(ii) $x^n = x^{n+1}\bar{x}$ for some positive integer n,

(iii) $\bar{x} = \bar{x}^2 x$.

Munn in [8] showed that a semigroup S is pseudo-invertible if and only if some power of every element of S lies in a subgroup of S.

In [9] the author proves that a pseudo-invertible semigroup S of matrices has a completely simple minimal ideal consisting precisely of those elements of S of least possible rank. This fact together with (1.3) gives immediately the following theorem.

(1.4) THEOREM. *If S is an affine semigroup, then the elements of minimal rank in S form a completely simple minimal ideal.*

Primarily for notational purposes we shall adopt the language of representation theory: Let S be an affine semigroup over a field Φ. By a *representation* Γ *of* S *of degree* n *over* Φ is meant an *affine* homomorphism of S into the multiplicative semigroup of the full matrix algebra of degree n over Φ. If Γ is an isomorphism we shall say that Γ is *faithful*.

We shall now obtain a decomposition of S similar to the two-sided Peirce decomposition in the theory of algebras. This decomposition will in fact be the translation of the Peirce decomposition of $S-e$ in the enveloping algebra $[S]$ (see appendix) with respect to an idempotent e in S.

(1.5) Let e be any fixed idempotent in S, and define, for each x in S,

$$\left.\begin{aligned} x_1 &= exe, \\ x_2 &= xe-exe+e, \\ x_3 &= ex-exe+e, \\ x_4 &= x-ex-xe+exe+e. \end{aligned}\right\} \tag{1}$$

Now, if Γ is any representation of S, since inner automorphisms are affine, by choosing a suitable basis we may assume that

$$\Gamma(e) = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

where I_k is the identity matrix of degree k. We now partition all matrices $\Gamma(x)$ $(x \in S)$ by $n = k+(n-k)$ as in (2) and write

$$\Gamma(x) = \begin{pmatrix} \Gamma_1(x) & \Gamma_3(x) \\ \Gamma_2(x) & \Gamma_4(x) \end{pmatrix}. \tag{3}$$

One easily obtains from (1), (2) and (3) that

$$\left.\begin{aligned} \Gamma(x_1) &= \begin{pmatrix} \Gamma_1(x) & 0 \\ 0 & 0 \end{pmatrix}, & \Gamma(x_3) &= \begin{pmatrix} I_k & \Gamma_3(x) \\ 0 & 0 \end{pmatrix}, \\ \Gamma(x_2) &= \begin{pmatrix} I_k & 0 \\ \Gamma_2(x) & 0 \end{pmatrix}, & \Gamma(x_4) &= \begin{pmatrix} I_k & 0 \\ 0 & \Gamma_4(x) \end{pmatrix}. \end{aligned}\right\} \tag{4}$$

(1.6) LEMMA. *If* Γ *is the representation in* (1.5) *of the affine semigroup* S, *where* e *is any idempotent in* S, *then*

(i) *if* $x, y, z, w \in S$, *there exists* $u \in S$ *such that*

$$\Gamma(u) = \begin{pmatrix} \Gamma_1(x) & \Gamma_3(z) \\ \Gamma_2(y) & \Gamma_4(w) \end{pmatrix}; \tag{5}$$

(ii) $\Gamma_2(S)$ *and* $\Gamma_3(S)$ *are vector spaces;*

(iii) $\Gamma_4(S)$ *is an algebra.*

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Proof. Using the notation of (1), we let

$$u = x_1 + y_2 + z_3 + w_4 - 3e.$$

It is easily seen from (4) that (5) holds. $\Gamma_2(S)$ and $\Gamma_3(S)$ are obviously linear varieties and are vector spaces since $0 = \Gamma_i(e) \in \Gamma_i(S)$ for $i = 1, 2$, where 0 denotes the zero matrix of the appropriate dimensions. It is likewise clear that $\Gamma_4(S)$ is a subspace. Therefore to prove (iii) it suffices to prove that $\Gamma_4(S)$ is a semigroup. But this is clear since from (1) and (4) we see that

$$\Gamma_4(x)\Gamma_4(y) = \Gamma_4(x_4 y_4) \in \Gamma_4(S)$$

for all x, y in S.

(1.7) By an *affine group* is meant an affine semigroup which is a group. Affine groups have a particularly simple representation:

Let G be an affine group of non-singular matrices of degree n. One may easily verify that the vector space $G - I_n$, where I_n denotes the identity matrix, is a radical algebra since G is a group. Since it is finite-dimensional it is a nilpotent algebra [5, p. 390] and hence, by [5, p. 202], the elements of $G - I_n$ may be simultaneously triangulated with only zeros on the main diagonal. It follows that for a proper choice of basis all the elements of G will be triangular with ones on the main diagonal.

The reader will also note that the correspondence between G and $G - I_n$ is a one-one correspondence between affine groups and finite dimensional nilpotent algebras.

(1.8) THEOREM. *If Γ is a faithful representation of the affine semigroup S and $\Gamma(x)$ $(x \in S)$ is partitioned as above with the idempotent e being taken from the kernel K of S, then*

(i) *$\Gamma_1(S)$ is an affine group of non-singular matrices of degree $k = \rho(e)$;*

(ii) *$x \in K$ if and only if*

$$\Gamma_4(x) = \Gamma_2(x)\Gamma_1(x)^{-1}\Gamma_3(x). \tag{6}$$

Proof. (i) follows from the well-known fact [4, Ex. 14, p. 84] that for all idempotents f in K we have $H(f, S) = fSf$. By (1.4), the kernel of an affine semigroup consists of the elements of minimal rank. Since we know by (i) that $\Gamma_1(x)$ is non-singular for all x in S, $\Gamma(x)$ has rank k if and only if the last $n-k$ columns are linear combinations of the first k, i.e. if and only if

$$\Gamma_3(x) = \Gamma_1(x)A, \quad \Gamma_4(x) = \Gamma_2(x)A \tag{7}$$

for some $k \times (n-k)$ matrix $A = A(x)$. Now if $x \in K$, $\Gamma(x)$ has rank k and therefore (7) holds. (6) is immediate from (7). Conversely, if (6) holds, (7) holds with $A = \Gamma_1(x)^{-1}\Gamma_3(x)$; hence $\Gamma(x)$ has rank k and x is therefore in K.

(1.9) The *direct product* of two affine semigroups S and T is the set of all ordered pairs (s, t) with coordinate-wise multiplication and the natural affine structure induced by the direct sum of the containing vector spaces. It will be denoted by $S \times T$.

(1.10) COROLLARY. *If the kernel of the affine semigroup S is a group, then S is isomorphic to $A \times G$, where A is an algebra and G is an affine semigroup which is isomorphic to the kernel K of S, and conversely.*

Proof. Let Γ be the representation in the above theorem, where e is taken to be the sole idempotent in K. e commutes with all elements of S, since xe and ex are both elements of $K = H(e)$, so that $ex = exe = xe$. Applying Γ to both sides of the equation $ex = xe$, we see that $\Gamma_2(x) = \Gamma_3(x) = 0$ for all x in S. Hence S is isomorphic to $\Gamma_1(S) \times \Gamma_4(S)$.

(1.11) COROLLARY. *If S is a commutative affine semigroup, then S is isomorphic to $A \times G$, where A is an algebra and G is an affine group.*

Proof. A completely simple semigroup whose idempotents commute must be a group (see the remark on p. 127 of [4]). The kernel of S is therefore a group; the result now follows from (1.10).

2. The kernel of an affine semigroup is not in general a linear variety, as was observed by Cohen and Collins [1]. They gave the following example for Φ the field of real numbers.

(2.1) *Example.* Let S be a three-dimensional vector space over a field Φ, with multiplication

$$(x, y, z)(a, b, c) = (x, b, xb),$$

where xb denotes the product of x and b in Φ. This semigroup may be faithfully represented as a semigroup of 3×3 matrices by letting (x, y, z) correspond to

$$\begin{pmatrix} 1 & y & 0 \\ 0 & 0 & 0 \\ x & z & 0 \end{pmatrix}.$$

It is not difficult to see that the kernel K of S is the set of all (x, y, z) such that $z = xy$. K is not however a linear variety, for if it were, it would be a subspace, since $(0, 0, 0) \in K$; but $(1, 0, 0)$ and $(1, 1, 1)$ are elements of K whose sum $(2, 1, 1)$ lies in K if and only if $2 \,.\, 1 = 1$, which is impossible in any field. (If Φ is the field of real numbers, then K forms a hyperbolic paraboloid, which is obviously not a linear variety.)

In this example we have $S = M(K)$, every element of K is idempotent, and $M(K)^2 = K$. We shall see later that the second of these two properties always implies the third. The next example shows, however, that it is not always true that $M(K)^2 = K$.

(2.2) *Example.* Let $\mathfrak{A}$ be a nilpotent algebra of $n \times n$ matrices which contains an element whose index of nilpotency is 5. Assume that the characteristic of Φ is not equal to two. Let S be the set of all $2n \times 2n$ matrices of the form

$$\begin{pmatrix} I_n + A_1 & A_3 \\ A_2 & A_4 \end{pmatrix} \quad (A_i \in \mathfrak{A}).$$

One easily sees that S is an affine semigroup and that the elements of the form $I_n + A_1$ $(A_1 \in \mathfrak{A})$ are a group, since $(I_n + A_1)^{-1} = I_n - A_1 + A_1^2 - A_1^3 + \ldots$. All elements of this group have rank n; hence the minimal rank of elements of S is n. Therefore, by (1.8), an element of S is in the kernel K if and only if $A_4 = A_2(I_n - A_1 + A_1^2 - \ldots)A_3$.

Letting A be an element which is nilpotent of index 5, we have

$$(\pm A)(I_n - A + A^2 - A^3 + A^4)(\pm A) = A^2 - A^3 + A^4.$$

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Hence the two matrices

$$X_1 = \begin{pmatrix} I_n + A & A \\ A & A^2 - A^3 + A^4 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} I_n + A & -A \\ -A & A^2 - A^3 + A^4 \end{pmatrix}$$

both lie in K. It follows that

$$Y = \tfrac{1}{2}X_1 + \tfrac{1}{2}X_2 = \begin{pmatrix} I_n + A & 0 \\ 0 & A^2 - A^3 + A^4 \end{pmatrix}$$

is an element of $M(K)$. Now

$$Y^2 = \begin{pmatrix} (I_n + A)^2 & 0 \\ 0 & A^4 \end{pmatrix},$$

and, since $A^4 \neq 0$, $Y^2 \notin K$. Hence $M(K)^2 \neq K$.

By the same argument, if $\mathfrak{A}$ has an element whose index of nilpotency is $2k+1$, then $M(K)^k \neq K$. It is true, however, that $M(K)^m = K$ if m exceeds the dimensions of $M(K)$. The proof of this fact will be the primary objective of the next five lemmas.

Suppose now that S is any affine semigroup with kernel K. As in (1.5), let e be an idempotent in K and let Γ be a faithful representation of S of degree n over Φ such that (2) and (3) hold. By (1.8) we know that $\Gamma_1(S)$ is a group of non-singular matrices of order k. Let $A = \Gamma_1(S) - I_k$. As noted in (1.7), A is a nilpotent algebra. We set $\Gamma_0(x) = \Gamma_1(x) - I_k$ for all x in S. Then

$$\Gamma_1(x) = I_k + \Gamma_0(x), \quad \Gamma_0(x) \in A, \text{ for all } x \in S. \tag{8}$$

We shall make constant use of this faithful representation Γ and the nilpotent algebra A throughout this section.

(2.3) LEMMA. *Let Γ be the above faithful representation of the affine semigroup S. Then $\Gamma_3(x)\Gamma_2(y) \in A$ for all x, y in S.*

Proof. By (1.6), there exists u in S such that

$$\Gamma(u) = \begin{pmatrix} I_k & \Gamma_3(x) \\ \Gamma_2(y) & 0 \end{pmatrix}.$$

Now clearly $\Gamma_1(u^2) = I_k + \Gamma_3(x)\Gamma_2(y)$, and hence, by (8), we have $\Gamma_3(x)\Gamma_2(y) = \Gamma_0(u^2) \in A$.

(2.4) LEMMA. *Let S be an affine semigroup with kernel K, and let Γ be as above. If $S = M(K)$, then $\Gamma_4(S)$ is a nilpotent algebra.*

Proof. Since $S = M(K)$, $x \in S$ implies that $x = \sum \alpha_i x_i$, where $x_i \in K$ and $\sum \alpha_i = 1$. Hence

$$\Gamma_4(x) = \sum \alpha_i \Gamma_4(x_i) \quad \text{for all } x \text{ in } S. \tag{9}$$

Since $x_i \in K$, by (1.8), we have

$$\Gamma_4(x_i) = \Gamma_2(x_i)\Gamma_1(x_i)^{-1}\Gamma_3(x_i) \tag{10}$$

for each i. Now, by (1.6) and (1.8), there exist u_i and v_i in S such that

$$\Gamma(u_i) = \begin{pmatrix} I_k & 0 \\ \Gamma_2(x_i) & 0 \end{pmatrix} \quad \text{and} \quad \Gamma(v_i) = \begin{pmatrix} \Gamma_1(x_i)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

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for each i. If we set $p_i = u_i v_i$, then we obtain

$$\Gamma_2(p_i) = \Gamma_2(x_i)\Gamma_1(x_i)^{-1}. \tag{11}$$

Let m be the index of nilpotency of A. We will show that $\Gamma_4(S)^{m+1} = 0$. By (9), (10), and (11), for each x in S, $\Gamma_4(x) = \sum \alpha_i \Gamma_2(p_i)\Gamma_2(x_i)$. Hence a product

$$\Gamma_4(x)\Gamma_4(y) \dots \Gamma_4(w)$$

of $m+1$ elements of $\Gamma_4(S)$ will be a sum of terms of the form

$$\Gamma_2(p_i)\Gamma_3(x_i)\Gamma_2(q_j)\Gamma_3(y_j) \dots \Gamma_2(s_1)\Gamma_3(w_1).$$

This may be written

$$\Gamma_2(p_i)a_1 a_2 \dots a_m \Gamma_3(w_1),$$

where $a_1 = \Gamma_3(x_i)\Gamma_2(q_j)$, etc. Each a_t ($t = 1, 2, \dots, m$) is contained in A by (2.3) and, since $A^m = 0$, we may clearly infer that $\Gamma_4(S)^{m+1} = 0$.

(2.5) Lemma. *Let S be an affine semigroup such that $S = M(K)$, where K is the kernel of S. Then some power of every element of S lies in K.*

Proof. Since some power of every element of S belongs to a subgroup of S by (1.3), it suffices to show that every idempotent of S lies in K. Let f be any idempotent of S. Identifying S with $\Gamma(S)$, by (8) and (2.4) we may assume that

$$f = \begin{pmatrix} I_k + A & C \\ B & D \end{pmatrix},$$

where A and D are nilpotent matrices. Since f is idempotent, it follows that

$$AC + CD = 0 \tag{12}$$

and

$$BC + D^2 = D. \tag{13}$$

From (12) we obtain $BA^iC = (-1)^i BCD^i$, for $i = 1, 2, \dots$. By (13), $BC = D - D^2$, and so

$$BA^iC = (-1)^i D^{i+1} + (-1)^{i+1} D^{i+2}. \tag{14}$$

From (14), we find, by summing over i, that

$$B(I_k - A + A^2 - \dots + (-A)^i)C = D - D^{i+2}. \tag{15}$$

Since A is nilpotent, $(I_k + A)^{-1} = I_k - A + A^2 - A^3 + \dots$. Hence, if we let i exceed the index of nilpotency of both A and D in (15), we obtain $B(I_k + A)^{-1}C = D$, and therefore, by (1.8), $f \in K$.

(2.6) Lemma. *Let S be any semigroup and I be an ideal of S. Assume that the following two properties hold:*

(i) *There exists an integer $N > 0$ such that, for any $x_1, x_2, \dots, x_n$ in S, at most N of the following inequalities are proper:*

$$S \supseteq Sx_n \supseteq Sx_{n-1}x_n \supseteq \dots \supseteq Sx_1 x_2 \dots x_n. \tag{16}$$

(ii) *Some power of every element of S is contained in the ideal I.*

Then $S^{N+1} \subseteq I$.

Proof. Let $y = x_1 x_2 \ldots x_n \in S^n$, where $n = N+1$. Then in (16) at least one inequality is an equality. For convenience let $z = x_{n-i}x_{n-i+1} \cdots x_n$, and assume that $Sz = Sx_{n-i-1}z$. Then $x_{n-i-1}z = sx_{n-i-1}z$ for some s in S; hence $x_{n-i-1}z = s^j x_{n-i-1}z$ for $j = 1, 2, \ldots$. By (ii), s^j belongs to I for some j; hence $x_{n-i-1}z \in I$. Consequently $y = x_1 x_2 \ldots x_{n-i-1}z$ is contained in I.

(2.7) THEOREM. *If S is an affine semigroup with kernel K, then $M(K)^n = K$ if $n > \dim M(K)$.*

Proof. Apply (2.6) with $I = K$ and $S = M(K)$. Since, for every y in S, Sy is a linear variety, the finite dimensionality of S clearly implies that (i) is satisfied with $N = \dim M(K)$. (ii) holds by (2.5).

Let $E = E_S$ denote the set of idempotents of the semigroup S.

(2.8) THEOREM. *If S is an affine semigroup with kernel $K \subseteq E$, then $M(K)^2 = K$.*

Proof. Let e be an idempotent in K and let Γ be the faithful representation used in (2.3) and (2.4). Since $K \subseteq E$ and $\Gamma_1(S)$ is isomorphic to a subgroup of K, we must have $\Gamma_1(x) = I_k$ for all x in S. Hence $A = \Gamma_1(S) - I_k = 0$ and, by 2.3, $\Gamma_3(y)\Gamma_2(z) = 0$ for all $y, z \in S$.

Now if $x \in M(K)$, $x = \sum \alpha_i x_i$, with $x_i \in K$, and $\sum \alpha_i = 1$. Thus $\Gamma_4(x) = \sum \alpha_i \Gamma_4(x_i)$. By (1.8), $\Gamma_4(x_i) = \Gamma_2(x_i)\Gamma_3(x_i)$. Hence $\Gamma_4(x) = \sum \alpha_i \Gamma_2(x_i)\Gamma_3(x_i)$. Since $\Gamma_3(y)\Gamma_2(z) = 0$ for all y, z in S, we have $\Gamma_4(x)\Gamma_2(y) = 0$, $\Gamma_3(y)\Gamma_4(x) = 0$ and $\Gamma_4(x)\Gamma_4(y) = 0$ for all x, y in $M(K)$. Therefore, by multiplying the partitioned matrices $\Gamma(x)$ and $\Gamma(y)$, one finds that

$$\Gamma(xy) = \begin{pmatrix} I_k & \Gamma_3(y) \\ \Gamma_2(x) & \Gamma_2(x)\Gamma_3(y) \end{pmatrix}.$$

This shows that $\Gamma_4(xy) = \Gamma_2(xy)\Gamma_3(xy)$, which implies that $xy \in K$.

(2.9) LEMMA. *If $S \subseteq \mathscr{L}V$ ($\dim V < \infty$) is a semigroup (not necessarily affine) with a completely simple kernel K, then K is the set of elements of minimal rank in S.*

Proof. (See [**9**]).

If T is a compact convex subsemigroup of an affine semigroup on a locally convex linear topological space, then the kernel K of T is contained in $E = E_S$, the set of idempotents of S [**1**, Theorem 3]. The following theorem shows that this is also true for the affine semigroup $M(T)$ spanned by T.

(2.10) THEOREM. *Let S be an affine semigroup and let T be a subsemigroup of S such that $S = M(T)$. Suppose also that T has a kernel $K_T \subseteq E_T$. Then, denoting the kernel of S by K_S, we have*

$$K_S = M(K_T) \cap E_S$$

and

$$K_T = T \cap K_S.$$

Proof. Since K_T is a union of groups, by (2.9), K_T is the set of elements of T of minimal rank. Let $k = \rho(K_T)$. We shall show that $k \leqq \rho(x)$ for all x in S and hence, by (1.4), K_S is the set of elements in S of rank k.

Let e be an idempotent in K_T and choose a basis so that

$$e = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, since $eTe \subseteq K_T \subseteq E_T$, it follows easily that every element of T has the form

$$\begin{pmatrix} I_k & B \\ A & C \end{pmatrix}.$$

An affine combination of elements of this form will be of this form also. Therefore every element of $M(T) = S$ has rank no less than k, the rank of I_k. It is now clear that $K_T \subseteq K_S$.

Since $M(K_T)$ is an ideal of S, $K_S \subseteq M(K_T)$. To show that $K_S \subseteq E_S$, let $x \in H(e, K_S)$ for some e in K_T. Since $K_S \subseteq M(K_T)$ and $K_T \subseteq E_T$, we must have $x = \sum \alpha_i e_i$, where $\sum \alpha_i = 1$ and $e_i = e_i^2 \in K_T$. Now, since $e \in K_T$, $ee_i e = e$ for all i, and so

$$x = exe = \sum \alpha_i e e_i e = \sum \alpha_i e = e.$$

Therefore $H(e, K_S) = \{e\}$, if $e \in K_T$. Since all groups of K_S are isomorphic, we have $K_S \subseteq E_S$. Hence $K_S \subseteq M(K_T) \cap E_S$.

From the fact that $K_T \subseteq K_S \subseteq M(K_T)$, we have $M(K_T) \subseteq M(K_S) \subseteq M(K_T)$, and so

$$M(K_T) = M(K_S).$$

By (2.8), $M(K_S)^2 = K_S$; hence $M(K_T)^2 = K_S$ and

$$M(K_T) \cap E_S = (M(K_T) \cap E_S)^2 \subseteq M(K_T)^2 = K_S.$$

Therefore $M(K_T) \cap E_S = K_S$.

That $K_T = T \cap K_S$ is clear since both K_T and K_S consist of the elements of minimal rank in their respective semigroups. This completes the proof.

If $x, y \in V$ or $\mathscr{L}V$, we shall denote the 1-dimensional linear variety generated by x and y by $M(x, y)$. $M(x, y)$ consists of all elements $\alpha x + (1-\alpha)y$, where $\alpha \in \Phi$.

The following two lemmas are due to H. S. Collins [2].

(2.11) Lemma. *If S is an affine semigroup over a field of cardinality greater than two, then, for $e, f \in E$, the following are equivalent:*

(i) $M(e,f) \subseteq E$;

(ii) $e+f = ef+fe$;

(iii) $M(e,f) \cap E \neq \{e,f\}$.

(2.12) Lemma. *If S is an affine semigroup, and $M(e,f) \subseteq E$, then $gM(e,f)g = g$ for all g in $M(e,f)$.*

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(2.13) THEOREM. *Let S be an affine semigroup over a field of characteristic different from two with kernel $K \subset E$. If $K \neq M(K)$, then $M(K)$ contains a semigroup isomorphic to the semigroup T on $\Phi \oplus \Phi \oplus \Phi$ with multiplication*

$$(x, y, z)(a, b, c) = (x, b, xb),$$

where xb denotes multiplication in Φ (see (2.1)).

Proof. A linear variety in a vector space V over a field of characteristic different from two may be characterized as a subset M of V such that $x, y \in M$ implies that $M(x, y) \subseteq M$.

Thus, since $K \neq M(K)$, there exist $e, f \in K \subseteq E$ such that $M(e, f)$ is not contained in K. Let $M = M(e, f, ef, fe)$. We shall show that M is a semigroup isomorphic to T. It is known [**4**, p. 77] that $gKg = H(g)$ for $g \in K \cap E$ and, since $K \subseteq E$, it is clear that $gKg = g$ for all $g \in K$. In particular $efe = e$ and $fef = f$. It is now apparent that $\{e, f, ef, fe\}$ is a semigroup. We shall now show that $\{e, f, ef, fe\}$ is an affine basis (see [**1**, p. 102]) for M. Note first that $M(e,f) \cap E = \{e, f\}$; for if not, by (2.11) and (2.12,) for all g in $M(e,f)$ we have $g = geg \in K$, contradicting the fact that $M(e,f)$ is not contained in K. Hence, if $ef \in M(e,f)$, then $ef \in M(e,f) \cap K$, which is contained in $M(e,f) \cap E = \{e, f\}$; this implies that $ef = e$ or $ef = f$. If $ef = e$, then $fe = fef = f$, which implies that $M(e,f)$ is a left zero semigroup and is therefore contained in K since it meets K. Thus the supposition that $ef = e$ leads to a contradiction. Similarly, $ef = f$ is impossible. Consequently, $ef \notin M(e,f)$.

Suppose however that $fe \in M(e, f, ef)$. Then $fe = \alpha e + \beta f + \gamma ef$, where $\alpha + \beta + \gamma = 1$; hence $fe = f^2 e = \alpha fe + \beta f^2 + \gamma fef = \alpha fe + (\beta + \gamma) f$, which implies that $(1-\alpha) fe = (\beta + \gamma) f$. As above, fe cannot be equal to f; hence $1 - \alpha = 0$. Now $fe = e + \beta f - \beta ef$. Multiplying this equation on the right by e and repeating the above procedure, we find that $\beta = 1$. Therefore $fe = e + f - ef$ and so $e + f = ef + fe$, which implies, by (2.11), that $M(e,f) \subseteq E$, a contradiction. Thus fe is not contained in $M(e, f, ef)$ and it follows that $\{e, f, ef, fe\}$ is an affine basis for M.

One may see without difficulty that the set $P = \{(1, 0, 0), (0, 1, 0), (1, 1, 1), (0, 0, 0)\}$ is an affine basis for the affine semigroup T. Moreover, P is a semigroup isomorphic to the semigroup $\{e, f, ef, fe\}$ under the correspondence:

$$e \longleftrightarrow (1, 0, 0),$$

$$f \longleftrightarrow (0, 1, 0),$$

$$ef \longleftrightarrow (1, 1, 1),$$

$$fe \longleftrightarrow (0, 0, 0).$$

It is now obvious that M is isomorphic to T. This completes the proof.

3. Appendix. In [**1**], Haskell Cohen and H. S. Collins introduced the notion of an affine semigroup in a locally convex real linear topological space V. They defined an affine semigroup to be a *convex* subset S of V endowed with an associative product for which the translations $x \to xy$ and $x \to yx$ are affine mappings for all y in S. Thus in the sense of Cohen and

Collins an affine semigroup need not be a linear variety, but merely a convex set. Furthermore, it is not obvious from their definition that an affine semigroup may always be embedded in an algebra. They did point out, however, that the multiplication on S can always be extended uniquely to the linear variety $M(S)$ generated by S so that the affine property of the multiplication holds throughout $M(S)$.

We shall show here that an affine semigroup (in the sense of Cohen and Collins) may always be embedded in a uniquely determined enveloping algebra. To do this it is clear from the above remarks that we need only consider affine semigroups which are linear varieties. Hence by an *affine semigroup* S we shall henceforth mean a linear variety S (see (0.2) above) in a possibly infinite-dimensional vector space over an arbitrary field Φ which is endowed with an associative product whose left and right translates are affine mappings. (The reader will note that, in these more general terms, the main body of this paper is concerned with finite-dimensional affine semigroups which, due to (3.3) below, are merely linear varieties of matrices which are closed under matrix multiplication.)

In what follows we shall at most suggest proofs, most of which are similar in nature to those of Theorem 7 and Theorem 8 of [**1**].

If S is an affine semigroup with zero z, one may without difficulty see that the vector space $S-z$ may be endowed with the structure of an algebra isomorphic to S under the affine mapping $x \to x-z$, thus obtaining

(3.1) LEMMA. *If S is an affine semigroup with zero, then S is isomorphic to the multiplicative semigroup of an algebra.*

Clearly we may without loss of generality assume that S does not contain the origin of the containing vector space V. As in the proof of Theorem 7 of [**1**], one may then easily extend the multiplication on S to the vector space $[S]$ spanned by S in V to obtain the following result:

(3.2) LEMMA. *Any affine semigroup S can be embedded in an algebra $[S]$, unique to within isomorphism, such that $0 \notin S$ and S generates $[S]$.*

(3.3) COROLLARY. *If S is an affine semigroup of finite dimension n, then S can be faithfully represented as a semigroup of matrices of degree $n+2$ with entries from the field Φ.*

(3.4) The algebra $[S]$ of (3.2) will be referred to as the *enveloping algebra* of S. If the affine semigroup S is isomorphic to a subsemigroup S' of an algebra A and if $0 \notin S'$, then $[S]$ is clearly isomorphic to the subalgebra generated by S' in A.

We shall say that a proper two-sided ideal M in an algebra A over Φ is a *hyper-ideal* if there exists an element e in A such that e and M generate A and e is an identity for A modulo M. Note that an ideal M is a hyper-ideal of A if and only if A/M is isomorphic to Φ.

It is now easy to deduce from the above the following theorem:

(3.5) THEOREM. *If M is a hyper-ideal of the algebra A and e is an identity for A modulo M, then $M+e$ is a subsemigroup of the multiplicative semigroup of A and hence an affine semigroup.*

Conversely, if S is any affine semigroup, then there exist A, M and e as above such that S is isomorphic to $M+e$. Moreover, A and M are unique to within isomorphism. We may, in fact, take $A=[S]$, e any element of S, and $M=S-e$.

If A is any algebra, a new multiplication $\circ$ may be defined on A by setting $a \circ b = a+b-ab$ for all $a, b \in A$, where ab denotes the multiplication in A. This is the so-called *circle composition*. One may quickly verify that $(A, \circ)$ is an affine semigroup with identity the zero of A.

(3.6) COROLLARY. *If an affine semigroup S has an identity, then S is isomorphic to $(A, \circ)$ for some algebra A.*

An algebra A over Φ, together with a Φ-epimorphism $\phi : A \to \Phi$ is called a *supplemented algebra* [**10**, p. 182]. The epimorphism ϕ is called the *augmentation epimorphism*. In this terminology Theorem (3.5) becomes

(3.7) THEOREM. *If A is a supplemented algebra over Φ with augmentation epimorphism ϕ, then $S = \phi^{-1}(1)$ is an affine semigroup.*

Conversely, if S is any affine semigroup, then there exists a supplemented algebra A and an augmentation epimorphism ϕ such that S is isomorphic to $\phi^{-1}(1)$.

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